How to Determine the Boundary Condition of a Strongly Degenerate Parabolic Equation

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Abstract: By reviewing Fichera-Oleǐnik theory, the portion of the boundary on which we should give the boundary value is determined, the corresponding initial-boundary value problem of the strongly degenerate parabolic equation

$$\frac{\partial u}{\partial t} = \Delta A(u) + \mathrm{div}(b(u,x,t)), \ \ (x,t) \in \Omega \times (0,T),$$

is considered. By introducing a new kind of entropy solution, we are able to get the existence and the stability of the solutions.

Key–Words: Initial-boundary value problem, boundary condition, Fichera-Oleǐnik theory, entropy solution, existence, stability.

1 Introduction

It is well-known that Tricomi [1] elicited interest in the general study of elliptic equations degenerating on the boundary of the domain firstly. Then Keldyš [2] made a great progress in developing the theory, he brought to light the fact that in the case of elliptic equations degenerating on the boundary, under definite assumptions, a portion of the boundary may be free from the prescription of boundary conditions. Later, Fichera [3-4] and Oleǐnik [5-6] developed and perfected the general theory of second order equation with nonnegative characteristic form, which, in particular contains those degenerating on the boundary. We can call the theory as Fichera-Oleinik theory in what follows. In details, if one wants to consider the boundary value problem of a linear degenerate elliptic equation,

$$\sum_{r,s=1}^{N+1} a^{rs}(x)\frac{\partial^2 u}{\partial x_r \partial x_s} + \sum_{r=1}^{N+1} b_r(x)\frac{\partial u}{\partial x_r} + c(x)u$$
$$= f(x), x \in \widetilde{\Omega} \subset \mathbb{R}^{N+1} \tag{1}$$

it needs and only needs to give partial boundary condition. Let us give a briefly explanation. Let $\{n_s\}$ be the unit inner normal vector of $\partial\widetilde{\Omega}$ and denote that

$$\Sigma_2 = \{x \in \partial\widetilde{\Omega} : a^{rs}n_r n_s = 0, (b_r - a^{rs}_{x_s})n_r < 0\}, \tag{2}$$

$$\Sigma_3 = \{x \in \partial\widetilde{\Omega} : a^{rs}n_s n_r > 0\}. \tag{3}$$

Then, to ensure the posedness of equation (1), Fichera-Oleinik theory tells us that the suitable boundary condition is

$$u|_{\sum_2 \bigcup \sum_3} = g(x). \tag{4}$$

In particular, if the matrix (a^{rs}) is positive definite, (4) is just the usual Dirichlet boundary condition.

In our paper, we shall use Fichera-Oleǐnik theory to consider equation of the form

$$\frac{\partial u}{\partial t} = \Delta A(u) + \mathrm{div}(b(u,x,t)), \ \ (x,t) \in Q_T, \tag{5}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain and the boundary $\partial\Omega = \Sigma$ is appropriately smooth, $Q_T = \Omega \times (0,T)$, and

$$A(u) = \int_0^u a(s)ds, \ \ a(s) \geq 0,$$
$$a(0) = 0. \tag{6}$$

If we want to consider the initial-boundary value problem of (5), the initial value condition is always necessary

$$u(x,0) = u_0(x), \ x \in \Omega. \tag{7}$$

But whether can we require the Dirichlet homogeneous boundary condition

$$u(x,t) = 0, \ (x,t) \in \partial\Omega \times (0,T) = \Sigma \times (0,T), \tag{8}$$

as usual?

For example, let us consider a simple equation

$$u_t = \triangle A(u(x,t)), (x,t) \in \Omega \times (0,T), \quad (9)$$

with the existence of A^{-1}. In other words, equation (9) is weakly degenerate, let $v = A(u), u = A^{-1}(v)$. Then

$$\triangle v - (A^{-1}(v))_t = 0. \quad (10)$$

According to Fichera-Oleinik theory, we know that we can give the Dirichlet homogeneous boundary condition (8). But, if equation (9) is strongly degenerate, then A^{-1} is not existential, we can not deal with it as equation (10).

For another example, let us consider the following equation

$$w^2 w_{\eta\eta} - w_\tau - \eta U w_\xi + A w_\eta + Bw = 0,$$

$$(\xi, \eta, \tau) \in \Omega \times (0,T), \quad (11)$$

which arises in the boundary layer theory of Prandtl system, where A, B are two known functions, one can refer to [7] for details. Clearly, this is a strongly degenerate parabolic equation, we also can not give the boundary condition (8) generally. In fact,Oleinik considered the domain $\Omega = \{0 < \tau < T, 0 < \xi < X, 0 < \eta < 1\}$, then she compared equation (11) with equation (1), and quoted the partial boundary condition of equation (11) as following

$$w|_{\tau=0} = w_0(\xi,\eta), w|_{\eta=1} = 0,$$

$$(\nu w w_\eta - v_0 w + c(\tau,\xi))|_{\eta=0} = 0, \quad (12)$$

where ν is the viscous coefficient, v_0 and $c(\tau,\xi)$ are known functions.

Now, we can rewrite equation (5) as

$$\frac{\partial u}{\partial t} = a(u)\Delta u + a'(u)|\nabla u|^2 + b_i'(u,x,t)\frac{\partial u}{\partial x_i}$$

$$+\frac{\partial b_i(u,x,t)}{\partial x_i}, \text{ in } Q_T, \quad (13)$$

where $b_i'(u,x,t) = \frac{\partial b_i(u,x,t)}{\partial u}$. Noticing that the domain is a cylinder $\Omega \times (0,T)$, if we let $t = x_{N+1}$ and regard the strongly degenerate parabolic equation (13) as the form of a "linear" degenerate elliptic equation as follows: when $i,j = 1,2,\cdots,N$, $a^{ii}(x,t) = a(u(x,t))$, $a^{ij}(x,t) = 0, i \neq j$, then

$$(\widetilde{a}^{rs})_{(N+1)\times(N+1)} = \begin{pmatrix} a^{ij} & 0 \\ 0 & 0 \end{pmatrix}.$$

If $a(0) = 0$, then equation (13) is not only strongly degenerate in the interior of Ω, but also on the boundary $\partial\Omega$. Now, $\sum_3$ is an empty set. Whereas

$$\widetilde{b}_s(x,t) = \begin{cases} b_i'(u,x,t) + a'(u)\frac{\partial u}{\partial x_i}, & 1 \le s \le N, \\ -1, & s = N+1, \end{cases}$$

Under this observation, according to Fichera-Oleǐnik theory, the initial value (7) is always needed, but on the lateral boundary $\partial\Omega \times (0,T)$, by $a(0) = 0$, the part of boundary on which we should impose the boundary value is

$$\Sigma_p = \{x \in \partial\Omega : (b_i'(0,x,t)$$

$$+a'(0)\frac{\partial u}{\partial x_i}|_{x\in\partial\Omega} - a'(0)\frac{\partial u}{\partial x_i}|_{x\in\partial\Omega})n_i < 0\}$$

$$= \{x \in \partial\Omega : b_i'(0,x,t)n_i < 0\}. \quad (14)$$

where $\{n_i\}$ be the unit inner normal vector of $\partial\Omega$.

However, the above calculations is just in form. Due to the strongly degenerate property of a, (13) generally only has a weak solution. In our paper, we consider the solution of (13) in BV sense, which is a kind of weak solution, and we can not define the trace of $\frac{\partial u}{\partial x_i}$ on $\partial\Omega$, it means that we can not define

$$\Sigma_p = \{x \in \partial\Omega : (b_i'(0,x,t) + a'(0)\frac{\partial u}{\partial x_i}|_{x\in\partial\Omega}$$

$$-a'(0)\frac{\partial u}{\partial x_i}|_{x\in\partial\Omega})n_i < 0\},$$

too. Fortunately, only if $b_i(s,x,t)$ is derivable, then

$$\Sigma_p = \{x \in \partial\Omega : b_i'(0,x,t)n_i < 0\}. \quad (15)$$

has a definite sense. We will show that, to assure the posedness of the solutions to the strongly degenerate parabolic equation (5), only a partial boundary condition is needed, and actually, we will show that Σ_p defined in (15) is just the portion can be imposed the boundary value condition in some weak sense. This is the most contribution of our paper.

2 The definition of the entropy solution

The paper is to investigate the solvability of equation (5)(equivalently, (13)) in $BV(Q_T)$. The definition of BV function and its properties are to be specified in the next section. It is well known that the BV functions are the weakest functions which have the traces.

The existence of the solution will be obtained as a limit point of the family $\{u_\varepsilon\}$ of solutions of regularized problem

$$\frac{\partial u}{\partial t} = \Delta A(u) + \varepsilon\Delta u + \frac{\partial b_i(u,x,t)}{\partial x_i}, \text{ in } Q_T, \quad (16)$$

with initial-boundary value conditions (7)-(8).

Since for the limit function u of certain subsequence of $\{u_\varepsilon\}$, $\widehat{a(u)}\frac{\partial u}{\partial x_j}$ generally can not define the trace $\gamma(\widehat{a(u)}\frac{\partial u}{\partial x_j})$ on $\sum$, we have to make a detour to avoid $\gamma(\widehat{a(u)}\frac{\partial u}{\partial x_j})$ in defining the solution of equation (5), where $\widehat{a(u)}$ is the composite means function of BV function $a(u)$, its definition is to be put forward in the next section.

