

# Applications of Steiner symmetrization to some extremal problems in geometric function theory

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*Abstract:* In this paper we investigate properties of the Steiner symmetrization in the complex plane. We use two recursive dynamic processes in order to derive some inequalities on analytic functions in the unit disk. We answer a question that was asked by Albert Baernstein II, regarding the coefficients of circular symmetrization functions. We mostly deal with the Steiner symmetrization  $G$  of an analytic function  $f$  in the unit disk  $U$ . We pose few problems we can not solve. An intriguing one is that of the inequality

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |G(re^{i\theta})|^p d\theta, \quad 0 < p < \infty$$

which is true for  $p = 2$  but can not be true for too large  $p$ . What is the largest such exponent or its supremum?

*Key-Words:* circular symmetrization, Steiner symmetrization, extremal problems.

## 1 Some extremal problems

**Definition 1** Let  $2 \leq p \leq \infty$ ,  $0 < \alpha < \infty$ . We define:

$$S(p, \alpha) = \{f \mid f \in H(U), f \text{ is univalent in } U,$$

$$f(0) = 0, 1 \leq |f'(0)|, \alpha \leq \|f\|_p\}$$

where  $H(U)$  is the space of all the functions that are holomorphic in the unit disc  $U = \{z \in \mathbb{C} \mid |z| < 1\}$ , and

$$\|f\|_p^p = \lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.$$

So  $\forall f \in S(p, \alpha)$ , the  $f$ -image  $f(U)$  is simply connected and  $f(U) \neq \mathbb{C}$ .

**Definition 2** Let  $2 \leq p \leq \infty$ ,  $0 < \alpha < \infty$ . We define:

$$N(p, \alpha) = \inf_{f \in S(p, \alpha)} \int_0^{2\pi} |f'(e^{i\theta})| d\theta.$$

**Proposition 3** If  $2 \leq p \leq \infty$ ,  $0 < \alpha < \infty$ , then there exists a function  $f \in S(p, \alpha)$  such that:

$$N(p, \alpha) = \int_0^{2\pi} |f'(e^{i\theta})| d\theta.$$

### Proof.

Since  $g(z) = (\alpha + 1)z \in S(p, \alpha)$  and  $\int_0^{2\pi} |g'(e^{i\theta})| d\theta = 2\pi(\alpha + 1)$ , it follows that  $N(p, \alpha) \leq 2\pi(\alpha + 1)$ . So it will suffice to consider the following subfamily  $B(p, \alpha) = \{f \mid f \in S(p, \alpha), \|f\|_\infty \leq \pi(\alpha + 1)\}$  of  $S(p, \alpha)$ . The subfamily  $B(p, \alpha)$  of  $S(p, \alpha)$  is a normal family (because it is uniformly bounded). Moreover,  $B(p, \alpha)$  is a compact family. For if  $f_n \in B(p, \alpha)$  and  $f_n \rightarrow f$  uniformly on compact subsets of  $U$  then  $f \in B(p, \alpha)$  or  $f \equiv 0$  ( $f(0) = 0$ ). But the condition  $1 \leq |f'(0)|$  prohibits the second alternative.  $\diamond$

**Remark 4** If in the definition of  $S(p, \alpha)$  the condition  $1 \leq |f'(0)|$  would have been dropped out, then the claim in Proposition 3 would have been false. Here is an:

**Example 5** For  $p = \infty$  it is clear that  $2\alpha \leq N(p, \alpha)$  because for every compatible function  $f$  we have  $0 \in f(U)$  and there is a point  $\omega \in \overline{f(U)}$  that satisfies  $|\omega| = \alpha$  (because  $\alpha \leq \|f\|_\infty$ ). Now, consider a sequence of simply connected domains  $\Omega_n$  that satisfy  $0, \alpha \in \Omega_n$ , such that these domains become narrow as  $n \rightarrow \infty$  and converge to the closed inter-

val on the  $X$ -axis,  $[0, \alpha]$ , and have smooth boundaries  $\partial\Omega_n$ . By the Riemann Mapping Theorem for each  $n$  there exists a conformal and onto mapping  $f_n : U \rightarrow \Omega_n$  such that  $f_n(0) = 0$ . Clearly we have  $\lim \int_0^{2\pi} |f'_n(e^{i\theta})| d\theta = 2\alpha$  and hence if the condition  $1 \leq |f'(0)|$  would have been dropped out from the definition of  $S(p, \alpha)$  we would have had  $N(\infty, \alpha) = 2\alpha$ , but there were no extremal function. Note that in our construction  $f_n \rightarrow 0$  and clearly the function 0 is not a compatible function.

**Proposition 6** *If  $2 \leq p \leq \infty$ ,  $0 < \alpha < \infty$  and if  $f \in S(p, \alpha)$  was extremal for  $N(p, \alpha)$ , then the simply connected domain  $f(U)$  can have no slits.*

**Proof.**

Let us assume to the contrary that  $f(U)$  had a slit  $\Gamma$ . Let  $D = f(U) \cup \Gamma$ , i.e.  $D$  is the simply connected domain we obtain from  $f(U)$  by erasing the slit  $\Gamma$ . By the Riemann Mapping Theorem there exists a conformal mapping  $F(z)$  defined on  $U$  such that  $F(U) = D$  and  $F(0) = 0$ . We define the standard mapping  $\phi : U \rightarrow U$ , by the formula,  $\phi(z) = F^{-1}(f(z))$ . Then  $|\phi(z)| \leq |z|$  and  $\forall z \in U f(z) = F(\phi(z))$ . Thus  $f \prec F$ , i.e.  $f$  is subordinated to  $F$ . This implies that the following three conditions hold true:

1.  $F(0) = 0$ ,  $F$  conformal.
2.  $1 \leq |f'(0)| \leq |F'(0)|$ , by the Schwarz Lemma.
3.  $\alpha \leq \|f\|_p \leq \|F\|_p$ , by Littlewood Subordination Theorem, [15] or page 422 in [13].

These imply that  $F \in S(p, \alpha)$ . But clearly we have  $\int_0^{2\pi} |F'(e^{i\theta})| d\theta < \int_0^{2\pi} |f'(e^{i\theta})| d\theta$ , a contradiction to the fact that  $f$  is extremal for  $N(p, \alpha)$ .  $\diamond$

We can strengthen the Proposition 6:

**Proposition 7** *If  $2 \leq p \leq \infty$ ,  $0 < \alpha < \infty$  and if  $f \in S(p, \alpha)$  was extremal for  $N(p, \alpha)$ , then the simply connected domain  $f(U)$  is a convex domain.*

**Proof.**

Let us assume to the contrary that  $f(U)$  is not a convex domain. By Proposition 6 it follows that there are points  $\omega_1, \omega_2 \in \partial f(U)$  such that  $\omega_1 \neq \omega_2$  and such that the open non-degenerated segment  $\Gamma$  between  $\omega_1$  and  $\omega_2$  lies in  $\mathbb{C} - \overline{f(U)}$ . Let  $D$  be the simply connected domain we get by the union of  $f(U)$  and the bounded domain whose boundary is the segment  $[\omega_1, \omega_2]$  and the corresponding part of  $\partial f(U)$  between  $\omega_1$  and  $\omega_2$ . From this point the proof proceeds as that of Proposition 6. Namely, by the Riemann Mapping Theorem there exists a conformal mapping  $F$  defined on  $U$  so that  $F(U) = D$  and  $F(0) = 0$ . Then  $f \prec F$  and so we have the same three conditions:

1.  $F(0) = 0$ ,  $F$  conformal.
2.  $1 \leq |f'(0)| \leq |F'(0)|$ , by the Schwarz Lemma.
3.  $\alpha \leq \|f\|_p \leq \|F\|_p$ , by Littlewood Subordination Theorem, [15] or page 422 in [13].

