# The Homotopy Perturbation Sumudu Transform Method For Solving The Nonlinear Partial Differential Equations

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*Abstract:* In this paper, we use the homotopy perturbation sumudu transform method (HPSTM) to solve the Ramani and the generalized nonlinear Hirota Satsuma coupled KdV equations. The proposed scheme finds the solution without any discretization or restrictive assumptions and avoids the roundoff errors.

*Key–Words:* Homotopy perturbation sumudu transform method, Coupled nonlinear evaluation equations, Exact solutions

## 1 Introduction

The nonlinear coupled evolution equations have many wide array of applications of many fields, which described the motion of the isolated waves, localized in a small part of space, in many fields such as physics, mechanics, biology, hydrodynamics, plasma physics, etc.. To further explain some physical phenomena, searching for exact solutions of nonlinear partial differential equations is very important. Up to now, many researches in mathematical physics have paid attention to these topics, and a lot of powerful methods have been presented such as the modified extended tanh-function method [8], generalized F-expansion method [1,25], homotopy analysis method [3], varational iteration method[10,14,23], extended and modified extended tanh method [2,7,8], the tanh-hyperbolic function method [20-21]. References [9,12,22] developed the homotopy perturbation method (HPM) by merging the standard homotopy and perturbation for solving various physical problems. It is worth mentioning that the HPM is applied without any discretization, restrictive assumption or transformation and is free from round off errors.The homotopy pertrbation sumudu transform method( HPSTM) [24] provides the solution in a rapid convergent series which may lead to the solution in a closed form. It is worth mentioning that HPSTM is an elegant combination of the sumudu transform method, the homotopy perturbation method and Hes polynomials. The advantage of this method is its capability of combining two powerful methods for obtaining exact and approximate solutions for nonlinear equations. Inspired and motivated by the ongoing research in this area. In this paper we use the HPSTM to solve the (1+1)-dimensional Ramani equations and the (1+1)- dimensional generalized nonlinear Hirota Satsuma coupled KdV equations

the (1+1)-dimensional Ramani equations [29]

$$\begin{aligned} &u_{6x} + 15u_{xx}u_{3x} + 15u_x u_{4x} + 45u_x^2 u_{xx} - \\ &5(u_{3xt} + 3u_{xx}u_t + 3u_x u_{xt}) - 5u_{tt} + 18v_x = 0, \\ &v_t - v_{3x} - 3v_x u_x - 3v u_{xx} = 0, \end{aligned} \tag{1}$$

and the (1+1)- dimensional generalized nonlinear Hirota Satsuma coupled KdV equations [13]

$$\begin{aligned} &u_t - \frac{1}{2}u_{xxx} + 3uu_x - 3(vw)_x = 0, \\ &v_t + v_{xxx} - 3uv_x = 0, \\ &w_t + w_{xxx} - 3uw_x = 0, \end{aligned} \tag{2}$$

In early 90s, Watugala (1998) [26] introduced a new integral transform, named the sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. The sumudu transform is defined over the set of functions

$$A = \{f(t)|\, \exists\, M,\ \tau_1, \tau_2 > 0,$$

$$|f(t)|\ < M\, e^{|t|\tau_j},\ \text{if} t \in (-1)^j \times [0, \infty)\}.$$

by the following formula

$$\bar{f}(u) = S[f(t)] = \int_0^\infty f(ut)e^{-t}dt, u \in (-\tau_1, \tau_2). \tag{3}$$

For further detail and properties of this transform, see [15,26-28] Some of the properties were established by Weerakoon in Kilicman et al. [27] and Weerakoon [28]. In Asiru [4-5], further fundamental properties of this transform were also established. Similarly, this transform was applied to the one-dimensional neutron transport equation in Kadem [16]. In fact it was shown that there is strong relationship between Sumudu and other integral transform (see Kilicman and Eltayeb [18]. In particular the relation between Sumudu transform and Laplace transforms was proved in Kilicman and Eltayeb. Further, in Eltayeb et al. [19], the Sumudu transform was extended to the distributions and some of their properties were also studied in Kilicman et al. (2010). Recently, this transform is applied to solve the system of differential equations (see Kilicman et al. [17]. Note that a very interesting fact about Sumudu transform is that the original function and its Sumudu transform have the same Taylor coefficients except the factor n (see Zhang [30]). Thus if $f(t) = \sum_{i=0}^{\infty} a_n t^n$ then$F(u) = \sum_{i=0}^{\infty} n!a_n u^n$ see Kilicman and Eltayeb. Similarly, the Sumudu transform sends combinations, C(m, n), into permutations, P(m,n) and hence it will be useful in the discrete systems.