Let $S_\eta(s) = \int_0^s h_\eta(\tau)d\tau$ for small $\eta > 0$. Here $h_\eta(s) = \frac{2}{\eta}(1-\frac{|s|}{\eta})_+$. Obviously $h_\eta(s) \in C(\mathbb{R})$, and

$$h_\eta(s) \geq 0, \ \mid sh_\eta(s) \mid \leq 1, \ \mid S_\eta(s) \mid \leq 1;$$
$$\lim_{\eta\to 0} S_\eta(s) = \text{sgn}s, \ \lim_{\eta\to 0} sS'_\eta(s) = 0. \quad (17)$$

For any $\eta > 0$, any given $t \in (0,T)$, let

$$\Sigma_{1\eta k} = \{x \in \Sigma, \ S_\eta(k)[b_i(0,x,t) -b_i(k,x,t)]n_i(x) > 0\}, \quad (18)$$

$$\Sigma_{2\eta k} = \{x \in \Sigma, \ S_\eta(k)[b_i(0,x,t) -b_i(k,x,t)]n_i(x) \leq 0\}, \quad (19)$$

here and in what follows, $\{n_i\}_{i=1}^N$ is the inner normal vector of Ω. Clearly, $\Sigma = \Sigma_{1\eta k}\bigcup\Sigma_{2\eta k}$. Let

$$\Sigma_1 = \bigcup_{\forall\eta\geq 0,\forall k\in\mathbb{R}} \Sigma_{1\eta k}. \quad (20)$$

Instead of the usual Dirichlet homogeneous boundary value condition (8), the homogeneous boundary condition we use in what follows is

$$\gamma u \mid_{\Sigma_1\times(0,T)} = 0. \quad (21)$$

In fact, by the definition of $\Sigma_{1\eta k}$, we know that

$$0 < S_\eta(k)[b_i(0,x,t) - b_i(k,x,t)]n_i(x) =$$
$$-kS_\eta(k)b'_i(\zeta,x,t)n_i(x),$$

where $\zeta \in (k,0)$. If we let $\eta \to 0$. Then

$$b'_i(\zeta,x,t)n_i(x) < 0.$$

Let $k \to 0$. We know that

$$b'_i(0,x,t)n_i(x) < 0, \quad (22)$$

which is in accordance with that (14).

At the same time, if the equation (5) is completely degenerate, $A(u) \equiv 0$, then it becomes the conservation law equation, and it is well known that only under the suitable entropy condition, the uniqueness of the solutions is true. By this fact, combining some ideas of references [8-9], we give the following definition of the entropy solution.

Definition 1 *A function u is said to be the entropy solution of equation (5) with the initial value (7) and the homogeneous boundary condition (21), if*

1. u satisfies that

$$u \in BV(Q_T)\cap L^\infty(Q_T), \ \frac{\partial}{\partial x_i}\int_0^u \sqrt{a(s)}ds \in L^2(Q_T).$$

2. For any $\varphi_1, \ \varphi_2 \in C^2(\overline{Q}_T)$, $\varphi_1 \geq 0$, $\nabla\varphi_1 \mid_{\sum} = 0$, $\varphi_1 \mid_{\partial\Omega\times[0,T]} = \varphi_2 \mid_{\partial\Omega\times[0,T]}$, and supp$\varphi_2$, supp$\varphi_1 \subset \overline{\Omega}\times(0,T)$, for any $k \in \mathbb{R}$, for any small $\eta > 0$, u satisfies

$$\iint_{Q_T}[I_\eta(u-k)\varphi_{1t} - B^i_\eta(u,x,t,k)\varphi_{1x_i} + A_\eta(u,k)\Delta\varphi_1$$
$$-S'_\eta(u-k) \mid \nabla\int_0^u \sqrt{a(s)}ds \mid^2 \varphi_1]dxdt$$
$$+S_\eta(k)\iint_{Q_T}[u\varphi_{2t} - (b_i(u,x,t) - b_i(0,x,t))\varphi_{2x_i}$$
$$+\frac{\partial b_i(0,x,t)}{\partial x_i}\varphi_2 + A(u)\Delta\varphi_2]dxdt$$
$$+S_\eta(k)\int_0^T\int_{\Sigma_{1\eta k}}[b_i(0,x,t) - b_i(k,x,t)]n_i\varphi_1 dtd\sigma \geq 0. \quad (23)$$

3. In the sense of the trace,

$$\gamma u \mid_{\Sigma_1\times(0,T)} = 0. \quad (24)$$

4. The initial condition is satisfied as follows

$$\lim_{t\to 0}\int_\Omega \mid u(x,t) - u_0(x) \mid dx = 0. \quad (25)$$

Here the pairs of equal indices imply a summation from 1 up to N, and

$$B^i_\eta(u,x,t,k) = \int_k^u b'_i(s,x,t)S_\eta(s-k)ds, \ A_\eta(u,k)$$
$$= \int_k^u a(s)S_\eta(s-k)ds, I_\eta(u-k)$$
$$= \int_0^{u-k} S_\eta(s)ds.$$

To explain the reasonableness of Definition 1, in one way, if equation (5) has a classical solution u. Multiplying (5) by $\varphi_1 S_\eta(u-k)$ and integrating over Q_T, we are able to show that u satisfies Definition 1. In another way, let $\eta \to 0$ in (23). One has

$$\iint_{Q_T} [|u-k|\varphi_{1t}$$
$$-\text{sgn}(u-k)(b_i(u,x,t)-b_i(k,x,t))\varphi_{1x_i}$$
$$+\text{sgn}(u-k)(A(u)-A(k))\Delta\varphi_1]dxdt$$
$$+\text{sgn}(k)\iint_{Q_T}(u\varphi_{2t}-(b_i(u,x,t)-b_i(0,x,t))\varphi_{2x_i}$$
$$+\frac{\partial b_i(0,x,t)}{\partial x_i}\varphi_2+A(u)\Delta\varphi_2)dxdt$$
$$-S_\eta(k)\int_0^T\int_{\Sigma_{1k\eta}}[(b_i(0,x,t)-b_i(k,x,t)]n_i\varphi_1 dtd\sigma \geq 0.$$

and let $\varphi_2 = 0$ and so $\varphi_1 |_\Sigma = 0$.

$$\iint_{Q_T} [|u-k|\varphi_{1t}$$
$$-\text{sgn}(u-k)(b_i(u,x,t)-b_i(k,x,t))\varphi_{1x_i}$$
$$+\text{sgn}(u-k)(A(u)-A(k))\Delta\varphi_1]dxdt \geq 0. \quad (26)$$

Thus if u is the entropy solution in Definition 1, then u is a entropy solution defined in [10],[11], [12] et al. In fact, the author has been interested in the posedness of the solutions to the strong degenerate parabolic equations for a long time, one can refer to [22-28].

Certainly, we have also noticed that Kobayasi and Ohwa [13] studied the well-posedness for anisotropic degenerate parabolic equations (1) with inhomogeneous boundary condition on a bounded rectangle by using the kinetic formulation which was introduced in [14]. Li and Wang [15] considered the entropy solutions of the homogeneous Dirichlet boundary value problem of (23) in an arbitrary bounded domain. Since the entropy solutions defined in [13], [15] are only in L^∞ space, the existence of the traditional trace (see Remark 8 in what follows), which was called the strong trace in [15], on the boundary is not guaranteed, the appropriate definition of entropy solutions are quoted, and the trace of the solution on the boundary is defined in an integral formula sense, which was called the weak trace in [17]. So, not only Definition 1 is different from the definitions of entropy solutions in [13], [15], but also the trace of the solution in our paper is in the traditional way.

The main results of our paper are the following theorems.

Theorem 2 *Suppose that $A(s)$ and $b_i(s,x,t)$ are smooth enough, and $u_0(x) \in L^\infty(\Omega)$, and suppose that*

$$A'(0) = a(0) = 0.$$

Then equation (5) with the initial-boundary value conditions (7)(21) has a entropy solution in the sense of Definition 1.

Theorem 3 *Suppose that $A(s)$, $b_i(s,x,t)$ is smooth enough. If $\Sigma_1 \neq \emptyset$ is a subset of Σ, let u, v be solutions of equation (1) with the different initial values $u_0(x)$, $v_0(x) \in L^\infty(\Omega)$ respectively. Suppose that*

$$\gamma u(x,t) = f(x,t),\ \gamma v = g(x,t),\ (x,t) \in \Sigma\times(0,T), \quad (27)$$

and in particular,

$$\gamma u = \gamma v = 0,\ x \in \Sigma_1. \quad (28)$$

Suppose that the distance function $d(x) = dist(x,\Sigma) < \lambda$ satisfies that

$$|\triangle d| \leq c, x \in \Omega \setminus \Omega_\lambda, \quad (29)$$

where λ is a small enough constant, and $\Omega_\lambda = \{x \in \Omega, d(x,\partial\Omega) > \lambda\}$. Then

$$\int_\Omega |\ u(x,t)-v(x,t)\ |\ dx \leq \int_\Omega |\ u_0 - v_0\ |\ dx$$
$$+ess \sup_{(x,t)\in\Sigma_2\times(0,T)} |f(x,t)-g(x,t)|, \quad (30)$$

where $(x,t) \in \mathbb{R}^{N+1}$, $ess\sup_{(x,t)\in\Sigma_2\times(0,T)} |f(x,t)-g(x,t)|$ is in the sense of $N-$dimensonal Hausdorff measure.