These imply that  $F \in S(p, \alpha)$ . But clearly we have  $\int_0^{2\pi} |F'(e^{i\theta})| d\theta < \int_0^{2\pi} |f'(e^{i\theta})| d\theta$ , a contradiction to the fact that  $f$  is extremal for  $N(p, \alpha)$ .  $\diamond$

**Proposition 8** *If  $2 \leq p \leq \infty$ ,  $0 < \alpha < \infty$  and if  $f \in S(p, \alpha)$  was extremal for  $N(p, \alpha)$ , and if  $|a| < 1$  then either:*

$$|f'(a)| \leq \frac{1}{1 - |a|^2},$$

or

$$\int_0^{2\pi} |f(e^{i\theta}) - f(a)|^p \left( \frac{1 - |a|^2}{|e^{i\theta} - a|^2} \right) d\theta \leq \alpha^p.$$

*In particular for  $a = 0$ : either  $|f'(0)| = 1$  or  $\int_0^{2\pi} |f(e^{i\theta})|^p d\theta = \|f\|_p^p = \alpha^p$ .*

**Proof.**

If

$$\phi(z) = \frac{z + a}{1 + \bar{a}z}$$

then

$$\phi'(z) = \frac{1 - |a|^2}{(1 + \bar{a}z)^2}.$$

We define  $F(z) = f(\phi(z)) - f(a)$ . Then we have:

1.  $\int_0^{2\pi} |F'(e^{i\theta})| d\theta = \int_0^{2\pi} |f'(e^{i\theta})| d\theta$  because the images  $F(U)$  and  $f(U)$  are congruent.
2.  $F'(0) = (1 - |a|^2)f'(a)$ .

We also have the identity:

$$\int_0^{2\pi} |F(e^{i\psi})|^p d\psi = \int_0^{2\pi} \left| f \left( \frac{e^{i\psi} + a}{1 + \bar{a}e^{i\psi}} \right) - f(a) \right|^p d\psi.$$

We make a change of the integration variable:

$$e^{i\theta} = \frac{e^{i\psi} + a}{1 + \bar{a}e^{i\psi}}, \quad d\psi = \frac{1 - |a|^2}{|e^{i\theta} - a|^2} d\theta,$$

and we rewrite the above identity as follows:

3.

$$\int_0^{2\pi} |F(e^{i\theta})|^p d\theta = \int_0^{2\pi} |f(e^{i\theta}) - f(a)|^p \left( \frac{1 - |a|^2}{|e^{i\theta} - a|^2} \right) d\theta.$$

Now 1, 2 and 3 above imply that:

4. If we have both:  $1 \leq (1 - |a|^2)|f'(a)|$  and

$$\alpha^p \leq \int_0^{2\pi} |f(e^{i\theta}) - f(a)|^p \left( \frac{1 - |a|^2}{|e^{i\theta} - a|^2} \right) d\theta,$$

then  $F \in S(p, \alpha)$  and hence  $F$  is extremal for  $N(p, \alpha)$ . If both the inequalities in 4 are sharp (none of them is an equality), then there is an  $M > 1$  such that  $M^{-1}F(z) \in S(p, \alpha)$ , which contradicts the fact that  $F(z)$  is extremal for  $N(p, \alpha)$ . Thus at least one of the two inequalities in 4 is in fact an equality and the proposition follows.  $\diamond$

## 2 Facts about symmetrizations

Motivated by the desire to solve the family of the extremal problems  $N(p, \alpha)$ , we will discuss in this section properties of symmetrizations of functions. Specifically we will consider symmetrizations that were introduced by Pölya and by Steiner. We will recall results from the paper [2].

**Definition 9** Let  $D$  be a domain in the Riemann sphere  $\mathbb{C} \cup \{\infty\}$ . The circular symmetrization of  $D$  is the domain  $D^*$  that is defined as follows: for each  $t \in (0, \infty)$  we define  $D(t) = \{\theta \in [0, 2\pi] \mid te^{i\theta} \in D\}$ . If  $D(t) = [0, 2\pi]$  then the intersection of  $D^*$  with the circle  $|z| = t$  is the full circle. If  $D(t) = \emptyset$  then the intersection of  $D^*$  with the circle  $|z| = t$  is the empty set  $\emptyset$ . If  $D(t)$  is a non trivial subset of  $[0, 2\pi]$  which has the measure  $|D(t)| = \alpha'$ , then the intersection of  $D^*$  with the circle  $|z| = t$  is the unique circular arc given by  $\{te^{i\theta} \mid |\theta| < \alpha'/2\}$ . Finally  $D^*$  contain the point  $0 (\infty)$  if and only if  $D$  contains the point  $0 (\infty)$ .

Section (j) of the paper [2] includes a proof of an important principle in symmetrization:

Let  $f \in H(U)$  and let us denote  $D = f(U)$ . Let  $D_0$  be a simply connected domain that contains  $D^*$ , and let us assume that  $D_0$  is not the full complex plane ( $\mathbb{C}$ ). Let  $F$  be a conformal mapping of  $U$  onto  $D_0$  that satisfies  $F(0) = |f(0)|$ . The following result is proved in [2]:

**Theorem 6. ([2])** If  $\Phi$  is a convex non-decreasing function on  $(-\infty, \infty)$ ,  $f \in H(U)$  and  $F$  as above, then for all  $0 \leq r < 1$  we have:

$$\int_{-\pi}^{\pi} \Phi(\log |f(re^{i\theta})|)d\theta \leq \int_{-\pi}^{\pi} \Phi(\log |F(re^{i\theta})|)d\theta.$$

If we choose in Theorem 6 above,  $\Phi(x) = e^{2x}$  and assume that we have the following expansions:  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  and  $F(z) = \sum_{n=0}^{\infty} A_n z^n$ , then we obtain the inequality  $\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \leq \sum_{n=0}^{\infty} |A_n|^2 r^{2n}$  for  $0 \leq r < 1$ . By the definition of  $F$  we have  $|A_0| = |a_0|$ , thus if we subtract  $|A_0|^2$  from both sides of the inequality and divide by  $r^2$  and then take  $r \rightarrow 0^+$  we obtain  $|f'(0)| \leq |F'(0)|$ , a classical result of Walter Hayman. If  $f$  is one-to-one in  $U$  then both

$D$  and  $D^*$  are simply connected and we can take  $F$  to be a conformal mapping from  $U$  onto  $D^*$  for which  $F(0) = |f(0)|$ .

At the end of section (k) in [2] the author asks if the following is true for all  $n$ :  $|a_n| \leq |A_n|$ ? Is the following weaker set of inequalities true:  $\sum_{k=0}^n |a_k|^2 \leq \sum_{k=0}^n |A_k|^2$ ? A. Baernstein II, remarks that these last inequalities if true, would prove a conjecture of Littlewood: If  $f$  is one-to-one and analytic in  $U$  and if  $f(z) \neq 0$ , for  $z \in U$ , then for each  $n > 1$  we have:  $a_n \leq 4n|a_0|$ .

(We give a proof for this assertion, for the reader's convenience. Assume  $a_0 = 1$ . We define  $g(z) = \sqrt{f(z)}$  and  $h(z) = -g(z)$ . Then  $g, h \in H(U)$  (because  $f(z)$  never vanishes in  $U$ ). Both  $g$  and  $h$  are one-to-one in  $U$  (because  $g(z_1) = g(z_2) \Rightarrow f(z_1) = f(z_2) \Rightarrow z_1 = z_2$ ). Let us suppose that  $g(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ , where  $\alpha_0 = 1$ . We denote  $D = g(U)$ . If  $\xi_0 \in D$  then  $-\xi_0 \notin D$  (because  $g(U) \cap h(U) = \emptyset$ ). Hence for any  $0 < t < \infty$  we must have  $|D(t)| \leq \pi$ . We conclude that for the symmetrization we have:  $D^* \subseteq \{z \in \mathbb{C} \mid \Re\{z\} \geq 0\}$ . Let  $G(z)$  be a conformal mapping from  $U$  onto  $D^*$  such that  $G(0) = g(0) = 1$ . Then for all  $|z| < 1$  we have  $\Re\{z\} \geq 0$  and hence if  $G(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$  then by a result of Constantine Carathéodory we have  $|B_n| \leq 2$  for all  $n = 1, 2, 3, \dots$ . We conclude that  $\sum_{k=1}^n |B_k|^2 \leq 4n$  which implies (assuming Baernstein's assumption above) that  $\sum_{k=0}^n |\alpha_k|^2 \leq 4n$ . We recall that  $f = g^2$  and hence:

$$\begin{aligned} |a_n|^2 &= |\alpha_0 \alpha_n + \alpha_1 \alpha_{n-1} + \dots + \alpha_n \alpha_0|^2 \leq \\ &\leq (2|\alpha_n| + |\alpha_1| |\alpha_{n-1}| + \dots + |\alpha_{n-1}| |\alpha_0|)^2 \leq \\ &(4 + 4(n-1)) \cdot 4n = (4n)^2. \end{aligned}$$

$\diamond$ )

The Bieberbach conjecture implies Littlewood's conjecture, [9]. So by now we know that both conjectures are true, [4].