## 2 Homotopy perturbation sumudu transform method

In this section the basic idea of the homotopy perturbation sumudu transform method (HPSTM) [24] is introduced. To show this basic idea, let us consider the following nonlinear partial differential equation in a general form

$$\begin{aligned} &DU(x,t) + RU(x,t) + NU(x,t) = g(x,t), \\ &U(x,0) = h(x), U_t(x,0) = f(x), \end{aligned} \tag{4}$$

where D is the second order linear differential operator $\frac{\partial^2}{\partial t^2}$ , R is the linear differential operator of less order than D, N represents the general nonlinear differential operator and g( x,t)is the source term. Taking the sumudu transform on both sides of Equation (2), we get

$$\begin{aligned} &S[DU(x,t)] + S[RU(x,t)] + S[NU(x,t)] \\ &= S[g(x,t)]. \end{aligned} \tag{5}$$

Using the differentiation property of the sumudu transform and above initial conditions, we have

$$\begin{aligned} &S[U(x,t)] = u^2 S[g(x,t)] + h(x) + uf(x) - \\ &u^2 S[RU(x,t) + NU(x,t)]. \end{aligned} \tag{6}$$

Now, applying the inverse sumudu transform on both sides of equation (6), we get

$$S[U(x,t)] = G(x,t) - S^{-1}[u^2 S[RU(x,t) + NU(x,t)]]. \tag{7}$$

where G(x , t) represents the term arising from the source term and the prescribed initial conditions. Now, we apply the homotopy perturbation method

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t) = U_0 + pU_1 + p^2 U_2 + p^3 U_3 + .... \tag{8}$$

and the nonlinear term can be decomposed as

$$NU(x,t) = \sum_{n=0}^{\infty} p^n H_n(U). \tag{9}$$

where $H_n's$ are the so-called He's polynomials [9,12,22 ], which can be calculated by using the formula

$$\begin{aligned} &H_n(U_0, U_1, ..., U_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \{N(\sum_{i=0}^{n} p^i U_i)\}_{p=0}, \\ &n = 0, 1, 2, .. \end{aligned} \tag{10}$$

Substituting Equations (9) and (10) in Equation (7), we get

$$\begin{aligned} &\sum_{n=0}^{\infty} p^n U_n(x,t) = G(x,t) - \\ &p\{S^{-1}[u^2 S[R \sum_{n=0}^{\infty} p^n U_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(U)]]\} \end{aligned} \tag{11}$$

which is the coupling of the sumudu transform and the homotopy perturbation method using He's polynomials. By comparing the coefficient of like powers of p, the following approximations are obtained

$$\begin{aligned} &p^0 : U_0(x,t) = G(x,t), \\ &p^1 : U_1(x,t) = -S^{-1}[u^2 S[RU_0(x,t) + H_0(U)]], \\ &P^2 : U_2(x,t) = -S^{-1}[u^2 S[RU_1(x,t) + H_1(U)]], \\ &p^3 : U_3(x,t) = -S^{-1}[u^2 S[RU_0(x,t) + H_0(U)]], \\ &. \\ &. \\ &. \end{aligned} \tag{12}$$

## 3 Applications

The homotopy perturbation sumudu transform method is used to solve the (1+1)-dimensional Ramani equations, and the (1+1)-dimensional generalized nonlinear Hirota Satsuma coupled KdV equations.

### 3.1 Solving the (1+1)-dimensional Ramani equations using HPSTM

In this subsection, we find the solutions $U(x,t)$ and $V(x,t)$ satisfying the coupled nonlinear Ramani equations (1) with the following initial conditions [29]:

$$\begin{aligned}&U(x,0)=a_0+2\alpha\coth(\alpha x),\\&U_t(x,0)=2\beta\alpha^2\text{csch}^2(\alpha x),\\&V(x,0)=-\frac{4}{9}\beta\alpha^4-\frac{16}{27}\alpha^6+\frac{5}{9}\beta^2\alpha^2-\\&\frac{5}{54}\beta^3+(\frac{20}{9}\beta\alpha^4+\frac{16}{9}\alpha^6-\frac{5}{9}\beta^2\alpha^2)\coth^2(\alpha x).\end{aligned} \tag{13}$$

These initial conditions follow by setting $t=0$ in the following exact solutions of Eqs. (1):

$$\begin{aligned}&U(x,t)=a_0+2\alpha\coth(\alpha\zeta),\\&V(x,t)=-\frac{4}{9}\beta\alpha^4-\frac{16}{27}\alpha^6+\frac{5}{9}\beta^2\alpha^2-\\&\frac{5}{54}\beta^3+(\frac{20}{9}\beta\alpha^4+\frac{16}{9}\alpha^6-\frac{5}{9}\beta^2\alpha^2)\coth^2(\alpha\zeta),\end{aligned} \tag{14}$$

where $\zeta=(x-\beta t)$, $a_0$, $\beta$ and $\alpha$ are arbitrary constants . Let us now solve the initial value problem (1) with the initial conditions (13) using the HPSTM. Taking the sumudu transform on both sides of equation (1) subject to the initial condition [13], we have

$$\begin{aligned}&5S[U(x,t)]=a_0+2\alpha\coth(\alpha x)+2t\beta\alpha^2\text{csch}^2(\alpha x)\\&+u^2S[U_{6x}+15U_{xx}u_{3x}+15U_xu_{4x}+45U_x^2U_{xx}-\\&5(U_{3xt}+3U_{xx}U_t+3U_xU_{xt})+18V_x],\\&S[V(x,t)]=-\frac{4}{9}\beta\alpha^4-\frac{16}{27}\alpha^6+\frac{5}{9}\beta^2\alpha^2-\frac{5}{54}\beta^3+\\&(\frac{20}{9}\beta\alpha^4+\frac{16}{9}\alpha^6-\frac{5}{9}\beta^2\alpha^2)\coth^2(\alpha x)-\\&uS[-V_{3x}-3V_xU_x-3VU_{xx}],\end{aligned} \tag{15}$$