3 BV function

Let us first introduce the concept of BV function according to ref. [16].

Definition 4 *Let $\Omega \subset \mathbb{R}^m$ be an open set and let $f \in L^1(\Omega)$. Define*

$$\int_\Omega |Df| = \sup\{\int_\Omega f divg dx :$$

$$g = (g_1, g_2, \cdots, g_N) \in C_0^1(\Omega;\mathbb{R}^m), |g(x)| \leq 1, x \in \Omega\},$$

where $divg = \sum_{i=1}^m \frac{\partial g_i}{\partial x_i}$.

Definition 5 *A function of $f \in L^1(\Omega)$ is said to have bounded variation in Ω if*

$$\int_\Omega |Df| < \infty.$$

We define $BV(\Omega)$ as the space of all functions in $L^1(\Omega)$ with bounded variation.

This is equivalent to that the generalized derivatives of every function in $BV(\Omega)$ are regular measures on Ω. Under the norm

$$\|f\|_{BV} = \|f\|_{L^1} + \int_\Omega |Df|,$$

$BV(\Omega)$ is a Banach space.

Proposition 6 (Semicontinuity) *Let $\Omega \subseteq \mathbb{R}^m$ be an open set and $\{f_j\}$ a sequence of functions in $BV(\Omega)$ which converge in $L^1_{loc}(\Omega)$ to a function f. Then*

$$\int_\Omega |Df| \leq \lim_{j\to\infty} \inf \int_\Omega |Df_j|.$$

Proposition 7 (Integration by part) *Let*

$$C_R^+ = \mathscr{B}(0,R) \times (0,R) = \mathscr{B}_R \times (0,R)$$

and $f \in BV(C_R^+)$. Then there exists a function $f^+ \in L^1(\mathscr{B}_R)$ such that for H_{n-1}-almost all $y \in \mathscr{B}_R$,

$$\lim_{\rho\to 0} \rho^{-m} \int_{C_\rho^+(y)} |f(z) - f^+(y)| dz = 0.$$

Moreover, if $C_R = \mathscr{B}_R \times (-R,R)$, then for every $g \in C_0^1(C_R;\mathbb{R}^m)$,

$$\int_{C_R^+} f divg dx = -\int_{C_R^+} \langle g, Df\rangle + \int_{\mathscr{B}_R} f^+ g dH_{n-1},$$

where $\mathscr{B}_\rho = \{x \in \mathbb{R}^m; \mid x \mid < \rho\}$.

Remark 8 *The function f^+ is called the trace of f on $\mathscr{B}_R$ and obviously*

$$f^+(y) = \lim_{\rho\to 0} \frac{1}{|C_\rho^+(y)|} \int_{C_\rho^+(y)} f(z)dz.$$

In our paper, we consider the solution of equation (5) in $BV(Q_T)$, where $Q_T = \Omega \times (0,T)$, Ω is bounded domain, and the dimension of Q_T is $m = N+1$.

Let Γ_u be the set of all jump points of $u \in BV(Q_T)$, v the normal of Γ_u at $X = (x,t)$, $u^+(X)$ and $u^-(X)$ the approximate limits of u at $X \in \Gamma_u$ with respect to $(v, Y-X) > 0$ and $(v, Y-X) < 0$ respectively. For continuous function $p(u,x,t)$ and $u \in BV(Q_T)$, define

$$\widehat{p}(u) = \int_0^1 p(\tau u^+ + (1-\tau)u^-)d\tau, \tag{31}$$

which is called the composite mean value of p. For a given t, we denote Γ_u^t, $H^t, (v_1^t, \cdots, v_N^t)$ and $u_\pm^t$ as all jump points of $u(\cdot,t)$, Housdorff measure of Γ_u^t, the unit normal vector of Γ_u^t, and the asymptotic limit of $u(\cdot,t)$ respectively. Moreover, if $f(s) \in C^1(\mathbb{R})$, $u \in BV(Q_T)$, then $f(u) \in BV(Q_T)$ and

$$\frac{\partial f(u)}{\partial x_i} = \widehat{f'}(u)\frac{\partial u}{\partial x_i}, \ i = 1,2,\cdots,N,N+1, \tag{32}$$

where $x_{N+1} = t$ as usual.

To obtain the uniqueness of the solutions, we need the following lemma.

Lemma 9 *Let u be a solution of (5). Then*

$$a(s) = 0, \ s \in I(u^+(x,t), u^-(x,t)), \tag{33}$$

a.e. on Γ_u, where $I(\alpha,\beta)$ denote the closed interval with endpoints α and β, and (33) is in the sense of Hausdorff measure $H_N(\Gamma_u)$.

We can prove this lemma as that of Lemma 2 in [8], we omit the details here.

4 The existence of the solution

Lemma 10[19] *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and let $f_k, f \in L^q(\Omega)$, as $k \to \infty$, $f_k \rightharpoonup f$ weakly in $L^q(\Omega), 1 \leq q < \infty$. Then*

$$\lim_{k\to\infty} \inf \ \| f_k \|^q_{L^q(\Omega)} \geq \| f \|^q_{L^q(\Omega)} . \tag{34}$$

Consider the following regularized problem

$$\frac{\partial u}{\partial t} = \Delta A(u) + \varepsilon\Delta u + \frac{\partial b_i(u,x,t)}{\partial x_i}, \text{ in } Q_T, \tag{35}$$

with the initial-boundary value conditions (7)-(8). It is well known that there are classical solutions $u_\varepsilon \in C^2(\overline{Q_T}) \bigcap C^3(Q_T)$ of this problem provided that A, b_i satisfy the assumptions in Theorem 2, one can refer to the reference [20] or the eighth chapter of [21] for this fact.

Firstly, since $u_0(x) \in L^\infty(\Omega)$, by the maximum principle, we have

$$\mid u_\varepsilon \mid \leq \| u_0 \|_{L^\infty} \leq M. \tag{36}$$

Secondly, let's make the BV estimates of u_ε. To the end, we begin with the local coordinates of the boundary Σ.

Let $\delta_0 > 0$ be small enough that

$$E^{\delta_0} = \{x \in \bar{\Omega}; dist(x,\Sigma) \leq \delta_0\} \subset \bigcup_{\tau=1}^n V_\tau,$$

where V_τ is a region, on which one can introduce local coordinates

$$y_k = F_\tau^k(x)(k=1,2,\cdots,N), y_N\mid_\Sigma = 0,$$

with F_τ^k appropriately smooth and $F_\tau^N = F_l^N$, such that the y_N-axes coincides with the inner normal vector.

Lemma 11[20] *Let u_ε be the solution of (35) with (7), (8). If $A(s), b_i(s,x,t)$ and u_0 are as in Theorem 2, then*

$$\varepsilon\int_\Sigma |\frac{\partial u_\varepsilon}{\partial n}|d\sigma \le c_1 + c_2(|gradu_\varepsilon|_{L^1(\Omega)} + |\frac{\partial u_\varepsilon}{\partial t}|_{L^1(\Omega)}). \tag{37}$$

with constants $c_i, i=1,2$ independent of ε.

We have the following important estimates of the solutions u_ε of (35) with the initial boundary conditions (7), (8) .

Theorem 12 *Let u_ε be the solution of (35) with (7), (8). If $A(s), b_i(s,x,t)$ and u_0 are as in Theorem 2, then*

$$|gradu_\varepsilon|_{L^1(\Omega)} \le c. \tag{38}$$

where $|gradu|^2 = \sum_{i=1}^N |\frac{\partial u}{\partial x_i}|^2 + |\frac{\partial u}{\partial t}|^2$, c is independent of ε.