Concerning the first question posed by Albert Baernstein II (above): Let  $f$  be a conformal function defined on  $U$ . We assume that  $f(U)$  has a finite area. Let us denote  $D = f(U)$ , and let  $F$  be a conformal mapping of  $U$  onto the symmetrization  $D^*$  such that  $F(0) = |f(0)|$ . Let us denote by  $S(D)$  and by  $S(D^*)$  the areas of the respective domains. We will use  $t d\phi \cdot dt$  for the area element in polar coordinates. Then we have the identities:

$$\begin{aligned} S(D) &= \int_0^{\infty} \int_{D(t)} t d\phi \cdot dt = \int_0^{\infty} t |D(t)| dt, \\ S(D^*) &= \int_0^{\infty} t |D^*(t)| dt. \end{aligned}$$

By the definition of  $D^*$  it follows that for all  $0 \leq t < \infty$  we have  $D(t) = D^*(t)$  and hence  $S(D) = S(D^*)$  (the well-known fact that circular symmetrization is an area preserving transformation). On the other hand we have  $S(D) = \int_0^1 \int_0^{2\pi} r |f'(re^{i\theta})|^2 d\theta dr$  and if  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  and  $F(z) = \sum_{n=0}^{\infty} A_n z^n$  then we obtain the well known formulas:  $S(D) = \pi \sum_{n=0}^{\infty} n |a_n|^2$ , and  $S(D^*) = \pi \sum_{n=0}^{\infty} n |A_n|^2$ . We conclude that  $\sum_{n=0}^{\infty} n |a_n|^2 = \sum_{n=0}^{\infty} n |A_n|^2$ . We recall that by the definition of  $F$  we have  $A_0 = |a_0|$  and by Hayman's result  $|a_1| \leq |A_1|$  and so either  $|a_n| = |A_n|$  for  $n = 0, 1, 2, \dots$  or there exist  $1 \leq n_1, n_2$  so that  $|a_{n_1}| < |A_{n_1}|$ , and  $|A_{n_2}| < |a_{n_2}|$ . We proved the following:

**Theorem 10** *If  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  is analytic, one-to-one in  $U$  and  $f(U)$  has a finite area. If  $F(z) = \sum_{n=0}^{\infty} A_n z^n$  is the circular symmetrization of  $f(z)$ , then we have:  $\sum_{n=0}^{\infty} n |a_n|^2 = \sum_{n=0}^{\infty} n |A_n|^2$  and either for all  $n = 0, 1, 2, \dots$  we have  $|a_n| = |A_n|$ , or there exist  $1 \leq n_1, n_2$  such that  $|a_{n_1}| < |A_{n_1}|$  and  $|A_{n_2}| < |a_{n_2}|$ .*

Theorem 10 answers the problem mentioned above that was raised by Albert Baernstein II. The answer is negative.

Next, let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ,  $z \in U$ , be analytic and one-to-one, and assume that  $\partial D = \partial f(U)$  is rectifiable.

**Remark 11** *Let  $0 < a < b$ ,  $0 < h$ . Then the sum of the lengths of the legs of a trapezoidal with bases of lengths  $a$  and  $b$  and height of length  $h$  is minimal, if and only if the legs are equal to one another. The same fact is true also for circular a trapezoidal which has bases (of lengths  $a$  and  $b$ ) located on two concentric circles of radii  $R$  and  $R + h$ .*

Let  $\gamma$  be a rectifiable curve. Let the (finite) length of  $\gamma$  be denoted by  $l(\gamma)$ . We chose a finite number of points on  $\gamma$  and join successive points by straight segments. The result is a polygonal curve that is composed of the straight segments  $\gamma_1, \gamma_2, \dots, \gamma_n$  (we assume they come geometrically one after the other). The length of the polygonal curve is the finite sum  $l_n(\gamma) = \sum_{j=1}^n l(\gamma_j)$ . When we refine the division points and take  $n \rightarrow \infty$  we obtain  $l(\gamma) = \lim_{n \rightarrow \infty} l_n(\gamma)$ . By the definition of  $D^*$  and by remark 11 it follows that in order to compute the lengths  $l(\partial D)$  and  $l(\partial D^*)$ , if at each approximation step we take our division points to be the intersection points of  $\partial D$  ( $\partial D^*$ ) with sets of concentric circles centered at the origin and of radii  $0 < t_1 < t_2 < \dots < t_n$ , so that  $0 < t_{j+1} - t_j < \epsilon_n$ ,  $j = 1, 2, \dots, n - 1$ ,  $\epsilon_n \rightarrow_{n \rightarrow \infty} 0^+$ , we obtain the inequalities:  $l_n(\partial D^*) \leq l_n(\partial D)$ . Hence when  $n \rightarrow \infty$

we obtain the well known fact  $l(\partial D^*) \leq l(\partial D)$  (circular symmetrization reduces the perimeter). On the other hand we have the two identities:

$$l(\partial D) = \int_0^{2\pi} |f'(e^{i\theta})| d\theta, \quad l(\partial D^*) = \int_0^{2\pi} |F'(e^{i\theta})| d\theta.$$

Here  $f$  is a conformal mapping defined on  $U$  with the image  $D = f(U)$  and  $F$  is a conformal mapping defined on  $U$  with the image  $F(U) = D^*$  (which is simply connected as well).

**Definition 12** *Let  $D$  be a domain in the Riemann sphere  $\mathbb{C} \cup \{\infty\}$ . The Steiner symmetrization of  $D$  is the domain  $D_S^*$  that is defined as follows: for each  $t \in (-\infty, \infty)$  we define  $D(t) = \{t + iy \in \mathbb{C} \mid t + iy \in D\}$ . If  $D(t) = \emptyset$  then the intersection of  $D_S^*$  with the vertical line  $H(t) = \{z \in \mathbb{C} \mid \Re\{z\} = t\}$  is the empty set  $\emptyset$ . If  $D(t) = D \cap H(t)$  is a non-empty and has the measure  $|D(t)| = \alpha'$ , then the intersection of  $D_S^*$  with the line  $H(t)$  is the unique segment or line given by  $\{t + iy \in \mathbb{C} \mid |y| < \alpha'/2\}$ .*

Let  $G$  be a conformal mapping defined on  $U$  with the image  $G(U) = D_S^*$  (which is simply connected). We assume as above that the boundary curves  $\partial D = \partial f(U)$  and  $\partial D_S^* = \partial G(U)$  are rectifiable. Then the remark above shows that we also have  $l_n(\partial D_S^*) \leq l_n(\partial D)$ , where this time we take our division points to be the intersection points of  $\partial D$  ( $\partial D_S^*$ ) with sets of parallel vertical lines of  $x$ -coordinates  $-\infty < t_1 < t_2 < \dots < t_n < \infty$ , so that  $0 < t_{j+1} - t_j < \epsilon_n$ ,  $j = 1, 2, \dots, n - 1$ ,  $\epsilon_n \rightarrow_{n \rightarrow \infty} 0^+$ . We have the identity:

$$l(\partial D_S^*) = \int_0^{2\pi} |G'(e^{i\theta})| d\theta.$$

We just proved the well known:

**Theorem 13** *If  $f \in H(U)$ ,  $f$  is one-to-one, and the boundary curve  $\{f(e^{i\theta}) \mid 0 \leq \theta < 2\pi\}$  is rectifiable. If  $F$  is the circular symmetrization (Pölya) of  $f$  and if  $G$  is the Steiner symmetrization of  $f$  ( $F(0) = G(0) = |f(0)|$ ), then the boundary curves  $\{F(e^{i\theta}) \mid 0 \leq \theta < 2\pi\}$  and  $\{G(e^{i\theta}) \mid 0 \leq \theta < 2\pi\}$  are rectifiable, and we have the following two inequalities:*

$$\int_0^{2\pi} |G'(e^{i\theta})| d\theta \leq \int_0^{2\pi} |f'(e^{i\theta})| d\theta,$$

$$\int_0^{2\pi} |F'(e^{i\theta})| d\theta \leq \int_0^{2\pi} |f'(e^{i\theta})| d\theta.$$

**Remark 14** *We comment the two items below in connection to the second question of Baernstein and the possible proof of the conjecture of Littlewood on*

non-vanishing one-to-one analytic functions in  $U$ . We supply proofs to these well known facts, for the convenience of the reader:

1. If  $F(z)$  is analytic and one-to-one in  $U$  and satisfies  $0 < \Re\{F(z)\}$ , then  $(F(z))^2$  is one-to-one in  $U$ .