The inverse of sumudu transform implies that

$$\begin{aligned}&5U(x,t)=a_0+2\alpha\coth(\alpha x)+2t\beta\alpha^2\\&\text{csch}^2(\alpha x)+S^{-1}[u^2S[U_{6x}+\\&15U_{xx}u_{3x}+15U_xu_{4x}+45U_x^2U_{xx}\\&-5(U_{3xt}+3U_{xx}U_t+3U_xU_{xt})+18V_x]],\\&V(x,t)=-\frac{4}{9}\beta\alpha^4-\frac{16}{27}\alpha^6+\frac{5}{9}\beta^2\alpha^2-\frac{5}{54}\beta^3+\\&(\frac{20}{9}\beta\alpha^4+\frac{16}{9}\alpha^6-\frac{5}{9}\beta^2\alpha^2)\coth^2(\alpha x)-\\&S^{-1}[uS[-V_{3x}-3V_xU_x-3VU_{xx}]],\end{aligned} \tag{16}$$

Now, applying the homotopy perturbation method, we get

$$\begin{aligned}&5\sum_{n=0}^{\infty}p^nU_n=a_0+2\alpha\coth(\alpha x)+2\beta\alpha^2\text{t csch}^2(\alpha x)+\\&pS^{-1}[u^2S[(\sum_{n=0}^{\infty}p^nU_n)_{6x}+15(\sum_{n=0}^{\infty}p^nU_n)_{xx}(\sum_{n=0}^{\infty}p^nU_n)_{3x}\\&+15(\sum_{n=0}^{\infty}p^nU_n)_x(\sum_{n=0}^{\infty}p^nU_n)_{4x}+45(\sum_{n=0}^{\infty}p^nU_n)_x^2\\&\times(\sum_{n=0}^{\infty}p^nU_n)_{xx}-5[(\sum_{n=0}^{\infty}p^nU_n)_{(3x)t}+\\&3(\sum_{n=0}^{\infty}p^nU_n)_{xx}(\sum_{n=0}^{\infty}p^nU_n)_t+\\&3(\sum_{n=0}^{\infty}p^nU_n)_x(\sum_{n=0}^{\infty}p^nU_n)_{xt}]+18(\sum_{n=0}^{\infty}p^nV_n)_x]],\end{aligned} \tag{17}$$

$$\begin{aligned}&\sum_{n=0}^{\infty}p^nV_n=-\frac{4}{9}\beta\alpha^4-\frac{16}{27}\alpha^6+\frac{5}{9}\beta^2\alpha^2-\frac{5}{54}\beta^3+\\&(\frac{20}{9}\beta\alpha^4+\frac{16}{9}\alpha^6-\frac{5}{9}\beta^2\alpha^2)\coth^2(\alpha x)-\\&pS^{-1}[uS[(\sum_{n=0}^{\infty}p^nV_n)_{3x}+3(\sum_{n=0}^{\infty}p^nV_n)_x(\sum_{n=0}^{\infty}p^nU_n)_x+\\&3(\sum_{n=0}^{\infty}p^nV_n)(\sum_{n=0}^{\infty}p^nU_n)_{xx}]],\end{aligned} \tag{18}$$

using eqs. (17) and (18) to compare the coefficient of like powers of $p$ then we have

$$\begin{aligned}&p^0:U_0(x,t)=a_0+2\alpha\coth(\alpha x)-2\beta\alpha^2\text{t csch}^2(\alpha x),\\&p^1:U_1(x,t)=S^{-1}[u^2S[\frac{1}{5}[(U_0)_{6x}+15(U_0)_{xx}(U_0)_{3x}+\\&15(U_0)_x(U_0)_{4x}+45(U_0)_x^2(U_0)_{xx}-5\{(U_0)_{(3x)t}+\\&3(U_0)_{xx}(U_0)_t+\ 3(U_0)_x(U_0)_{xt}\}+18(V_0)_x]]],\\&p^2:U_2(x,t)=S^{-1}[u^2S[\frac{1}{5}[(U_1)_{6x}+15\{(U_0)_{xx}(U_1)_{3x}+\\&(U_1)_{xx}(U_0)_{3x}\}+15\{(U_0)_x(U_1)_{4x}+(U_1)_x(U_0)_{4x}\}+\\&45\{(U_0)_x^2(U_1)_{xx}+2(U_1)_x(U_0)_x(U_0)_{xx}\}-\\&\quad 5\{(U_1)_{(3x)t}+3\{(U_0)_{xx}(U_1)_t+(U_1)_{xx}(U_0)_t\}+\\&3\{(U_0)_x(U_1)_{xt}+(U_1)_x(U_0)_{xt}\}\}\ +\ 18(V_1)_x]]],\end{aligned} \tag{19}$$

$$\begin{aligned}&p^0:V_0(x,t)=-\frac{4}{9}\beta\alpha^4-\frac{16}{27}\alpha^6+\frac{5}{9}\beta^2\alpha^2-\frac{5}{54}\beta^3+\\&(\frac{20}{9}\beta\alpha^4+\frac{16}{9}\alpha^6-\frac{5}{9}\beta^2\alpha^2)\coth^2(\alpha x),\\&p^1:V_1(x,t)=S^{-1}[uS[(V_0)_{3x}+\\&3(V_0)_x(U_0)_x+\ 3V_0(U_0)_{xx}]],\end{aligned}$$