Proof Differentiate (35) with respect to x_s, $s = 1,2,\cdot,N,N+1$, $x_{N+1}=t$, and sum up for s after multiplying the resulting relation by $u_{\varepsilon x_s}\frac{S_\eta(|\text{grad}u_\varepsilon|)}{|\text{grad}u_\varepsilon|}$. In what follows, we simply denote u_ε by u. Integrating over Ω yields

$$\int_\Omega \frac{\partial u_{x_s}}{\partial t}u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}dx$$

$$= \int_\Omega \frac{\partial}{\partial t}\int_0^{|\text{grad}u|} S_\eta(\tau)d\tau dx = \frac{d}{dt}\int_\Omega I_\eta(|\text{grad}u|dx,$$

here and in what follows, pairs of the indices of s imply a summation from 1 to $N+1$, pairs of the indices of i,j imply a summation from 1 to N, $\{n_i\}_{i=1}^N$ is the inner normal vector of Ω as before.

$$\int_\Omega \triangle(a(u)u_{x_s})u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}dx$$

$$= \int_\Omega \frac{\partial}{\partial x_i}[a'(u)u_{x_i}u_{x_s} + a(u)u_{x_ix_s}]u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}dx$$

$$= \int_\Omega \frac{\partial}{\partial x_i}(a'(u)u_{x_i}u_{x_s})u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}dx$$

$$+ \int_\Omega \frac{\partial}{\partial x_i}(a(u)u_{x_ix_s})u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}dx, \tag{39}$$

$$\int_\Omega \frac{\partial}{\partial x_i}(a'(u)u_{x_i}u_{x_s})u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}dx$$

$$= \sum_{s=1}^{N+1}\int_\Omega \frac{\partial}{\partial x_i}(a'(u)u_{x_i})u_{x_s}^2\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}dx$$

$$+ \int_\Omega a'(u)u_{x_i}\frac{\partial}{\partial x_i}I_\eta(|\text{grad}u|)dx$$

$$= \int_\Omega \frac{\partial}{\partial x_i}(a'(u)u_{x_i})|\text{grad}u|S_\eta(|\text{grad}u|)dx$$

$$+ \int_\Sigma a'(u)u_{x_i}n_iI_\eta(|\text{grad}u|)d\sigma$$

$$- \int_\Omega I_\eta(|\text{grad}u|)\frac{\partial}{\partial x_i}(a'(u)u_{x_i})dx$$

$$= \int_\Omega \frac{\partial}{\partial x_i}(a'(u)u_{x_i})[|\text{grad}u|S_\eta(|\text{grad}u|) - I_\eta(|\text{grad}u|)]dx$$

$$- \int_\Sigma a'(u)u_{x_i}n_iI_\eta(|\text{grad}u|)d\sigma, \tag{40}$$

$$\int_\Omega \frac{\partial}{\partial x_i}(a(u)u_{x_ix_s})u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}dx$$

$$= \int_\Omega \frac{\partial}{\partial x_i}(a(u)u_{x_ix_s})\frac{\partial}{\partial \xi_s}I_\eta(|\text{grad}u|)dx$$

$$= -\int_\Sigma a(u)u_{x_ix_s}n_i\frac{\partial}{\partial \xi_s}I_\eta(|\text{grad}u|)d\sigma$$

$$- \int_\Omega a(u)\frac{\partial^2 I_\eta(|\text{grad}u|)}{\partial\xi_s\partial\xi_p}u_{x_sx_i}u_{x_px_i}dx, \tag{41}$$

where $\xi_s = u_{x_s}$.

$$\varepsilon\int_\Omega \triangle u_{x_s}u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}dx$$

$$= -\varepsilon\int_\Sigma \frac{\partial I_\eta(|\text{grad}u|)}{\partial x_i}n_id\sigma$$

$$-\varepsilon\int_\Omega \frac{\partial^2 I_\eta(|\text{grad}u|)}{\partial\xi_s\partial\xi_p}u_{x_sx_i}u_{x_px_i}dx. \tag{42}$$

$$\int_\Omega \frac{\partial}{\partial x_i}[b_i'(u,x,t)u_{x_s} + b_{ix_s}(u,x,t)]u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}dx$$

$$= \int_\Omega \frac{\partial(b_i'(u,x,t)u_{x_s)}}{\partial x_i}u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}dx$$

$$+ \sum_{i=1}^N\int_\Omega \frac{\partial b_{ix_s}(u,x,t)}{\partial x_i}u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}dx,$$

where $b_{ix_s}(u,x,t)=\frac{\partial b_i(u,x,t)}{\partial x_s}$.

$$\int_\Omega \frac{\partial(b_i'(u,x,t)u_{x_s})}{\partial x_i}u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}dx$$
$$=\int_\Omega \frac{\partial}{\partial x_i}(b_i'(u,x,t))|\text{grad}u|S_\eta(|\text{grad}u|)dx$$
$$+\sum_{i=1}^N\int_\Omega b_i'(u,x,t)\frac{\partial I_\eta(|\text{grad}u|)}{\partial x_i}dx$$
$$=\int_\Omega \frac{\partial}{\partial x_i}(b_i'(u,x,t))$$
$$\cdot\left[|\text{grad}u|S_\eta(|\text{grad}u|)-I_\eta(|\text{grad}u|)\right]dx$$
$$-\int_\Sigma b_i'(u,x,t)I_\eta(|\text{grad}u|)n_i d\sigma. \tag{43}$$

From (39)-(43), $a(0)=0$, we have

$$\frac{d}{dt}\int_\Omega I_\eta(|\text{grad}u|dx$$
$$=\int_\Omega \frac{\partial}{\partial x_i}(a'(u)u_{x_i})\left[|\text{grad}u|S_\eta(|\text{grad}u|)-I_\eta(|\text{grad}u|)\right]dx$$
$$-\int_\Omega a(u)\frac{\partial^2 I_\eta(|\text{grad}u|)}{\partial\xi_s\partial\xi_p}u_{x_sx_i}u_{x_px_i}dx$$
$$-\varepsilon\int_\Omega \frac{\partial^2 I_\eta(|\text{grad}u|)}{\partial\xi_s\partial\xi_p}u_{x_sx_i}u_{x_px_i}dx$$
$$+\int_\Omega \frac{\partial}{\partial x_i}(b_i'(u,x,t))$$
$$\cdot\left[|\text{grad}u|S_\eta(|\text{grad}u|)-I_\eta(|\text{grad}u|)\right]dx$$
$$+\sum_{i=1}^N\int_\Omega \frac{\partial^2 b_i(u,x,t)}{\partial x_i\partial x_s}u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}dx$$
$$-[\int_\Sigma a'(0)u_{x_i}n_iI_\eta(|\text{grad}u|)d\sigma$$
$$+\int_\Sigma b_i'(0,x,t)I_\eta(|\text{grad}u|)n_id\sigma$$
$$+\varepsilon\int_\Sigma \frac{\partial I_\eta(|\text{grad}u|)}{\partial x_i}n_id\sigma]. \tag{44}$$

Observing that on Σ,

$$-b_i'(u,x,t)\frac{\partial u}{\partial n}n_i=\varepsilon\triangle u+\triangle A(u)+b_{ix_i}(u,x,t),$$
$$u=0, \tag{45}$$

then the surface integrals in (44) can be rewritten as

$$S=-[\int_\Sigma b_i'(0,x,t)I_\eta(|\text{grad}u|)n_id\sigma$$
$$+\varepsilon\int_\Sigma \frac{\partial I_\eta(|\text{grad}u|)}{\partial x_i}n_id\sigma$$
$$+\int_\Sigma a'(0)u_{x_i}n_iI_\eta(|\text{grad}u|)d\sigma]$$
$$=\int_\Sigma b_{ix_i}(0,x,t)\frac{I_\eta(|\text{grad}u|)}{\frac{\partial u}{\partial n}}d\sigma$$
$$-\varepsilon\int_\Sigma\left[\frac{\partial I_\eta(|\text{grad}u|)}{\partial x_i}n_i-\triangle u\frac{I_\eta(|\text{grad}u|)}{\frac{\partial u}{\partial n}}\right]d\sigma$$
$$-\int_\Sigma a(0)\left[\frac{\partial I_\eta(|\text{grad}u|)}{\partial x_i}n_i-\triangle u\frac{I_\eta(|\text{grad}u|)}{\frac{\partial u}{\partial n}}\right]d\sigma$$
$$=\int_\Sigma b_{ix_i}(0,x,t)\frac{I_\eta(|\text{grad}u|)}{\frac{\partial u}{\partial n}}d\sigma$$
$$-\varepsilon\int_\Sigma\left[\frac{\partial I_\eta(|\text{grad}u|)}{\partial x_i}n_i-\triangle u\frac{I_\eta(|\text{grad}u|)}{\frac{\partial u}{\partial n}}\right]d\sigma$$

Since that

$$u_{x_{N+1}}|_\Sigma=u_t|_\Sigma=0,$$
$$\lim_{\eta\to0}S=\int_\Sigma b_{ix_i}(0,x,t)\text{sgn}(\frac{\partial u}{\partial n})d\sigma$$
$$-\varepsilon\int_\Sigma \text{sgn}(\frac{\partial u}{\partial n})(u_{x_ix_j}n_jn_i-\triangle u)d\sigma. \tag{46}$$

Using the local coordinates on $V_\tau,\tau=1,2,\cdots,n$:

$$y_k=F_\tau^k(x),k=1,2,\cdots,N,\ y_m|_\Sigma=0.$$

By elementary computations (refer to [20]), we obtain on $\Sigma\bigcap V_\tau$,

$$u_{x_ix_j}=\sum_{k=1}^N u_{y_Ny_k}F^N_{x_i}F^k_{x_j}+\sum_{k=1}^{N-1}u_{y_Ny_k}F^N_{x_i}F^k_{x_j}$$
$$+u_{y_m}F^m_{x_ix_j}$$
$$u_{x_ix_j}n_jn_i=\frac{\sum_{k=1}^N u_{y_Ny_k}F^N_{x_i}F^k_{x_j}F^N_{x_j}F^N_{x_i}}{|\text{grad}F^N|^2}$$
$$+\sum_{k=1}^{N-1}u_{y_Ny_k}F^k_{x_i}F^N_{x_j}+\frac{u_{y_m}F^m_{x_ix_j}F^N_{x_j}F^N_{x_i}}{|\text{grad}F^N|^2}$$

in which $F^k=F^k_\tau$. By the fact of that the inner normal vector is

$$\vec{n}=-(\frac{\partial F^N}{\partial x_1},\cdots,\frac{\partial F^N}{\partial x_N})=-\text{grad}F^N,$$

then

$$u_{x_ix_j}n_jn_i-\triangle u=u_{y_m}\left(\frac{F^m_{x_ix_j}F^N_{x_j}F^N_{x_i}}{|\text{grad}F^N|^2}-F^m_{x_ix_i}\right).$$

Since $b_{ix_i}(0,x,t)$ is bounded, by (46) and Lemma 11, we see that $\lim_{\eta\to 0} S$ can be estimated by $|\text{grad}u|_{L_1(\Omega)}$.