**Proof.**

$(F(z_1))^2 = (F(z_2))^2 \Rightarrow F(z_1) = F(z_2)$  or  $F(z_1) = -F(z_2)$ . But if  $F(z_1) = -F(z_2)$  then  $\Re\{F(z_1)\} \cdot \Re\{F(z_2)\} < 0$  which proves that  $F(z_1) = F(z_2)$  and hence  $z_1 = z_2$ .  $\diamond$

2. If  $f \in H(U)$  is one-to-one in  $U$  and  $f(z) \neq 0$  in  $U$  and if  $F(z)$  is the circular symmetrization of  $\sqrt{f(z)}$ , then  $(F(z))^2$  is the circular symmetrization of  $f(z)$  (and vice versa).

**Proof.**

By remark 1 above  $(F(z))^2$  is one-to-one which implies the claim.  $\diamond$

We need one more well known fact, this time on the Steiner symmetrization. This fact is a consequence of Baernstein's symmetrization inequality for Green function. See [2], and Theorem 2.5 in [3]. We mention (although we do not make a use of it) that the case of equality for the Green functions was not treated by Baernstein. It was later on considered by Essén and Shea in [10]. Our proof uses the fact that we can regard Steiner symmetrization as a limiting case (at infinity) of circular (Pölya) symmetrization. We will outline it below.

**Theorem 15** *If  $f$  is analytic and one-to-one in  $U$  and if  $G$  is the Steiner symmetrization of  $f$ , then for any  $r$ ,  $0 \leq r < 1$  we have the inequality:*

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leq \int_0^{2\pi} |G(re^{i\theta})|^2 d\theta.$$

**Proof.**

Let  $0 < M$ . Let  $F_M(z)$  be the circular symmetrization of the shifted function  $M + f(z)$ . Let us denote  $G_M(z) = F_M(z) - M$ . We will use the theorem of Baernstein ([2], Theorem 6) cited above. For any  $0 < p < \infty$ :

$$\int_0^{2\pi} |M + f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |F_M(re^{i\theta})|^p d\theta.$$

This can be re-written as follows:

$$\int_0^{2\pi} |M + f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |M + G_M(re^{i\theta})|^p d\theta.$$

which proves the following:

$$\begin{aligned} & \int_0^{2\pi} \left( 1 + \frac{p}{M} \Re\{f(re^{i\theta})\} + \frac{1}{M^2} \left\{ \frac{p^2}{4} |f(re^{i\theta})|^2 + \right. \right. \\ & \quad \left. \left. + 2 \binom{p/2}{2} \Re\{(f(re^{i\theta}))^2\} \right\} \right) d\theta \leq \\ & \leq \int_0^{2\pi} \left( 1 + \frac{p}{M} \Re\{G_M(re^{i\theta})\} + \frac{1}{M^2} \left\{ \frac{p^2}{4} |G_M(re^{i\theta})|^2 + \right. \right. \\ & \quad \left. \left. + 2 \binom{p/2}{2} \Re\{(G_M(re^{i\theta}))^2\} \right\} \right) d\theta + \\ & \quad + o\left(\frac{1}{M^2}\right). \end{aligned}$$

But  $\lim_{M \rightarrow \infty} G_M = G$  the Steiner symmetrization of  $f$ , uniformly on compact subsets of  $U$ . This proves (on taking  $M \rightarrow \infty$ ) that:

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leq \int_0^{2\pi} |G(re^{i\theta})|^2 d\theta. \quad \diamond$$

**Remark 16** *The analog of the theorem of Baernstein (Theorem 6, quoted above) to Steiner symmetrization is false, i.e. the statement  $\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |G(re^{i\theta})|^p d\theta$  for  $0 < p < \infty$  is false. It is easy to construct a counterexample. Thus  $p = 2$  is an exception. This naturally raises a question.*

**A problem (the means of the Steiner symmetrization):**

*Find the largest possible  $2 \leq p_0$  for which  $\forall 0 \leq r < 1$  and  $\forall p \leq p_0$  (or  $\forall p < p_0$ ) we have the inequality:*

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |G(re^{i\theta})|^p d\theta,$$

for any  $f \in H(U)$ , one-to-one in  $U$ , with  $f(0) = 0$ , where  $G$  is the Steiner symmetrization of  $f$ .

By Theorem 15 above we know that  $2 \leq p_0$  and that the inequality is valid for  $p = 2$ .

We conclude this section with a simple demonstration of the type of reductions we can make so far concerning the solution of the family of problems  $N(p, \alpha)$ .

**Proposition 17** *If  $2 \leq p \leq \infty$ ,  $0 < \alpha < \infty$ , and if  $f \in S(p, \alpha)$  is an extremal function for  $N(p, \alpha)$  then we may assume that the domain  $f(U)$  is circular symmetric (Pölya symmetric).*

**Proof.**

Let the function  $F$  be the Pölya symmetrization of the extremal function  $f$ . Then the following three properties hold true:

1.  $F \in H(U)$ ,  $F(0) = 0$ ,  $F$  is one-to-one in  $U$ .
2.  $1 \leq |f'(0)| \leq |F'(0)|$ , by a classical result of Walter Hayman.
3.  $\alpha \leq \|f\|_p \leq \|F\|_p$ , by Littlewood Subordination Theorem, [15] or page 422 in [13].

As in Theorem 13, it follows that:

$$\int_0^{2\pi} |F'(e^{i\theta})|d\theta \leq \int_0^{2\pi} |f'(e^{i\theta})|d\theta.$$

But by the assumption, the function  $f$  is an extremal function for the problem  $N(p, \alpha)$ , and so it follows that also the function  $F$  is extremal for the problem  $N(p, \alpha)$ .  $\diamond$

### 3 A solution of the problem $N(2, \alpha)$

We will present a solution of the simple case  $N(2, \alpha)$ . Already here we will conclude a few interesting conclusions. For example, we will be able to prove the convergence of certain infinite products of geometric parameters which we can not easily explicitly compute. The main argument will be based on the compactness of the family  $S(2, \alpha)$ . Let  $f \in S(2, \alpha)$ ,  $0 < \alpha < \infty$ . We will outline a simple recursive process that constructs in each phase a function  $g$  which satisfies the two properties:

1.  $g \in S(2, \alpha)$ .
2.  $\int_0^{2\pi} |g'(e^{i\theta})|d\theta \leq \int_0^{2\pi} |f'(e^{i\theta})|d\theta$ .

In fact the 1-norm of  $g'(e^{i\theta})$  will be smaller than or equals to the same norm of the function constructed in the previous step.

**Step 1:** We choose an angle  $\phi$  and define  $\bar{f}_\phi(z) = e^{i\phi} f(z)$ .

**Step 2:** We compute the function  $g_\phi(z)$ , the Steiner symmetrization of  $\bar{f}_\phi$ .

We note that  $g_\phi$  has the following 4 properties:

- $g_\phi \in H(U)$ ,  $g_\phi$  is one-to-one and  $g_\phi(0) = 0$ .
- $1 \leq |\bar{f}'_\phi(0)| \leq |g'_\phi(0)|$ , by the result of Walter Hayman mentioned before.
- $\alpha \leq \|\bar{f}_\phi\|_2 \leq \|g_\phi\|_2$ , by Theorem 15.
- $\int_0^{2\pi} |g'_\phi(e^{i\theta})|d\theta \leq \int_0^{2\pi} |\bar{f}'_\phi(e^{i\theta})|d\theta$ , by Theorem 2.5.

**Step 3:** We compute the following number:

$$c_\phi = \max \left( \frac{1}{|g'_\phi(0)|}, \frac{\alpha}{\|g_\phi\|_2} \right),$$

and then we have  $0 < c_\phi \leq 1$ . We compute  $g(z) = c_\phi g_\phi(z)$ .