$$p^2 : V_2(x,t) = S^{-1}[uS[(V_1)_{3x} + 3\{(V_0)_x(U_1)_x + (V_1)_x(U_0)_x\} + 3\{V_0(U_1)_{xx} + V_1(U_0)_{xx}\}]], \quad (20)$$

the other components can be found similarly. After some reduction, we have

$$U_0(x,t) = a_0 + 2\alpha\coth(\alpha x) + 2t\beta\alpha^2\text{csch}^2(\alpha x),$$
$$U_1(x,t) = -2\alpha^3\beta^2t^2\coth(\alpha x)\text{csch}^2(\alpha x)$$
$$U_2(x,t) = \frac{2}{3}\alpha^4\beta^3t^3\{2\coth^2(\alpha x)\text{csch}^2(\alpha x) + \text{csch}^4(\alpha x)\}, \quad (21)$$

$$V_0(x,t) = -\frac{4}{9}\beta\alpha^4 - \frac{16}{27}\alpha^6 + \frac{5}{9}\beta^2\alpha^2 - \frac{5}{54}\beta^3 + (\frac{20}{9}\beta\alpha^4 + \frac{16}{9}\alpha^6 - \frac{5}{9}\beta^2\alpha^2)\coth^2(\alpha x),$$
$$V_1(x,t) = 2t\alpha\beta(\frac{20}{9}\beta\alpha^4 + \frac{16}{9}\alpha^6 - \frac{5}{9}\beta^2\alpha^2)\coth(\alpha x)\text{csch}^2(\alpha x),$$
$$V_2(x,t) = -(\frac{20}{9}\beta\alpha^4 + \frac{16}{9}\alpha^6 - \frac{5}{9}\beta^2\alpha^2) \times t^2\{2\alpha^2\beta^2\coth^2(\alpha x)\text{csch}^2(\alpha x) + \text{csch}^4(\alpha x)\}. \quad (22)$$

Therefore, using the eqs.(21),(22) then approximate solutions of the system of equations of eqs. (1) take the following forms:

$$U(x,t) = a_0 + 2\alpha\coth(\alpha x) + 2t\beta\alpha^2\text{csch}^2(\alpha x) - 2\alpha^3\beta^2t^2\coth(\alpha x)\text{csch}^2(\alpha x) + \frac{2}{3}\alpha^4\beta^3t^3 \{2\coth^2(\alpha x)\text{csch}^2(\alpha x) + \text{csch}^4(\alpha x)\} + ..., \quad (23)$$

$$V(x,t) = -\frac{4}{9}\beta\alpha^4 - \frac{16}{27}\alpha^6 + \frac{5}{9}\beta^2\alpha^2 - \frac{5}{54}\beta^3 + (\frac{20}{9}\beta\alpha^4 + \frac{16}{9}\alpha^6 - \frac{5}{9}\beta^2\alpha^2)\{\coth^2(\alpha x) + 2t\alpha\beta\coth(\alpha x)\text{csch}^2(\alpha x) - 2t^2\alpha^2\beta^2\coth^2(\alpha x) \times\text{csch}^2(\alpha x) - t^2\text{csch}^4(\alpha x)\} + ... \quad (24)$$

The accuracy of the homotopy sumudu perturbation method for the eqs. (1) under conditions (13) is controllable and the absolute errors are very small with the present choice of $x, t$. These results are listed in Tables 1, 2 and Figures 1-4. It is also clear that when more terms for HPSTM are computed, the numerical results are much more closer to the corresponding exact solution.

Table 1. The HPSTM results of $U(x,t)$ for the first three approximation in comparison with the exact solution if $a_0 = 1$, $\beta = \alpha = .01$,and $t = 20$ for the solution of the system (1) with the initial conditions(13).

| $x$ | $U_{Exact}$ | $U_{HPSTM}$ | $\lvert U_{Exact} - U_{HPSTM}\rvert$ |
|---|---|---|---|
| -50 | 0.956868 | 0.956869 | 1.27251607E-6 |
| -40 | 0.947597 | 0.9476 | 2.48972102E-6 |
| -30 | 0.931774 | 0.93178 | 5.90319012E-6 |
| -20 | 0.899647 | 0.899667 | 1.98912333E-5 |
| -10 | 0.803242 | 0.8034 | 1.58430305E-4 |
| 10 | 1.20473 | 1.20457 | 1.58430305E-4 |
| 20 | 1.10233 | 1.10231 | 2.00989070E-5 |
| 30 | 1.06909 | 1.06908 | 5.94270457E-6 |
| 40 | 1.05288 | 1.05287 | 2.50222741E-6 |
| 50 | 1.04343 | 1.04343 | 1.27764171E-6 |

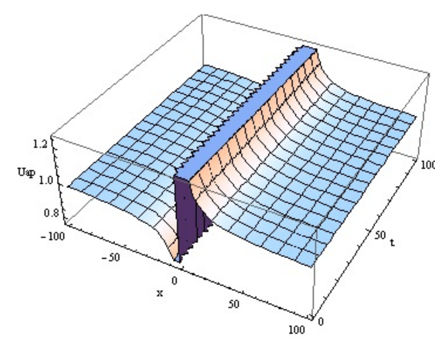

Figure 1: The exact solution of U(x,t) for the equations (1) if $a_0 = 1$, $\beta = \alpha = .01$.