Thus, noticing that

$$\lim_{\eta\to 0}[|\text{grad}u|S_\eta(|\text{grad}u|)-I_\eta(|\text{grad}u|)]=0,$$

we have

$$\frac{d}{dt}\int_\Omega |\text{grad}u|dx\le c_1+c_2\int_\Omega |\text{grad}u|dx,$$

and by Gronwall Lemma,

$$\int_\Omega |\text{grad}u|dxdt\le c. \tag{47}$$

By (35), (47), we have

$$\iint_{Q_T}(a(u_\varepsilon)+\varepsilon)\mid \nabla u_\varepsilon\mid^2 dxdt\le C \tag{48}$$

Then there is a subsequence $\{u_{\varepsilon_n}\}$ of u_ε and a function $u\in BV(Q_T)\cap L^\infty(Q_T)$ such that $u_{\varepsilon_n}\to u$ a.e. on Q_T.

We now prove that u is a generalized solution of (5)-(7)-(21). For any $\varphi(x,t)\in C_0^1(Q_T)$,

$$\iint_{Q_T}[\frac{\partial}{\partial x_i}\int_0^{u_\varepsilon}\sqrt{a(s)}ds-\frac{\partial}{\partial x_i}\int_0^{u}\sqrt{a(s)}ds]\varphi(x,t)dxdt$$

$$=-\iint_{Q_T}[\int_0^{u_\varepsilon}\sqrt{a(s)}ds-\int_0^{u}\sqrt{a(s)}ds]\varphi_{x_i}(x,t)dxdt$$

By a limiting process, we know the above equality is also true for any $\varphi(x,t)\in L^2(Q_T)$. By Hölder inequality, from (48), we have

$$\frac{\partial}{\partial x_i}\int_0^{u_\varepsilon}\sqrt{a(s)}ds\rightharpoonup\frac{\partial}{\partial x_i}\int_0^{u}\sqrt{a(s)}ds \text{ weakly in}$$

$L^2(Q_T), i=1,2,\cdots,N$. This implies that

$$\frac{\partial}{\partial x_i}\int_0^{u}\sqrt{a(s)}ds\in L^2(Q_T), i=1,2,\cdots,N.$$

Thus u satisfies (1) in Definition 1.

Let $\varphi\in C^2(\bar{Q}_T),\ \varphi_1\ge 0$, supp$\varphi\subset\bar\Omega\times(0,T)$, $\nabla\varphi_1\mid_\Omega=0$ and $\{n_i\}$ be the inner normal vector of Ω. Multiplying (35) by $\varphi_1 S_\eta(u_\varepsilon-k)$, and integrating over Q_T, we obtain

$$\iint_{Q_T}\frac{\partial u_\varepsilon}{\partial t}\varphi_1 S_\eta(u_\varepsilon-k)dxdt$$

$$=\iint_{Q_T}\Delta A(u_\varepsilon)\varphi_1 S_\eta(u_\varepsilon-k)dxdt$$

$$+\varepsilon\iint_{Q_T}\Delta u_\varepsilon\varphi_1 S_\eta(u_\varepsilon-k)dxdt$$

$$+\sum_{i=1}^N\iint_{Q_T}\frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}\varphi_1 S_\eta(u_\varepsilon-k)dxdt. \tag{49}$$

Let's calculate every term in (49) by the part integral method.

$$\iint_{Q_T}\frac{\partial u_\varepsilon}{\partial t}\varphi_1 S_\eta(u_\varepsilon-k)dxdt$$

$$=-\iint_{Q_T}I_\eta(u_\varepsilon-k)\varphi_{1t}dxdt. \tag{50}$$

$$\varepsilon\iint_{Q_T}\Delta u_\varepsilon\varphi_1 S_\eta(u_\epsilon-k)dxdt$$

$$=-\varepsilon\int_0^T\int_\sum\nabla u_\varepsilon\cdot\overrightarrow{n}\varphi_1 S_\eta(u_\epsilon-k)dtd\sigma$$

$$-\varepsilon\iint_{Q_T}\nabla u_\varepsilon(S_\eta(u_\varepsilon-k)\nabla\varphi_1$$

$$+\varphi_1 S'_\eta(u_\varepsilon-k)\nabla u_\varepsilon)dxdt$$

$$=\varepsilon S_\eta(k)\int_0^T\int_\sum\nabla u_\varepsilon\cdot\overrightarrow{n}\varphi_1 dtd\sigma$$

$$-\varepsilon\iint_{Q_T}\nabla u_\varepsilon S_\eta(u_\varepsilon-k)\nabla\varphi_1 dxdt$$

$$-\varepsilon\iint_{Q_T}\mid\nabla u_\varepsilon\mid^2 S'_\eta(u_\varepsilon-k)\varphi_1 dxdt, \tag{51}$$

$$\iint_{Q_T}\Delta A(u_\varepsilon)\varphi_1 S_\eta(u_\varepsilon-k)dxdt$$

$$=S_\eta(k)\int_0^T\int_\sum\nabla A(u_\varepsilon)\cdot\overrightarrow{n}\varphi_1 dtd\sigma$$

$$-\iint_{Q_T}\nabla A(u_\varepsilon)(S_\eta(u_\varepsilon-k)\nabla\varphi_1$$

$$+\varphi_1 S'_\eta(u_\varepsilon-k)\nabla u_\varepsilon)dxdt$$

$$=S_\eta(k)\int_0^T\int_\sum\nabla A(u_\varepsilon)\cdot\overrightarrow{n}\varphi_1 dtd\sigma$$

$$-\iint_{Q_T}\nabla A(u_\varepsilon)S_\eta(u_\varepsilon-k)\nabla\varphi_1 dxdt$$

$$-\iint_{Q_T}a(u_\varepsilon)\mid\nabla u_\varepsilon\mid^2 S'_\eta(u_\varepsilon-k)\varphi_1 dxdt, \tag{52}$$

$$-\iint_{Q_T}\nabla A(u_\varepsilon)S_\eta(u_\varepsilon-k)\nabla\varphi_1 dxdt$$

$$=\iint_{Q_T}A_\eta(u_\varepsilon,k)\triangle\varphi_1 dxdt$$

$$+\int_0^T\int_{\sum}\nabla\varphi_1\cdot\overrightarrow{n}A_\eta(u_\varepsilon,k)dtd\sigma, \quad (53)$$

$$\iint_{Q_T}\frac{\partial b_i(u_\varepsilon,x,t)}{\partial x_i}\varphi_1 S_\eta(u_\varepsilon-k)dxdt$$
$$=-\int_0^T\int_{\sum}[b_i(u_\varepsilon,x,t)-b(k,x,t)]$$
$$\cdot n_i\varphi_1 S_\eta(u_\varepsilon-k)dtd\sigma$$
$$-\iint_{Q_T}[b_i(u_\varepsilon,x,t)-b_i(k,x,t)]$$
$$\cdot[\frac{\partial\varphi_1}{\partial x_i}S_\eta(u_\varepsilon-k)$$
$$+\varphi_1 S'_\eta(u_\varepsilon-k)\frac{\partial u_\varepsilon}{\partial x_i}]dxdt$$
$$=-S_\eta(k)\int_0^T\int_{\sum}\varphi_1[b_i(0,x,t)-b_i(k,x,t)]$$
$$\cdot n_i d\sigma dt$$
$$-\iint_{Q_T}B^i_\eta(u_\varepsilon,x,t,k)\varphi_{1x_i}dxdt. \quad (54)$$