**Remark 18** *In less formality we can describe the process that was outlined above as follows:*

1. Rotate  $f(U)$  about 0 (in an angle  $\phi$ ).
2. Perform Steiner symmetrization of the rotated domain.
3. Shrink the domain that was obtained, by an optimal factor  $c_\phi$ , where  $0 < c_\phi \leq 1$ .

**Definition 19** *Let  $D$  be a domain that contains the origin, 0. The  $\phi$ -deformation of  $D$  is the domain  $D_\phi$  which we get by the Steiner symmetrization of  $e^{i\phi} D$ . In other words  $D_\phi$  is the resulting domain after steps 1 and 2 in the process we described above.*

**Definition 20** *If  $f \in S(2, \alpha)$  and if  $\phi$  is a real number, we will denote the function  $g$  which is obtained after executing the process above (steps 1, 2 and 3), by  $g = f_\phi$ . The number  $c_\phi$  will be called the shrinking factor.*

If  $\{\phi_n\}_{n=1}^\infty$  is a sequence of real numbers, then the corresponding sequence of shrinking factors will be defined to be the sequence of the shrinking factors we obtain by executing the following iterative process:

$$\{f, f_{\phi_1}, (f_{\phi_1})_{\phi_2}, ((f_{\phi_1})_{\phi_2})_{\phi_3}, \dots\}.$$

If  $\{D_n\}_{n=1}^\infty$  is a sequence of bounded domains, and if  $D$  is a bounded domain, then we will say that the sequence of the domains converges to to the domain  $D$  and write  $D_n \rightarrow D$ , if for any  $\epsilon > 0$  there exists a number  $N$  such that for all  $n > N$  we will have  $\text{distance}(\partial D_n, \partial D) < \epsilon$ . We have the following surprising fact:

**Theorem 21** *If  $0 < \alpha < \infty$ , and if  $f \in S(2, \alpha)$  and also  $\{\phi_n\}_{n=1}^\infty$  is any sequence of real numbers, then we have the double inequality:*

$$0 < \prod_{n=1}^\infty c_{\phi_n} \leq 1.$$

**Proof.**

For each  $n$ ,  $n = 1, 2, 3, \dots$ , we have  $0 < c_{\phi_n} \leq 1$ . Hence  $0 \leq \prod_{n=1}^\infty c_{\phi_n} \leq 1$ . If  $\prod_{n=1}^\infty c_{\phi_n} = 0$  then  $f_{\phi_n} \rightarrow 0$  uniformly on compact subsets of  $U$ . This, however, contradicts the compactness of the family  $S(2, \alpha)$ .  $\diamond$

**Theorem 22** *Let  $D$  be a bounded domain that contains the origin, 0. Then, there exists a disk  $B$  whose center is the origin, 0 and there exists a sequence of  $\phi_n$ -deformations of  $D$  that will be denoted by  $\{D_n\}_{n=1}^\infty$  so that  $D_n \rightarrow B$ , (The assumption means*

that  $D_1$  is the  $\phi_1$ -deformation of  $D$ , and  $D_{n+1}$  is the  $\phi_{n+1}$ -deformation of  $D_n$ . Moreover, the disk  $B$  is unique in the sense that if  $\{D'_n\}_{n=1}^\infty$  is the sequence of  $\phi'_n$ -deformations of  $D$  that satisfies  $D'_n \rightarrow B'$ , then  $d(B) = d(B')$  (equality of diameters).

**Proof.**

Firstly, we note that, if  $D$  is a bounded domain that contains the origin, 0, and if  $\phi$  is real number, then the  $\phi$ -deformation of  $D$ ,  $D_\phi$ , has its diameter smaller than or equals to the diameter  $D$ , i.e.  $d(D_\phi) \leq d(D)$ . We denote by  $A$  the set of all the domains  $\tilde{D}$  for which there exists a finite sequence of real numbers  $\{\alpha_n\}_{n=1}^N$  such that if  $D_1 = D_{\phi_1}$ ,  $D_{n+1} = (D_n)_{\alpha_{n+1}}$ ,  $1 \leq n \leq N - 1$ , then  $\tilde{D} = D_N$ . We define  $d = \inf\{d(\tilde{D}) \mid \tilde{D} \in A\}$ . We claim that  $0 < d < \infty$ . For  $0 \in D$  and hence there is a disk with a positive-radius  $\epsilon/2$  about the origin, 0 is contained in  $D$ . Hence  $\epsilon \leq d \leq d(D)$ . Moreover, there exists a bounded domain  $B$  that contains the origin, 0 such that  $d = d(B)$  and such that there is a sequence of  $\phi_n$ -deformations of  $D$ ,  $\{D_n\}_{n=1}^\infty$  that satisfy  $D_n \rightarrow B$ . This domain  $B$  is a disk centered at 0: The domain  $B$  has a diameter in every direction otherwise there is a  $\phi$ -deformation that properly shrinks diameters. Since  $B$  has a minimal diameter, there is no way to shrink its diameter properly using a  $\phi$ -deformation. A domain which has a diameter in every direction is a disk (for the triangle inequalities imply that any pair of diameters must intersect through their common mid-point). Finally, if  $B$  is a disk and if  $0 \in B$ , then using a single  $\phi$ -deformation we can bring  $B$  to be a disk (with the same diameter) whose center is the origin, 0.

Let us prove the uniqueness claim: we note that a  $\phi$ -deformation (just like Steiner symmetrization) is an area preserving transformation. Hence if  $2R = d(B')$  then  $\pi R^2 = \int \int_D dx dy$  and we conclude that  $d(B') = d(B) = 2\sqrt{\int \int_D dx dy / \pi}$ .  $\diamond$

We shall now solve the problem  $N(2, \alpha)$ .

**Theorem 23** *If  $0 < \alpha < \infty$ , and if  $G(z) = \max\{1, \alpha\}z$ , then  $G \in S(2, \alpha)$  and we have:*

$$N(2, \alpha) = \int_0^{2\pi} |G'(e^{i\theta})| d\theta = 2\pi \max\{1, \alpha\}.$$

**Proof.**

By Theorem 21 and Theorem 22 it follows that among those functions that minimize, there is one that conformally maps  $U$  onto a disk centered at the origin, 0. Such a mapping has the form  $cz$  and since it should belong to the family  $S(2, \alpha)$ , we necessarily have:  $\max\{1, \alpha\} \leq |c|$ . But by our assumption

the function solves (by minimizing) the extremal problem. We conclude that  $|c| = \max\{1, \alpha\}$ .  $\diamond$

**Remark 24** *Our definition of the family  $S(2, \alpha)$  requires the condition  $1 \leq |f'(0)|$  in order to obtain a compact family. If instead of that condition we had  $\epsilon \leq |f'(0)|$  for some fixed  $0 < \epsilon$ , we could have solved the corresponding problem  $N(2, \alpha)$  similarly, except that this time our multiplier had to be  $\max\{\epsilon, \alpha\}$ . In particular for small enough  $\epsilon$  the solution would have been  $G(z) = \alpha z$ . We conclude that if  $f \in H(U)$ ,  $f$  is one-to-one,  $f(0) = 0$ , and if we denote  $\alpha = \|f\|_2$  then if  $\alpha < \infty$  we can solve the problem  $N(2, \alpha)$  with a small enough  $\epsilon > 0$  (meaning  $\epsilon < \alpha$ ) and obtain a solution  $\alpha z$ . This helps in proving the following inequality:*

**Theorem 25** *Let  $f \in H(U)$ ,  $f(0) = 0$ . Then for each  $0 \leq r < 1$  we have the following inequality:*

$$\left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right)^{1/2} \leq \frac{r}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})| d\theta.$$

*In particular we have  $\|f\|_2 \leq \|f'\|_1$ . Both inequalities above are sharp.*

**Proof.**

Let us start by assuming that we already proved the first inequality for functions  $f \in H(U)$ , for which  $f(0) = 0$ , that are also one-to-one. Let  $g \in H(U)$  satisfy  $g(0) = 0$  and let  $0 < r < 1$ . We consider the conformal mapping  $f \in H(U)$  such that  $f(0) = 0$  that satisfies  $g(U) \subseteq f(U)$ . The function  $g$  is subordinate to the function  $f$  (which means that there exists a  $w \in H(U)$ ,  $w(0) = 0$ ,  $|w(z)| < 1$  so that  $g(z) = f(w(z))$ ). By a theorem of Littlewood, [15] or page 422 in [13], we have the following inequality:

$$\frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^2 d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta. \quad (1)$$

Since we assumed that  $f$  is conformal, it follows by our initial assumption (at the beginning of the proof) that the following inequality is true:

$$\left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right)^{1/2} \leq \frac{r}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})| d\theta. \quad (2)$$

Finally (by the extension theorem of the Riemann Theorem) the holomorphic mapping maps the boundary of the domain onto the boundary of range domain (the boundaries are smooth enough and we can use a theorem of Constantine Carathéodory). Since  $f$  is (by assumption) a conformal mapping it traces the outer boundary of  $\partial g(rU)$  once, while the mapping  $g$  traces

the same outer boundary at least once. We conclude that:

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |g'(re^{i\theta})|d\theta. \quad (3)$$

Equations (1),(2) and (3) prove the assertion of the theorem for a general holomorphic  $g$  (given that the assertion is known to be valid for one-to-one holomorphic mappings).