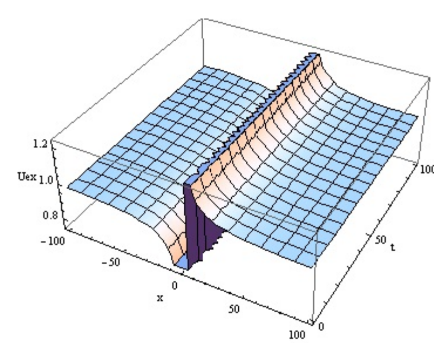

Figure 2: The approximate solution of U(x,t) for the first three approximation of the equations (1) if $a_0 = 1$, $\beta = \alpha = .01$.

Table 2. The HPSTM results of $V(x,t)$ for the first three approximation in comparison with the analytical solution if $a_0 = 1$, $\beta = \alpha = .01$,and $t = 20$

for the solution of the system (1) with the initial conditions(13).

| $x$ | $V_{Ex}$ | $V_{HPS}$ | $\lvert V_{Ex} - V_{HPS}\rvert$ |
|---|---|---|---|
| -50 | -1.1204856E-7 | -1.12232E-7 | 1.853690E-7 |
| -40 | -1.2401512E-7 | -1.24375E-7 | 1.855450E-7 |
| -30 | -1.4991042E-7 | -1.50764E-7 | 1.860388E-7 |
| -20 | -2.2395349E-7 | -2.26833E-7 | 1.880649E-7 |
| -10 | -6.2401173E-7 | -6.46982E-7 | 2.081564E-7 |
| 10 | -6.2401173E-7 | -6.00746E-7 | 1.619202E-7 |
| 20 | -2.2395349E-7 | -2.21054E-7 | 1.822860E-7 |
| 30 | -1.4991042E-7 | -1.49052E-7 | 1.843273E-7 |
| 40 | -1.2401512E-7 | -1.23653E-7 | 1.848238E-7 |
| 50 | -1.1204856E-7 | -1.11863E-7 | 1.850006E-7 |

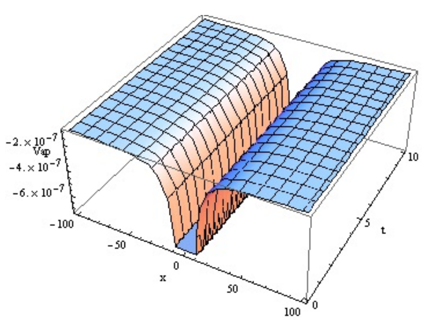


Figure 3: The exact solution of V(x,t) for the equations (1) if $a_0 = 1$, $\beta = \alpha = .01$.

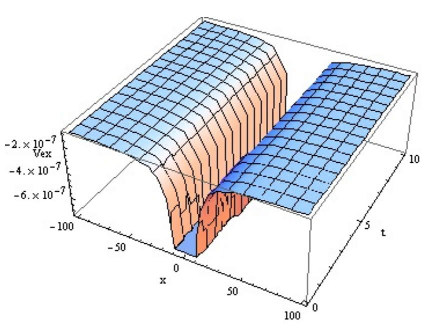


Figure 4: The approximate solution of V(x,t) for the first three approximation for the equations (1) if $a_0 = 1$, $\beta = \alpha = .01$.

### 3.2 On solving the generalized nonlinear Hirota Satsuma coupled KdV equations by HPSTM

In this subsection, we find the solutions $U(x,t)$ and $V(x,t)$, $W(x,t)$ satisfying the generalized nonlinear Hirota Satsuma coupled KdV system of equations (2) with the following initial conditions [13]:

$$U(x,0) = \frac{1}{3}(\beta - 2k^2) + 2k^2\tanh^2(kx),$$
$$V(x,0) = -\frac{4k^2c_0(\beta+k^2)}{3c_1^2} + \frac{4k^2(\beta+k^2)}{3c_1}\tanh(kx),$$
$$W(x,0) = c_0 + c_1\tanh(kx), \tag{25}$$

where $k$, $\beta, c_0$ and $c_1$ are arbitrary constants.
Taking the sumudu transform on both sides of equation (2) subject to the initial condition (25), we have

$$S[U(x,t)] = \frac{1}{3}(\beta - 2k^2) + 2k^2\tanh^2(kx) -$$
$$uS[-\frac{1}{2}U_{xxx} + 3UU_x - 3VW_x],$$
$$S[V(x,t)] = -\frac{4k^2c_0(\beta+k^2)}{3c_1^2} +$$
$$\frac{4k^2(\beta+k^2)}{3c_1}\tanh(kx) - uS[V_{xxx} - 3UV_x],$$
$$S[W(x,t)] = \frac{4k^2c_0(\beta+k^2)}{3c_1^2} + \frac{4k^2(\beta+k^2)}{3c_1}\tanh(kx) -$$
$$uS[W_{xxx} - 3UW_x], \tag{26}$$