From (49)-(54), we have

$$\iint_{Q_T}I_\eta(u_\varepsilon-k)\varphi_{1t}dxdt$$
$$+\iint_{Q_T}A_\eta(u_\varepsilon,k)\triangle\varphi_1 dxdt$$
$$+\iint_{Q_T}B^i_\eta(u_\varepsilon,x,t,k)\varphi_{1x_i}dxdt$$
$$-\varepsilon\iint_{Q_T}\nabla u_\varepsilon\cdot\nabla\varphi_1 S_\eta(u_\varepsilon-k)dxdt$$
$$-\varepsilon\iint_{Q_T}|\nabla u_\varepsilon|^2 S'_\eta(u_\varepsilon-k)\varphi_1 dxdt$$
$$-\iint_{Q_T}a(u_\varepsilon)|\nabla u_\varepsilon|^2 S'_\eta(u_\varepsilon-k\varphi_1 dxdt$$
$$+\varepsilon S_\eta(k)\int_0^T\int_{\sum}\nabla u_\varepsilon\cdot\overrightarrow{n}\varphi_1 dtd\sigma$$
$$+S_\eta(k)\int_0^T\int_{\sum}\nabla A(u_\varepsilon)\cdot\overrightarrow{n}\varphi_1 dtd\sigma$$
$$+S_\eta(k)\int_0^T\int_{\sum}\nabla\varphi_1\cdot\overrightarrow{n}A_\eta(0,k)dtd\sigma$$
$$+S_\eta(k)\int_0^T\int_{\sum_{1\eta k}}(b_i(0,x,t)-b_i(k,x,t))n_i\varphi_1 dtd\sigma$$
$$+S_\eta(k)\int_0^T\int_{\sum_{2\eta k}}(b_i(0,x,t)-b_i(k,x,t))n_i\varphi_1 dtd\sigma$$
$$=0. \quad (55)$$

Taking $\varphi_2\in C^2(\bar{Q}_T)$,

$$\varphi_1\mid_{\partial\Omega\times[0,T]}=\varphi_2\mid_{\partial\Omega\times[0,T]},$$

$\text{supp}\varphi_2\subset\bar{\Omega}\times(0,T)$,

$$S_\eta(k)\int_0^T\int_{\sum}\nabla A(u_\varepsilon)\cdot\overrightarrow{n}\varphi_1 dtd\sigma$$
$$+\varepsilon S_\eta(k)\int_0^T\int_{\sum}\nabla u_\varepsilon\cdot\overrightarrow{n}\varphi_1 dtd\sigma$$
$$S_\eta(k)[-\varepsilon\iint_{Q_T}\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial\varphi_2}{\partial x_i}dxdt$$
$$-\iint_{Q_T}\nabla A(u_\varepsilon)\cdot\nabla\varphi_2 dxdt$$
$$+\iint_{Q_T}\frac{\partial b_i(0,x,t)}{\partial x_i}\varphi_2 dxdt$$
$$-\iint_{Q_T}(b_i(u_\varepsilon,x,t)-b_i(0,x,t))\frac{\partial\varphi_2}{\partial x_i}dxdt$$
$$+\iint_{Q_T}u_\varepsilon\frac{\partial\varphi_2}{\partial t}dxdt], \quad (56)$$

$$\iint_{Q_T}\nabla A(u)\cdot\nabla\varphi_2 dxdtdxdt$$
$$=-\int_0^T\int_{\sum}A(0)\frac{\partial\varphi_2}{\partial x_i}n_i dtd\sigma$$
$$-\iint_{Q_T}A(u_\varepsilon)\Delta\varphi_2 dxdt$$
$$=-\iint_{Q_T}A(u_\varepsilon)\Delta\varphi_2 dxdt, \quad (57)$$

For $\nabla\varphi_1\mid_{\sum}=0$, $\varphi_1\mid_{\sum_{2k\eta}}=0$, and $a(0)=0$, from (55)-(57), we have

$$\iint_{Q_T}I_\eta(u_\varepsilon-k)\varphi_{1t}dxdt+\iint_{Q_T}A_\eta(u_\varepsilon,k)\triangle\varphi_1 dxdt$$
$$+\iint_{Q_T}B^i_\eta(u_\varepsilon,x,t,k)\varphi_{1x_i}dxdt$$
$$+S_\eta(k)\left[-\varepsilon\iint_{Q_T}\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial\varphi_2}{\partial x_i}dxdt+\iint_{Q_T}A(u_\varepsilon)\Delta\varphi_2 dxdt\right]$$
$$+S_\eta(k)[-\iint_{Q_T}(b_i(u_\varepsilon,x,t)-b_i(0,x,t))\frac{\partial\varphi_2}{\partial x_i}dxdt$$
$$+\iint_{Q_T}\frac{\partial b_i(0,x,t)}{\partial x_i}\varphi_2 dxdt+\iint_{Q_T}u_\varepsilon\frac{\partial\varphi_2}{\partial t}dxdt]$$
$$-\varepsilon\iint_{Q_T}\nabla u_\varepsilon\cdot\nabla\varphi_1 S_\eta(u_\varepsilon-k)dxdt$$

$$-\iint_{Q_T} a(u_\varepsilon) \mid \nabla u_\varepsilon \mid^2 S'_\eta(u_\varepsilon - k)\varphi_1 dxdt$$
$$+S_\eta(k)\int_0^T \int_{\Sigma_{1\eta k}} [(b_i(0,x,t) - b_i(k,x,t)]$$
$$\cdot n_i\varphi_1 dtd\sigma \geq 0. \quad (58)$$

By Lemma 10,

$$\liminf_{\varepsilon\to 0} \iint_{Q_T} S'_\eta(u_\varepsilon - k)a(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_i}\varphi_1 dxdt$$
$$\geq \iint_{Q_T} S'_\eta(u-k) \mid \nabla \int_0^u \sqrt{a(s)}ds \mid^2 \varphi_1 dxdt. \quad (59)$$

Let $\varepsilon \to 0$ in (58). By (59), we get (23) and (24) is naturally concealed in the limiting process.

The proof of (25) is similar to that in [8], [9], we omit the details here.

5 The uniqueness of the solutions

Proof of Theorem 3 Let u, v be two entropy solutions of equation (5) with the different initial values

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x).$$

respectively, and with the homogeneous boundary value $u(x,t) = v(x,t) = 0, (x,t) \in \Sigma_1 \times (0,T)$.

By Definition 1, for any $\varphi_1, \varphi_2 \in C^2(\bar{Q}_T)$, $\varphi_1 \geq 0$, $\varphi_1 \mid_{\partial\Omega\times[0,T]} = \varphi_2 \mid_{\partial\Omega\times[0,T]}$, supp$\varphi_2$, supp$\varphi_1 \subset \bar{\Omega} \times (0,T)$, $\eta > 0$, $k, l \in \mathbb{R}$, we have

$$\iint_{Q_T} [I_\eta(u-k)\varphi_{1t} - B^i_\eta(u,x,t,k)\varphi_{1x_i}$$
$$+A_\eta(u,k)\Delta\varphi_1 - S'_\eta(u-k) \mid \nabla \int_0^u \sqrt{a(s)}ds \mid^2 \varphi_1]dxdt$$
$$-S_\eta(k)[b_i(0,x,t) - b_i(k,x,t)]\int_0^T \int_{\Sigma_{1k\eta}} \varphi_1 n_i dtd\sigma$$
$$+S_\eta(k)\iint_{Q_T} [u\varphi_{2t} - b_i(u,x,t)\varphi_{2x_i} + \frac{\partial b_i(0,x,t)}{\partial x_i}\varphi_2$$
$$+A(u)\Delta_x\varphi_2]dxdt \geq 0, \quad (60)$$

$$\iint_{Q_T} [I_\eta(v-l)\varphi_{1\tau} - B^i_\eta(v,y,\tau,l)\varphi_{1y_i}$$
$$+A_\eta(v,l)\Delta\varphi_1 - S'_\eta(v-l) \mid \nabla \int_0^v \sqrt{a(s)}ds \mid^2 \varphi_1]dyd\tau$$
$$-S_\eta(l)[(b_i(0,y,\tau) - b_i(l,y,\tau)]\int_0^T \int_{\Sigma_{1l\eta}} \varphi_1 n_i d\tau d\sigma$$
$$+S_\eta(l)\iint_{Q_T} [v\varphi_{2\tau} - b_i(v,y,\tau)\varphi_{2y_i} + \frac{\partial b_i(0,y,\tau)}{\partial y_i}\varphi_2$$
$$+A(v)\Delta_y\varphi_2]dyd\tau \geq 0, \quad (61)$$

Especially, if $\varphi_1 \in C^2_0(Q_T)$, $\varphi_2 \equiv 0$, we have

$$\iint_{Q_T} [I_\eta(u-k)\varphi_{1t} - B^i_\eta(u,x,t,k)\varphi_{1x_i} + A_\eta(u,k)\Delta\varphi_1$$
$$-S'_\eta(u-k) \mid \nabla \int_0^u \sqrt{a(s)}ds \mid^2 \varphi_1]dxdt \geq 0, \quad (62)$$

$$\iint_{Q_T} [I_\eta(v-l)\varphi_{1\tau} - B^i_\eta(v,y,\tau,l)\varphi_{1y_i} + A_\eta(v,l)\Delta\varphi_1$$
$$-S'_\eta(v-l) \mid \nabla \int_0^v \sqrt{a(s)}ds \mid^2 \varphi_1]dyd\tau \geq 0. \quad (63)$$