Thus from now on we can assume that the mapping  $f$  in the statement of the theorem is one-to-one. We solve the problem  $N(2, \alpha)$  which is presented in remark 24. We will get the minimizing function  $\alpha z$ . We clearly have the identity:

$$\alpha^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta.$$

The perimeter of the minimal circle is  $2\pi\alpha$  while the perimeter of  $f(rU)$  is given by:

$$r \int_0^{2\pi} |f'(re^{i\theta})|d\theta.$$

We conclude the following inequality:

$$2\pi \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right)^{1/2} \leq r \int_0^{2\pi} |f'(re^{i\theta})|d\theta.$$

This proves our inequality. Finally, if we take  $f(z) = \alpha z$  the inequality becomes an equality. This proves that our inequality is, indeed sharp.  $\diamond$

### 4 The problems $N(p, \alpha)$ for values $2 < p < \infty$ of the parameter

**Remark 26** *The two inequalities of Theorem 25 were proven for the value  $p = 2$ , using results on conformal mappings. What can be said, at this point, on similar inequalities but for values of the parameter  $p$  different from 2? We will not tackle that problem directly. Instead, we will use convexity arguments in the form of interpolation theory of operators. Interpolation theorems rely on two estimates given for two different values of a parameter such as  $p$ , and extend them by giving estimates for all the values of that parameter that reside between the two first values. Not always, though the inequalities for the intermediate values of  $p$  are sharp. That might happen also in cases in which the two extreme estimates are sharp. At this point we have our inequality (which is sharp) for the value  $p = 2$ . Lemma 27 below provides the second (sharp) inequality for  $p = \infty$ . This case is much easier than the case  $p = 2$ .*

**Lemma 27** *Let the function  $f \in H(U)$  satisfy  $f(0) = 0$ . Then for any value of  $r, 0 \leq r < 1$  we have the following estimate:*

$$\max_{0 \leq \theta < 2\pi} |f(re^{i\theta})| \leq \frac{r}{2} \int_0^{2\pi} |f'(re^{i\theta})|d\theta.$$

*In particular we have the inequality  $\|f\|_\infty \leq \pi \|f'\|_1$ . Both inequalities above are sharp.*

**Proof.**

If the two inequalities of the lemma are true for  $f \in H(U)$ , satisfying  $f(0) = 0$  which are also one-to-one, then like in the first part of the proof of Theorem 25 it follows that the inequalities remain true in the general case, where  $f$  is not necessarily one-to-one. Thus we will assume from now and till the end of the proof that the mapping  $f$  in the statement of the lemma is also one-to-one. In this case we have as in the proof of Theorem 25 at our disposal elementary facts from plane geometry. The expression:

$$r \int_0^{2\pi} |f'(re^{i\theta})|d\theta,$$

is the perimeter of the domain  $f(rU)$ . Clearly we have:

$$2 \max_{0 \leq \theta < 2\pi} |f(re^{i\theta})| \leq r \int_0^{2\pi} |f'(re^{i\theta})|d\theta.$$

This last inequality is evident because  $0 \in f(rU)$  and because there exists a point  $w \in \overline{f(rU)}$  for which:  $w = \max_{0 \leq \theta < 2\pi} |f(re^{i\theta})|$ . The first inequality of the assertion follows. By remark 4 and by example 5 that follows it, we conclude that the inequality is, indeed sharp.  $\diamond$

**Theorem 28** *Let  $f \in H(U)$  satisfy  $f(0) = 0$ , and let  $2 \leq p \leq \infty$ . Then for each value of  $r, 0 \leq r < 1$  we have:*

$$\left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} \leq \frac{r}{2\pi^{2/p}} \int_0^{2\pi} |f'(re^{i\theta})|d\theta.$$

*In particular also the following inequality holds true:  $\|f\|_p \leq \pi^{(p-2)/p} \|f'\|_1$ .*

**Proof.**

The cases  $p = 2, \infty$  were proved in Theorem 25 and in Lemma 27, respectively. We choose a small  $\epsilon > 0$  and assume that  $2 < p < \infty$ . There exists an  $N, p < N$  such that for any  $N < q$  we have the following estimate:

$$\frac{(1/(2\pi)) \int_0^{2\pi} |f(re^{i\theta})|^q d\theta}{((r/(2\pi)) \int_0^{2\pi} |f'(re^{i\theta})|d\theta)^q} \leq \pi^{q-2+\epsilon}. \quad (4)$$



This follows from Lemma 27. In the band  $0 < \Re\{z\} < 1$  we define the following function:

$$F(z) = \frac{(1/(2\pi)) \int_0^{2\pi} |f(re^{i\theta})|^{2z+q(1-z)} d\theta}{((r/(2\pi)) \int_0^{2\pi} |f'(re^{i\theta})| d\theta)^{2z+q(1-z)}}.$$

Then  $F(z)$  is analytic in  $0 < \Re\{z\} \leq 1$  (for if  $c > 0$ , then  $c^z$  is non-zero analytic). Let us write  $z = t + is$ ,  $t, s \in \mathbb{R}$ ,  $0 \leq t \leq 1$ . By the triangle inequality we obtain:

$$|F(z)| \leq \frac{(1/(2\pi)) \int_0^{2\pi} |f(re^{i\theta})|^{2t+q(1-t)} d\theta}{((r/(2\pi)) \int_0^{2\pi} |f'(re^{i\theta})| d\theta)^{2t+q(1-t)}}.$$

Hence:

by equation (4) we have  $|F(is)| \leq \pi^{q-2+\epsilon}$ , and

by Theorem 25  $|F(1 + is)| \leq 1$ .

By Hadamard Convexity Theorem we deduce that:

$$|F(t + is)| \leq \pi^{(q-2+\epsilon)(1-t)}. \tag{5}$$

Now, suppose that  $p = 2t + q(1 - t)$ , then:

$$(q - 2 + \epsilon)(1 - t) = (p - 2) + \epsilon \left( \frac{p - 2}{q - 2} \right).$$

So by equation (5) we get:

$$\begin{aligned} & \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} \leq \\ & \leq \left( \frac{r}{2\pi^{2/p}} \int_0^{2\pi} |f'(re^{i\theta})| d\theta \right) \pi^{\epsilon(p-2)/(q-2)}. \end{aligned}$$

Now, we will take  $q \rightarrow \infty$  and obtain:

$$\left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} \leq \frac{r}{2\pi^{2/p}} \int_0^{2\pi} |f'(re^{i\theta})| d\theta.$$

The proof of the theorem is now complete.  $\diamond$

**Theorem 29** *If  $f \in H(U)$  and if  $\|f'\|_1 < \infty$ , then  $f \in H^p(U)$  for all  $p, 2 \leq p \leq \infty$  and the following inequality holds true:*

$$\|f\|_p \leq \pi^{(p-2)/p} \|f'\|_1 + |f(0)|.$$

**Proof.**

We use Theorem 28 with the function  $f(z) - f(0)$ , and than use the triangle inequality.  $\diamond$

**Remark 30** *The inequalities in Theorem 28 are sharp for  $p = 2, \infty$  and we get equalities for the extremal function  $f(z) = \alpha z$ , in the case  $p = 2$ . We do not*