The inverse of sumudu transform implies that

$$U(x,t) = \frac{1}{3}(\beta - 2k^2) + 2k^2\tanh^2(kx) -$$
$$S^{-1}[uS[-\frac{1}{2}U_{xxx} + 3UU_x - 3(VW)_x]],$$
$$V(x,t) = -\frac{4k^2c_0(\beta+k^2)}{3c_1^2} + \frac{4k^2(\beta+k^2)}{3c_1}\tanh(kx) -$$
$$S^{-1}[uS[V_{xxx} - 3UV_x],$$
$$W(x,t) = \frac{4k^2c_0(\beta+k^2)}{3c_1^2} + \frac{4k^2(\beta+k^2)}{3c_1}\tanh(kx) -$$
$$S^{-1}[uS[W_{xxx} - 3UW_x]], \tag{27}$$

applying the homotopy perturbation method, we get

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = \frac{1}{3}(\beta - 2k^2) + 2k^2\tanh^2(kx) -$$
$$pS^{-1}[uS[-\frac{1}{2}(\sum_{n=0}^{\infty} p^n U_n)_{xxx} +$$
$$3(\sum_{n=0}^{\infty} p^n U_n)(\sum_{n=0}^{\infty} p^n U_n)_x - 3(\sum_{n=0}^{\infty} p^n V_n \sum_{n=0}^{\infty} p^n W_n)_x]],$$

$$\sum_{n=0}^{\infty} p^n V_n(x,t) = (\beta + k^2)(-\frac{4k^2c_0}{3c_1^2} + \frac{4k^2}{3c_1}\tanh(kx))$$
$$-pS^{-1}[uS[(\sum_{n=0}^{\infty} p^n V_n)_{xxx} - 3\sum_{n=0}^{\infty} p^n U_n(\sum_{n=0}^{\infty} p^n V_n)_x]],$$
$$\sum_{n=0}^{\infty} p^n W_n(x,t) = (\beta + k^2)(\frac{4k^2c_0}{3c_1^2} + \frac{4k^2}{3c_1}\tanh(kx)) -$$
$$pS^{-1}[uS[(\sum_{n=0}^{\infty} p^n W_n)_{xxx} - 3\sum_{n=0}^{\infty} p^n U_n(\sum_{n=0}^{\infty} p^n W_n)_x]], \quad (28)$$

using eqs. (28)to compare the coefficient of like powers of $p$ then we have

$$p^0 : U_0(x,t) = \frac{1}{3}(\beta - 2k^2) + 2k^2\tanh^2(kx)$$
$$p^1 : U_1(x,t) = -S^{-1}[uS[-\frac{1}{2}(U_0)_{xxx} + 3(U_0)(U_0)_x - 3(V_0W_0)_x]],$$
$$p^2 : U_2(x,t) = -S^{-1}[uS[-\frac{1}{2}(U_1)_{xxx} + 3((U_1)(U_0)_x + (U_0)(U_1)_x - 3((V_1W_0)_x + (V_0W_1)_x)]], \quad (29)$$

$$p^0 : \ V_0(x,t) = (\beta + k^2)(-\frac{4k^2c_0}{3c_1^2} + \frac{4k^2}{3c_1}\tanh(kx))$$
$$p^1 : \ V_1(x,t) = -S^{-1}[uS[(V_0)_{xxx} - 3U_0(V_0)_x]],$$
$$p^2 \ V_2(x,t) = -S^{-1}[uS[(V_1)_{xxx} - 3(U_0(V_1)_x + U_1(V_0)_x)]], \quad (30)$$

$$p^0 : W_0(x,t) = \frac{4k^2c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)}{3c_1}\tanh(kx)$$
$$p^1 : W_1(x,t) = -S^{-1}[uS[(W_0)_{xxx} - 3U_0(W_0)_x]],$$
$$p^2 : \ W_2(x,t) = -S^{-1}[uS[(W_1)_{xxx} - 3(U_1(W_0)_x + U_0(W_1)_x)]], \quad (31)$$

The other components can be found similarly. Consequently, we have

$$U_0(x,t) = \frac{1}{3}(\beta - 2k^2) + 2k^2\tanh^2(kx),$$
$$U_1(x,t) = 4\beta t k^3 \text{sech}^2(kx)\tanh(kx),$$
$$U_2(x,t) = 2\beta^2 k^4 t^2 \text{sech}^4(kx)\{2 - \cosh(2kx)\},$$
$$U_3(x,t) = \frac{2}{3}\beta^3 k^5 t^3 \text{sech}^5(kx) \times\{-11\sinh(kx) + \sinh(3kx)\} \quad (32)$$

$$V_0(x,t) = -\frac{4k^2c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)}{3c_1}\tanh(kx),$$
$$V_1(x,t) = \frac{4}{3c_1}\beta t k^3(\beta + k^2)\text{sech}^2(kx),$$
$$V_2(x,t) = -\frac{4}{3c_1}\beta^2 k^4 t^2(\beta + k^2)\text{sech}(kx)\tanh(kx),$$
$$V_3(x,t) = \frac{4}{9c_1}\beta^3 k^5 t^3(\beta + k^2)\text{sech}^4(kx) \{-2 + \cosh(2kx)\}, \quad (33)$$

$$W_0(x,t) = c_0 + c_1\tanh(kx),$$
$$W_1(x,t) = c_1 t k\beta\text{sech}^2(kx),$$
$$W_2(x,t) = -\beta^2 c_1 k^2 t^2 \text{sech}^2(kx)\tanh(kx),$$
$$W_3(x,t) = \frac{1}{3}\beta^3 c_1 k^3 t^3 \text{sech}^4(kx)\{-2 + \cosh(2kx)\}. \quad (34)$$