Let $\psi(x,t,y,\tau) = \phi(x,t)j_h(x-y,t-\tau)$. Here $\phi(x,t) \geq 0,\ \phi(x,t) \in C^\infty_0(Q_T)$, and

$$j_h(x-y,t-\tau) = \omega_h(t-\tau)\Pi^N_{i=1}\omega_h(x_i - y_i), \quad (64)$$

$$\omega_h(s) = \frac{1}{h}\omega(\frac{s}{h}), \omega(s) \in C^\infty_0(R),\ \omega(s) \geq 0, \quad (65)$$

$$\omega(s) = 0\ if\ \mid s \mid > 1,\ \int_{-\infty}^{\infty} \omega(s)ds = 1. \quad (66)$$

We choose $k = v(y,\tau)$, $l = u(x,t)$, $\varphi_1 = \psi(x,t,y,\tau)$ in (62) (63), integrate over Q_T, to get

$$\iint_{Q_T}\iint_{Q_T} [I_\eta(u-v)(\psi_t + \psi_\tau)$$
$$-(B^i_\eta(u,x,t,v)\psi_{x_i} + B^i_\eta(v,y,\tau,u)\psi_{y_i})$$
$$+A_\eta(u,v)\Delta_x\psi + A_\eta(v,u)\Delta_y\psi]$$
$$-S'_\eta(u-v)\left(\mid \nabla_x \int_0^u \sqrt{a(s)}ds \mid^2 + \mid \nabla_y \int_0^v \sqrt{a(s)}ds \mid^2\right)$$
$$\cdot\psi dxdtdyd\tau. \quad (67)$$

Clearly,

$$\frac{\partial j_h}{\partial t} + \frac{\partial j_h}{\partial \tau} = 0,\ \frac{\partial j_h}{\partial x_i} + \frac{\partial j_h}{\partial y_i} = 0,\ i = 1, \cdots, N;$$

$$\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \tau} = \frac{\partial \phi}{\partial t}j_h,\ \frac{\partial \psi}{\partial x_i} + \frac{\partial \psi}{\partial y_i} = \frac{\partial \phi}{\partial x_i}j_h.$$

Noticing that

$$\lim_{\eta\to 0} B^i_\eta(u,x,t,v) = \text{sgn}(u-v)(b_i(u,x,t) - b_i(v,x,t)),$$

and

$$\lim_{\eta\to 0} B^i_\eta(v,y,\tau,u) = \text{sgn}(v-u)(b_i(v,y,\tau)-b_i(u,y,\tau)),$$

as $\eta \to 0$, we have,

$$\iint_{Q_T}\iint_{Q_T} [B^i_\eta(u,x,t,v)\psi_{x_i} + B^i_\eta(v,y,\tau,u)\psi_{y_i}]dxdtdyd\tau$$

$$\to \iint_{Q_T}\iint_{Q_T} \text{sgn}(u-v)[b_i(u,x,t)-b_i(v,x,t)] \cdot\psi_{x_i}dxdtdyd\tau$$

$$+\iint_{Q_T}\iint_{Q_T} \text{sgn}(v-u)[b_i(v,y,\tau)-b_i(u,y,\tau)] \psi_{y_i}dxdtdyd\tau.$$

and the right hand side of this formula can be dealt with as

$$\iint_{Q_T}\iint_{Q_T} \text{sgn}(u-v)[b_i(u,x,t)-b_i(v,x,t)] \psi_{x_i}dxdtdyd\tau$$

$$= \iint_{Q_T}\iint_{Q_T} \text{sgn}(u-v)[b_i(u,x,t)-b_i(v,x,t)] (\phi_{x_i}j_h+\phi j_{hx_i})dxdtdyd\tau$$

$$= \iint_{Q_T}\iint_{Q_T} \text{sgn}(u-v)[b_i(u,x,t)-b_i(v,y,\tau)] (\phi_{x_i}j_h+\phi j_{hx_i})dxdtdyd\tau$$

$$+\iint_{Q_T}\iint_{Q_T} \text{sgn}(u-v)[b_i(v,y,\tau)-b_i(v,x,t)] \cdot(\phi_{x_i}j_h+\phi j_{hx_i})dxdtdyd\tau.$$

$$\iint_{Q_T}\iint_{Q_T} \text{sgn}(v-u)[b_i(v,y,\tau) -b_i(u,y,\tau)]\psi_{y_i}dxdtdyd\tau$$

$$= -\iint_{Q_T}\iint_{Q_T} \text{sgn}(v-u)[b_i(v,y,\tau)-b_i(u,y,\tau)] \phi j_{hy_i}dxdtdyd\tau$$

$$= -\iint_{Q_T}\iint_{Q_T} \text{sgn}(v-u)[b_i(v,y,\tau)-b_i(u,x,t)] \phi j_{hy_i}dxdtdyd\tau$$

$$-\iint_{Q_T}\iint_{Q_T} \text{sgn}(v-u)[b_i(u,x,t)-b_i(u,y,\tau)] \phi j_{hy_i}dxdtdyd\tau.$$

So,

$$\lim_{\eta\to 0}\iint_{Q_T}\iint_{Q_T} [B^i_\eta(u,x,t,v)\psi_{x_i} + B^i_\eta(v,y,\tau,u)\psi_{y_i}] \cdot dxdtdyd\tau$$

$$= \iint_{Q_T}\iint_{Q_T} \text{sgn}(u-v)[b_i(u,x,t)-b_i(v,y,\tau)] \cdot\phi_{x_i}j_hdxdtdyd\tau$$

$$+\iint_{Q_T}\iint_{Q_T} \text{sgn}(u-v)[b_i(v,y,\tau)-b_i(v,x,t)] \cdot\phi_{x_i}j_hdxdtdyd\tau.$$

As $h \to 0$, we have

$$\lim_{h\to 0}\lim_{\eta\to 0}\iint_{Q_T}\iint_{Q_T} [B^i_\eta(u,x,t,v)\psi_{x_i} + B^i_\eta(v,y,\tau,u)\psi_{y_i}]dxdtdyd\tau$$

$$= \iint_{Q_T} \text{sgn}(u-v)[b_i(u,x,t)-b_i(v,x,t)]\phi_{x_i}dxdt. \tag{68}$$

For the third term in (67), we have

$$\iint_{Q_T} [A_\eta(u,v)\Delta_x\psi + A_\eta(v,u)\Delta_y\psi]dxdtdyd\tau$$

$$= \iint_{Q_T}\iint_{Q_T} \{A_\eta(u,v)(\Delta_x\phi j_h + 2\phi_{x_i}j_{hx_i} + \phi\Delta j_h) + A_\eta(v,u)\phi\Delta_y j_h\}dxdtdyd\tau$$

$$= \iint_{Q_T}\iint_{Q_T} \{A_\eta(u,v)\Delta_x\phi j_h + A_\eta(u,v)\phi_{x_i}j_{hx_i} + A_\eta(v,u)\phi_{x_i}j_{hy_i}\}dxdtdyd\tau$$

$$-\iint_{Q_T}\iint_{Q_T} \{a(u)\widehat{S_\eta(u}-v)\frac{\partial u}{\partial x_i} -\int_u^v \widehat{a(s)S'_\eta(s}-v)ds\frac{\partial u}{\partial x_i})\phi j_{hx_i}\}dxdtdyd\tau, \tag{69}$$

where definition (31) and formula (32) are used, i.e.

$$a(u)\widehat{S_\eta(u}-v)$$

$$= \int_0^1 a(su^+ + (1-s)u^-)S_\eta(su^+ + (1-s)u^- - v)ds,$$

$$\int_u^v a(s)\widehat{S'_\eta(s}-v)ds$$

$$= \int_0^1\int_{su^+ + (1-s)u^-}^v a(\sigma)S_\eta(\sigma - su^+ - (1-s)u^-)d\sigma ds.$$

Noticing that

$$\iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)$$
$$\cdot\left(\mid \nabla_x\int_0^u\sqrt{a(s)}ds\mid^2+\mid \nabla_y\int_0^v\sqrt{a(s)}ds\mid^2\right)$$
$$\cdot\psi dxdtdyd\tau$$
$$=\iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)$$
$$\cdot\left(\mid \nabla_x\int_0^u\sqrt{a(s)}ds\mid-\mid \nabla_y\int_0^v\sqrt{a(s)}ds\mid\right)^2$$
$$\cdot\psi dxdtdyd\tau$$
$$+2\iint_{Q_T}\iint_{Q_T} S'_\eta(u-v)\nabla_x\int_0^u\sqrt{a(s)}ds$$
$$\cdot\nabla_y\int_0^v\sqrt{a(s)}ds\psi dxdtdyd\tau, \tag{70}$$

and by Lemma 9

$$\iint_{Q_T}\iint_{Q_T}\nabla_x\nabla_y\int_v^u\sqrt{a(\delta)}\int_\delta^v\sqrt{a(\sigma)}$$
$$\cdot S'_\eta(\sigma-\delta)d\sigma d\delta\psi dxdtdyd\tau$$
$$=\iint_{Q_T}\iint_{Q_T}\int_0^1\int_0^1\sqrt{a(su^++(1-s)u^-)}$$
$$\sqrt{a(\sigma v^++(1-\sigma)v^-}$$
$$\times S'_\eta[\sigma v^++(1-\sigma)v^--su^+-(1-s)u^-]dsd\sigma$$
$$\cdot\nabla_xu\nabla_yvdxdtdyd\tau$$
$$=\iint_{Q_T}\iint_{Q_T}\int_0^1\int_0^1 S'_\eta$$
$$[\sigma v^++(1-\sigma)v^--su^+-(1-s)u^-]dsd\sigma$$
$$\widehat{\sqrt{a(u)}}\nabla_xu\widehat{\sqrt{a(v)}}\nabla_yvdxdtdyd\tau$$
$$=\iint_{Q_T}\iint_{Q_T}\int_0^1\int_0^1 S'_\eta(v-u)$$
$$\nabla_x\int_0^u\sqrt{a(s)}ds\nabla_y\int_0^v\sqrt{a(s)}dsdxdtdyd\tau. \tag{71}$$