*know if these inequalities are sharp for the other values of  $p$ , i.e.  $2 < p < \infty$ . In any event for these values of the parameter  $p$ , the function  $f(z) = \alpha z$  does not give us equality. Thus either the inequalities in Theorem 28 are not sharp, or they are sharp but for values  $2 < p < \infty$ , the function  $f(z) = \alpha z$  is not an extremal function. Theorem 28 belongs to the family of results that give estimates of the Hardy norms in terms of the areas of the images of the functions. Our inequality is not sharp at least for  $0 < p < 2 + (1/2)$ . For these values of  $p$  sharp inequality (with the best constant) can be obtained using Theorem 8.1 of [18], and the classical isoperimetric inequality. The extremal functions are indeed  $f(z) = c \cdot z$ . Our method does not give the best constants as expected when one uses interpolation techniques. However we deal not just with estimates of the Hardy norms, but with estimates that remain valid for the intermediate concentric circles, i.e.  $f(rz)$  for  $0 \leq r < 1$ . We mention two more papers that are closely related to estimating Hardy norms in terms of area of the image domain: [1], [14].*

## 5 Few open problems

Here are four natural problems for which we do not know the answer at this point:

1. Let  $f \in S$  and let  $F$  be the Steiner symmetrization of  $f$ . For which values of  $p, 2 \leq p \leq \infty$  we have the following inequality?

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |F(re^{i\theta})|^p d\theta.$$

We know that this inequality holds true for  $p = 2$  (by Theorem 15) and it is faulty for  $p = \infty$  (easy).

2. Let  $f \in S(2, \alpha)$  and let  $\{c_{\phi_n}\}_{n=1}^\infty$  be a sequence of shrinking factors that deform  $f(U)$  into the minimal disk. What is the value of the convergent infinite product  $\prod_{n=1}^\infty c_{\phi_n}$ ?

3. Are the inequalities in Theorem 28 and in Theorem 29 sharp? If not find the optimal constants (which should be at most  $\pi^{(p-2)/p}$ ) and find the optimal functions.

4. Are the extremal functions for the inequalities in Theorem 28 (with optimal constants) unique?

## 6 Inequalities on the real part of Steiner symmetrization

The results of this section might be already known to experts on Steiner symmetrization. The author was not able to find in the literature these results. We refer to the first sentence in section 4.11 on page 130 of the book [12]. Here it is: "Circular symmetrization is more powerful than Steiner symmetrization, and any result obtainable by the later method can also be obtained by the former on taking exponentials, though this may be less direct." The author did not check the validity of this declaration, however, the hint of connecting the two types of symmetrization by the exponential function was taken in order to try and resolve problem number 1 on the list of problems given in the previous section (section 5). We did not manage to solve that problem. However, this idea in W. K. Hayman's book,[12], produced a family of integral inequalities comparing  $\Re\{f(z)\}$  with the real part of the Steiner symmetrization  $\Re\{F(z)\}$  of  $f(z)$ . We remark that connecting the Circular symmetrization with the Steiner symmetrization by the exponential function is a very different idea than the geometric idea of connecting them by shifting the function to infinity. This idea was presented in the proof of Theorem 15 and gave us a positive answer to problem 1 above, in the case where  $p = 2$ .

Let  $f \in H(U)$  and let us denote  $D = f(U)$ . Let  $D^*$  be the Steiner symmetrization of  $D$ , with respect to the  $x$ -axis. Then  $D^*$  is a simply connected domain. We recall that for each  $a \in \mathbb{R}$ , we denote:  $l(a) = \text{meas}\{y \in \mathbb{R} \mid a + iy \in D\}$ , and we have by the definition of the Steiner symmetrization:  $D^* = \{a + iy \mid |y| < (1/2) \cdot l(a)\}$ . This means that for any  $a \in \mathbb{R}$  the intersection of the vertical line  $x = a$  with  $D^*$  is either an empty set or an open vertical interval (line segment) symmetric about the  $x$ -axis at the point of the intersection,  $(a, 0)$ :  $\{x = a\} \cap D^* = \{a + iy \mid |y| < (1/2) \cdot l(a)\}$ . This vertical interval might be a full line parallel to the  $y$ -axis, in the case that  $l(a) = \infty$ .

Let us apply (as suggested by Hayman) the exponential mapping to this line segment:

$$\exp(\{x = a\} \cap D^*) = \left\{ e^a e^{iy} \mid |y| < \frac{1}{2} l(a) \right\}. \quad (6)$$

This is a circular arc. This arc is a proper arc of the circle  $|z| = e^a$  provided that  $l(a) \leq 2\pi$ . If  $2\pi < l(a)$  then the arc visits some of the points of the circle  $|z| = e^a$  at least twice. This means that in any event the set  $\exp(D^*)$  is a circular symmetric domain. If for all  $a \in \mathbb{R}$  we have  $l(a) < 2\pi$  then the set  $\exp(D^*)$  is a simply connected circular symmetric domain. For

any  $a \in \mathbb{R}$  we have the following identity:

$$\{|z| = e^a\} \cap \exp(D) = \{z \mid |z| = e^w, w \in D, \Re\{w\} = a\}.$$

We note that the total length of the arcs that comprise the set  $\{|z| = e^a\} \cap \exp(D)$  is  $e^a \cdot l(a)$ , so when we form the Pölya symmetrization of  $\exp(D)$  we obtain:

$$\bigcup_{a \in \mathbb{R}} \left\{ e^a \cdot e^{i\theta} \mid |\theta| < \frac{1}{2} l(a) \right\},$$

and by equation (6) this is:

$$\bigcup_{a \in \mathbb{R}} \exp(\{x = a\} \cap D^*) = \exp(D^*).$$

This proves parts 1 and 2 of the following:

- Proposition 31**
1.  $\exp(D^*)$  is the circular symmetrization of  $\exp(D)$ , where  $D$  is a domain and  $D^*$  is the Steiner symmetrization of this domain.
  2. If for all  $a \in \mathbb{R}$ ,  $l(a) < 2\pi$ , then the intersection arcs  $\{e^a e^{i\theta} \mid |\theta| < (1/2)l(a)\}$  are simple (i.e. they do not pass through any point more than once).
  3. If for all  $a \in \mathbb{R}$ ,  $l(a) < 2\pi$ , then  $\exp(D^*)$  is a simply connected domain.

**Proof.**

We only need to prove part 3 (because part 1 and part 2 were proven above). However, part 3 follows at once by part 2 and by the fact that the Steiner symmetric domain  $D^*$  is a simply connected domain.  $\diamond$

Using the result of Albert Baernstein (Theorem 6 in [2]) we have the following:

**Theorem 32** Let  $f \in H(U)$  and let us denote  $D = f(U)$ , and assume that for any  $a \in \mathbb{R}$  we have  $l(a) < 2\pi$ . Also suppose that  $f(0) \geq 0$ . Let  $F \in H(U)$  be a conformal mapping of  $U$  onto  $D^*$ , where  $F(0) = |f(0)| = f(0)$  and where  $D^*$  is the Steiner symmetrization of  $D$ .

If  $\Phi$  is a convex non-decreasing function on  $(-\infty, \infty)$ , then for all  $r, 0 \leq r < 1$ , we have:

$$\int_{-\pi}^{\pi} \Phi(\Re\{f(re^{i\theta})\})d\theta \leq \int_{-\pi}^{\pi} \Phi(\Re\{F(re^{i\theta})\})d\theta.$$

**Proof.**

By Proposition 31(1):  $\exp(D^*)$  is the circular symmetrization of  $\exp(D)$ . By our assumption on  $f$  ( $\forall a \in \mathbb{R}, l(a) < 2\pi$ ), and by Proposition 31(2), the mapping:  $\exp : D^* \rightarrow \exp(D^*)$ , is injective and so conformal. We recall that the mapping:  $F : U \rightarrow D^*$  is conformal and  $F(0) = |f(0)|$ . Hence the composition:  $\exp(F) : U \rightarrow \exp(D^*)$ , is a conformal

mapping of  $U$  onto  $\exp(D^*)$  which is a simply connected domain (as should be the case), by Proposition 31(3). Also we have:  $\exp(F(0)) = \exp(|f(0)|) = \exp(f(0))$ , where the last equality follows by our assumption,  $f(0) \geq 0$ .