Therefore, the approximate series solution is given by

$$U(x,t) = \frac{1}{3}(\beta - 2k^2) + 2k^2\tanh^2(kx) + 4\beta t k^3\text{sech}^2(kx)\tanh(kx) + 2\beta^2 k^4 t^2\text{sech}^4(kx)\{2 - \cosh(2kx)\} + \frac{2}{3}\beta^3 k^5 t^3\text{sech}^5(kx)\{-11\sinh(kx) + \sinh(3kx)\} + ... \quad (35)$$

$$V(x,t) = -\frac{4k^2c_0(\beta + k^2)}{3c_1^2} + \frac{(\beta + k^2)}{3c_1}(4k^2\tanh(kx) + 4\beta t k^3\text{sech}^2(kx)) - \frac{4}{3c_1}\beta^2 k^4 t^2(\beta + k^2)\text{sech}(kx)\tanh(kx) + \frac{4}{9c_1}\beta^3 k^5 t^3(\beta + k^2)\text{sech}^4(kx)\{-2 + \cosh(2kx)\} + ..., \quad (36)$$

$$W(x,t) = c_0 + c_1\tanh(kx) + c_1 t k\beta\text{sech}^2(kx) - \beta^2 c_1 k^2 \times t^2\text{sech}^2(kx)\tanh(kx) + \frac{1}{3}\beta^3 c_1 k^3 t^3\text{sech}^4(kx) \times\{-2 + \cosh(2kx)\} + ... \quad (37)$$

With reference to [13], the exact solutions $U(x,t)$, $V(x,t)$ and $W(x,t)$ of the system of equations (2) take the following forms

$$U(x,t) = \frac{\beta - 2k^2}{3} + 2k^2\tanh^2[k(x + \beta t)],$$
$$V(x,t) = -\frac{4k^2c_0(\beta + k^2)}{3c_1^2} + \frac{4k^2(\beta + k^2)}{3c_1}\tanh[k(x + \beta t)],$$
$$W(x,t) = c_0 + c_1\tanh[k(x + \beta t)]. \quad (38)$$

Both the exact results and the approximate solutions are plotted in (Figures 5-10). The numerical results are much more closer to the corresponding exact solutions with the initial condition. The comparison between the exact solutions (38) and the approximate solutions (35), (36) and (37) are shown in tables 3.3, 3.4, 3.5 when $c_0 = 1.5,\ c_1 = .1, \beta = 1.5$ and $k = 0.1$. It seems that the errors are very small.

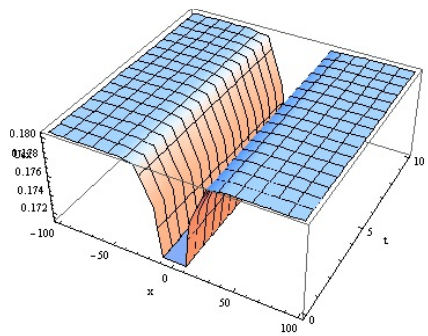


Figure 5: The approximate solution for $U(x,t)$ in Eq. (35) if $\beta = 1.5$ and $k = 0.1$.

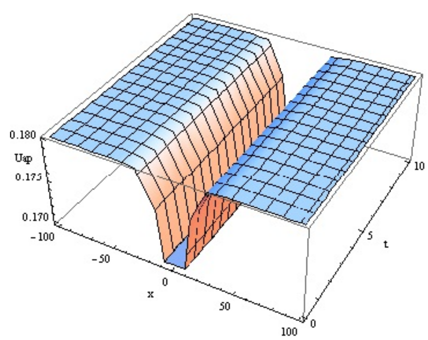


Figure 6: The exact solution for $U(x,t)$ in Eq. (36) if $\beta = 1.5$ and $k = 0.1$.

Table 3.3. The HPSTM results of $U(x,t)$ for the first four approximation in comparison with the exact solution if $\beta = 1.5$ , $k = .1$ and $t = .1$ for the solution of the system of equations (2) with the initial conditions (25).

| $x$ | $U_{Exact}$ | $U_{HPSTM}$ | $\lvert U_{Exact} - U_{HPSTM}\rvert$ |
|---|---|---|---|
| -50 | 0.51333 | 0.51333 | 1.2301271E-13 |
| -40 | 0.513306 | 0.513306 | 9.0127905E-13 |
| -30 | 0.51313 | 0.51313 | 6.2057026E-12 |
| -20 | 0.511879 | 0.511879 | 2.4218294E-11 |
| -10 | 0.504741 | 0.504741 | 2.3615165E-10 |
| 00 | 0.493338 | 0.493338 | 6.7491395E-10 |
| 10 | 0.505124 | 0.505124 | 2.3270352E-10 |
| 20 | 0.511961 | 0.511961 | 2.4189539E-11 |
| 30 | 0.513142 | 0.513142 | 6.1378679E-12 |
| 40 | 0.513307 | 0.513307 | 8.9084295E-13 |
| 50 | 0.51333 | 0.51333 | 1.2179146E-13 |

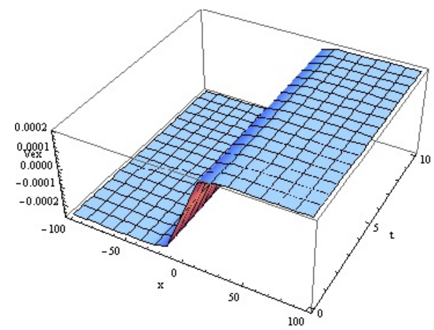


Figure 7: The approximate solution for $V(x,t)$ in Eq. (36) if $c_0 = 1.5,\ c_1 = .1, \beta = 1.5$ and $k = 0.1$.