$$\iint_{Q_T}\iint_{Q_T}\nabla_x\nabla_y\int_v^u\sqrt{a(\delta)}$$
$$\cdot\int_\delta^v\sqrt{a(\sigma)}S'_\eta(\sigma-\delta)d\sigma d\delta\psi dxdtdyd\tau$$
$$=\iint_{Q_T}\iint_{Q_T}\int_0^1\sqrt{a(su^++(1-s)u^-)}$$
$$\times\int_{su^++(1-s)u^-}^v\sqrt{a(\sigma)}S'_\eta(\sigma-su^+-(1-s)u^-)d\sigma ds$$
$$\frac{\partial u}{\partial x_i}j_{hx_i}\phi dxdtdyd\tau, \tag{72}$$

we have

$$\iint_{Q_T}\iint_{Q_T}(a(u)\widehat{S_\eta(u}-v)\frac{\partial u}{\partial x_i}$$
$$-\int_u^v\widehat{a(s)S'_\eta(s-u)}ds\frac{\partial u}{\partial x_i})j_{hx_i}\phi dxdtdyd\tau$$
$$+2\iint_{Q_T}\iint_{Q_T}S'_\eta(u-v)\nabla_x\int_0^u\sqrt{a(s)}ds$$
$$\cdot\nabla_y\int_0^v\sqrt{a(s)}ds\psi dxdtdyd\tau$$
$$=\iint_{Q_T}\iint_{Q_T}[\int_0^1 a(su^++(1-s)u^-)$$
$$\cdot S_\eta(su^++(1-s)u^--v)ds$$
$$-\int_0^1\int_{su^++(1-s)u^-}^v a(\sigma)S'_\eta(\sigma-su^+-(1-s)u^-)d\sigma ds$$
$$+2\int_0^1\sqrt{a(su^++(1-s)u^-)}\int_{su^++(1-s)u^-}^v\sqrt{a(\sigma)}$$
$$S'_\eta(\sigma-su^+-(1-s)u^-)d\sigma ds]\frac{\partial u}{\partial x_i}j_{hx_i}\phi dxdtdyd\tau$$
$$=-\iint_{Q_T}\iint_{Q_T}\int_0^1\int_{su^++(1-s)u^-}^v$$
$$[\sqrt{a(\sigma)}-\sqrt{a(su^++(1-s)u^-)}]$$
$$\times S'_\eta(\sigma-su^+-(1-s)u^-)d\sigma ds\frac{\partial u}{\partial x_i}j_{hx_i}\phi dxdtdyd\tau, \tag{73}$$

which tend to 0 as $\eta\to 0$.

Since

$$\lim_{\eta\to 0}A_\eta(u,v)=\lim_{\eta\to 0}A_\eta(v,u)=\text{sgn}(u-v)[A(u)-A(v)],$$

we have

$$\lim_{\eta\to 0}[A_\eta(u,v)\phi_{x_i}j_{hx_i}+A_\eta(u,v)\phi_{y_i}j_{hy_i}]=0. \tag{74}$$

Combing (68)-(74), and letting $\eta\to 0, h\to 0$ in (68), we get

$$\iint_{Q_T}[\mid u(x,t)-v(x,t)\mid\phi_t$$
$$-\text{sgn}(u-v)(b_i(u,x,t)-b_i(v,x,t))\phi_{x_i}$$
$$+|A(u)-A(v)|\Delta\phi]dxdt\geq 0. \tag{75}$$

Let $\omega_\lambda(x) \in C_0^2(\Omega)$ be defined as follows: for any given small enough $0 < \lambda, 0 \le \omega_\lambda \le 1, \omega|_{\partial\Omega} = 0$ and

$$\omega_\lambda(x) = 1, if\ d(x) = \mathrm{dist}(x, \partial\Omega) \ge \lambda.$$

when $0 \le d(x) \le \lambda$,

$$\omega_\lambda(d(x)) = 1 - \frac{(d(x) - \lambda)^2}{\lambda^2}.$$

Especially, we can choose ϕ in (75) by

$$\phi(x, t) = \omega_{\lambda\varepsilon}(x)\eta(t),$$

where $\eta(t) \in C_0^\infty(0, T)$,

$$\omega_{\lambda\varepsilon} = \omega_\lambda * \delta_\varepsilon(d),$$

is the mollified function of ω_λ. Then

$$\omega'_{\lambda\varepsilon}(d) = \int_{\{|s|<\varepsilon\}\bigcap\{0<d-s<\lambda\}} \omega'_\lambda(d - s)\delta_\varepsilon(s)ds$$

$$= -\int_{\{|s|<\varepsilon\}\bigcap\{0<d-s<\lambda\}} \frac{2(d - s - \lambda)}{\lambda^2}\delta_\varepsilon(s)ds$$

$$|\omega'_{\lambda\varepsilon}(d)| \le \frac{c}{\lambda}.$$

$$\omega''_{\lambda\varepsilon}(d) = -\frac{2}{\lambda^2}\int_{\{|s|<\varepsilon\}\bigcap\{0<d-s<\lambda\}} \delta_\varepsilon(s)ds.$$

Now,

$$\begin{aligned}\triangle\phi &= \eta(t)\triangle(\omega_{\lambda\varepsilon}(d(x)))\\ &= \eta(t)\nabla(\omega'_{\lambda\varepsilon}(d)\nabla d)\\ &= \eta(t)[\omega''_{\lambda\varepsilon}(d)|\nabla d|^2 + \omega'_{\lambda\varepsilon}(d)\triangle d]\end{aligned}$$

$$= \eta(t)[-\frac{2}{\lambda^2}\int_{\{|s|<\varepsilon\}\bigcap\{0<d-s<\lambda\}} \delta_\varepsilon(s)ds + \omega'_{\lambda\varepsilon}(d)\triangle d],$$

using the conditions (29), and using the fact of that $|\nabla d| = 1$, from (75), we have

$$\int_{Q_T} |u(x, t) - v(x, t)|\phi_t dxdt$$

$$+c\int_0^T \int_{\Omega_\lambda} \eta(t)|\omega'_{\lambda\varepsilon}(d)| \mid u - v \mid dxdt \ge 0, \quad (76)$$

where $\Omega_\lambda = \{(x \in \Omega : d(x) < \lambda\}$. According to the definition of the trace of BV functions (see [16]), by (27),(28), when $x \in \Sigma_1$, $\gamma u = \gamma v$, let $\lambda \to 0$ in (76). We have

$$\lim_{\lambda\to 0} |\int_0^T \int_{\Omega_\lambda} \eta(t)|\omega'_{\lambda\varepsilon}(d)| \mid u - v \mid dxdt|$$

$$\le c\lim_{\lambda\to 0}\int_0^T \eta(t)\frac{1}{\lambda}\int_{\Omega_\lambda} | \mid u - v \mid dxdt$$

$$= c\int_0^T \eta(t)|u - v|_{\partial\Omega}dt = c\int_0^T \eta(t)|u - v|_{\partial\Omega}dt$$

$$\le c\mathrm{ess}\sup_{\Sigma_2\times(0,T)} |f(x, t) - g(x, t)|,$$

and so

$$c\mathrm{ess}\sup_{\Sigma_2\times(0,T)} |f(x, t) - g(x, t)|$$

$$+\int_{Q_T} \mid u(x, t) - v(x, t) \mid \eta'_t dxdt \ge 0. \quad (77)$$

Let $0 < s < \tau < T$, and

$$\eta(t) = \int_{s-t}^{\tau-t} \alpha_\varepsilon(\sigma)d\sigma, \ \ \varepsilon < \min\{\tau, T - s\}.$$

Here $\alpha_\varepsilon(t)$ is the kernel of mollifier with $\alpha_\varepsilon(t) = 0$ for $t \notin (-\varepsilon, \varepsilon)$. Then

$$c\mathrm{ess}\sup_{\Sigma_2\times(0,T)} |f(x, t) - g(x, t)|$$

$$+\int_0^T [\alpha_\varepsilon(t - s) - \alpha_\varepsilon(t - \tau)]\, |u - v|_{L^1(\Omega)}dt \ge 0,$$

Let $\varepsilon \to 0$. Then

$$|u(x, \tau) - v(x, \tau)|_{L^1(\Omega)} \le |u(x, s) - v(x, s)|_{L^1(\Omega)}$$

$$+c\mathrm{ess}\sup_{\Sigma_2\times(0,T)} |f(x, t) - g(x, t)|$$

and the desired result follows by letting $s \to 0$.

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