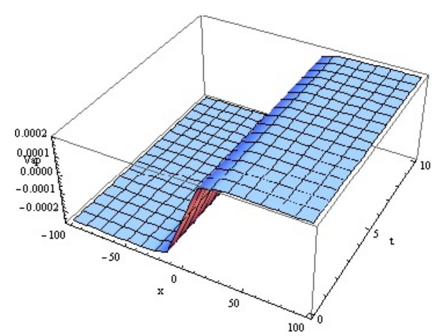


Figure 8: The exact solutions for $V(x,t)$ in Eq. (38) if $c_0 = 1.5,\ c_1 = .1, \beta = 1.5$ and $k = 0.1$.

Table 3.4. The MVIM results of $V(x,t)$ for the first three approximation in comparison with the exact solution if $c_0 = 1.5,\ c_1 = .1,\ \beta = 1.5,\ k = .1$ and $t = .1$ for the solution of the system of equations (1) with the initial conditions (25).

| $x$ | $V_{Exact}$ | $V_{HPSTM}$ | $\|V_{Exact} - V_{HPSTM}\|$ |
|---|---|---|---|
| -50 | -3.22131 | -3.22131 | 8.2854612E-11 |
| -40 | -3.22119 | -3.22119 | 6.1079052E-10 |
| -30 | -3.22031 | -3.22031 | 4.4357273E-9 |
| -20 | -3.21387 | -3.21387 | 2.8796159E-8 |
| -10 | -3.17205 | -3.17205 | 7.0109642E-8 |
| 00 | -3.01698 | -3.01698 | 2.2647961E-7 |
| 10 | -2.86541 | -2.86541 | 7.0674640E-8 |
| 20 | -2.8257 | -2.8257 | 2.8431448E-8 |
| 30 | -2.81963 | -2.81963 | 4.3709915E-9 |
| 40 | -2.8188 | -2.8188 | 6.017208E-10 |
| 50 | -2.81868 | -2.81868 | 8.162182E-11 |

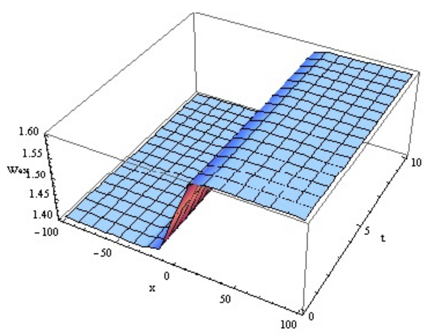


Figure 9: The approximate solution for $W(x,t)$ in Eq. (37) if $c_0 = 1.5,\ c_1 = .1, \beta = 1.5$ and $k = 0.1$.

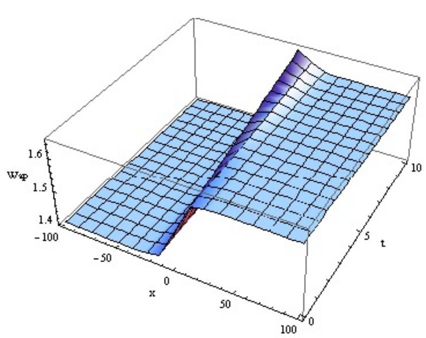


Figure 10: The exact solution for $W(x,t)$ in Eq. (38) if $c_0 = 1.5,\ c_1 = .1, \beta = 1.5$ and $k = 0.1$.

Table 3.5. The HPSTM results of $W(x,t)$ for the first three approximation in comparison with the exact solution if $c_0 = 1.5,\ c_1 = .1,\ \beta = 1.5$ , $k = .1$ and $t = .1$ for the solution of the system of equations (2) with the initial conditions (25).

| $x$ | $W_{Exact}$ | $W_{appr}$ | $\|W_{Exact} - W_{appr}\|$ |
|---|---|---|---|
| -50 | 1.40001 | 1.40001 | 4.11530809E-11 |
| -40 | 1.40007 | 1.40007 | 3.03372660E-10 |
| -30 | 1.40051 | 1.40051 | 2.203175641E-9 |
| -20 | 1.4037 | 1.4037 | 1.430272833E-8 |
| -10 | 1.42448 | 1.42448 | 3.482267030E-8 |
| 00 | 1.5015 | 1.5015 | 1.124898760E-7 |
| 10 | 1.57678 | 1.57678 | 3.510329849E-8 |
| 20 | 1.59651 | 1.59651 | 1.412158034E-8 |
| 30 | 1.59952 | 1.59952 | 2.171022472E-9 |
| 40 | 1.59993 | 1.59993 | 2.988682634E-10 |
| 50 | 1.59999 | 1.59999 | 4.054068192E-11 |

# 4 Conclusion

In the present paper, the homotopy perturbation sumudu transform method is used to find the solutions of the nonlinear coupled equations in the mathematical physics via the (1+1)-dimensional Ramani equations and the (1+1)-dimensional generalized nonlinear Hirota Satsuma coupled KdV equations together with the initial conditions. It can be concluded that the HPSTM is very powerful and efficient in finding the exact solutions for wide classes of problems. It is worth pointing out that the HPSTM presents rapid convergence solutions.

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