To sum up we have  $g(z) = \exp(f(z)) \in H(U)$  where by the above notations:  $g : U \rightarrow \exp(D) = g(U)$ . The mapping  $G(z) = \exp(F(z))$  is a conformal and onto mapping  $G : U \rightarrow \exp(D^*)$  that satisfies  $G(0) = \exp(F(0)) = \exp(f(0)) = g(0) = |g(0)|$ . The simply connected domain  $\exp(D^*)$  is the circular symmetrization of  $\exp(D) = g(U)$ . Thus the pair of mappings  $g, G$  satisfy all the assumptions of Albert Baernstein result, Theorem 6 in [2]. Using this theorem we obtain:

$$\int_{-\pi}^{\pi} \Phi(\log |g(re^{i\theta})|)d\theta \leq \int_{-\pi}^{\pi} \Phi(\log |G(re^{i\theta})|)d\theta,$$

for any convex and non-decreasing  $\Phi$  on  $(-\infty, \infty)$ , and any  $r, 0 \leq r < 1$ . Plugging in the expressions  $g(z) = \exp(f(z))$  and  $G(z) = \exp(F(z))$  we obtain:

$$\begin{aligned} & \int_{-\pi}^{\pi} \Phi(\log |\exp(f(re^{i\theta}))|)d\theta \leq \\ & \leq \int_{-\pi}^{\pi} \Phi(\log |\exp(F(re^{i\theta}))|)d\theta. \end{aligned}$$

Finally, since  $\log |\exp(\alpha)| = \Re\{\alpha\}$  for any  $\alpha \in \mathbb{C}$ , we get:

$$\int_{-\pi}^{\pi} \Phi(\Re\{f(re^{i\theta})\})d\theta \leq \int_{-\pi}^{\pi} \Phi(\Re\{F(re^{i\theta})\})d\theta,$$

where  $F$  is the Steiner symmetrization of  $f$ .  $\diamond$

If we take  $\Phi(x) = e^{px}$  for some  $p > 0$ , and take,

$$\Phi(x) = \begin{cases} x^p & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$$

for some  $p > 1$ , then we note that both functions are convex non-decreasing on  $(-\infty, \infty)$ , and we deduce from Theorem 32 the following:

**Corollary 33** *Let  $f \in H(U)$  satisfy  $l(a) < 2\pi$  for all  $a \in \mathbb{R}$  and  $f(0) \geq 0$ , and let  $F : U \rightarrow f(U)^*$  ( $f(U)^*$  is the Steiner symmetrization of  $f(U)$ ) be a conformal onto with  $F(0) = f(0)$ , then:*

1. For any  $0 < p$  we have:

$$\begin{aligned} & \int_{-\pi}^{\pi} \exp(p\Re\{f(re^{i\theta})\}) d\theta \leq \\ & \leq \int_{-\pi}^{\pi} \exp(p\Re\{F(re^{i\theta})\}) d\theta, \quad 0 \leq r < 1. \end{aligned}$$

2. For any  $1 < p$  we have:

$$\begin{aligned} & \int_{-\pi}^{\pi} \left(\Re\{f(re^{i\theta})\}\right)_+^p d\theta \leq \\ & \leq \int_{-\pi}^{\pi} \left(\Re\{F(re^{i\theta})\}\right)_+^p d\theta, \quad 0 \leq r < 1. \end{aligned}$$

Here if  $a \in \mathbb{R}$ , then we denote:

$$a_+ = \begin{cases} a & , \quad 0 \leq a \\ 0 & , \quad a < 0 \end{cases}$$

In particular if in 2 we take the  $p$ 'th root from both sides and then let  $p \rightarrow \infty$ , we obtain:

$$\sup_{\theta} \left(\Re\{f(re^{i\theta})\}\right)_+ \leq \sup_{\theta} \left(\Re\{F(re^{i\theta})\}\right)_+, \quad 0 \leq r < 1.$$

The last inequality in Corollary 33 is clear by the definition of the Steiner symmetrization, for the value  $r = 1$  (if that makes sense).

**Remark 34** 1. *In fact it follows by the definition of Steiner symmetrization that: if  $h(z) = \Re\{f(z)\}$  and if  $H(z) = \Re\{F(z)\}$ , then we have  $h(U) = H(U)$  and in particular:*

$$\lim_{r \rightarrow 1^-} \inf_{\theta} \Re\{f(re^{i\theta})\} = \lim_{r \rightarrow 1^-} \inf_{\theta} \Re\{F(re^{i\theta})\},$$

and also

$$\lim_{r \rightarrow 1^-} \sup_{\theta} \Re\{f(re^{i\theta})\} = \lim_{r \rightarrow 1^-} \sup_{\theta} \Re\{F(re^{i\theta})\}.$$

2. *Another example for a concrete inequality we can deduce from Theorem 32 is, for example, the following: we take for  $\Phi(x)$  the function  $\Phi(x) = \exp(x_+^p)$ ,  $p > 1$ . The first two derivatives on  $x > 0$  are,*

$$\Phi'(x) = px^{p-1} \exp(x^p) > 0,$$

$$\Phi''(x) = px^{p-2}(p-1 + px^p) \exp(x^p) > 0.$$

For  $x \leq 0$  we have  $\Phi(x) \equiv 1$ , a constant function. Using this  $\Phi(x)$ , we obtain by Theorem 6.2 the following result:

**Corollary 35** *Let  $f \in H(U)$  satisfy  $l(a) < 2\pi$  for all  $a \in \mathbb{R}$ , and  $f(0) \geq 0$ , and let  $F : U \rightarrow f(U)^*$  be conformal, onto with  $F(0) = f(0)$ , then for any  $p > 1$  we have the following inequality:*

$$\begin{aligned} & \int_{-\pi}^{\pi} \exp\left(\left(\Re\{f(re^{i\theta})\}\right)_+^p\right) d\theta \leq \\ & \leq \int_{-\pi}^{\pi} \exp\left(\left(\Re\{F(re^{i\theta})\}\right)_+^p\right) d\theta, \quad \text{for all } 0 \leq r < 1. \end{aligned}$$

**Remark 36** In fact Corollary 35 is a special case of the following more general statement:

If  $f(z)$  and  $F(z)$  are as in Corollary 35, and if  $\Psi(z)$  is any entire function with non-negative MaClaurin coefficients (i.e.  $\Psi^{(n)}(0) \geq 0$  for any  $n \in \mathbb{Z}^+ \cup \{0\}$ ), then for any  $p > 1$  and any  $0 \leq r < 1$  we have:

$$\int_{-\pi}^{\pi} \Psi \left( \left( \Re\{f(re^{i\theta})\} \right)_+^p \right) d\theta \leq \int_{-\pi}^{\pi} \Psi \left( \left( \Re\{F(re^{i\theta})\} \right)_+^p \right) d\theta.$$

**Proof.**

By the assumptions we have:  $\Psi(z) = \sum_{n=0}^{\infty} a_n z^n$ , with  $a_n \geq 0$  for  $n \in \mathbb{Z}^+ \cup \{0\}$ . Now the claim follows by Corollary 33(2). For by that corollary and the non-negativity of the coefficients  $a_n$ , we get:

$$\begin{aligned} & \int_{-\pi}^{\pi} a_n \left( \Re\{f(re^{i\theta})\} \right)_+^{pn} d\theta \leq \\ & \leq \int_{-\pi}^{\pi} a_n \left( \Re\{F(re^{i\theta})\} \right)_+^{pn} d\theta, \\ & \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} a_n \left( \left( \Re\{f(re^{i\theta})\} \right)_+^p \right)^n d\theta \leq \\ & \leq \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} a_n \left( \left( \Re\{F(re^{i\theta})\} \right)_+^p \right)^n d\theta, \\ & \int_{-\pi}^{\pi} \Psi \left( \left( \Re\{f(re^{i\theta})\} \right)_+^p \right) d\theta \leq \\ & \leq \int_{-\pi}^{\pi} \Psi \left( \left( \Re\{F(re^{i\theta})\} \right)_+^p \right) d\theta. \end{aligned}$$

◇

The change of order in the integration and the summation is easily justified.

**Remark 37** In the above results, the function  $f \in H(U)$  was assumed to satisfy  $l(a) < 2\pi, \forall a \in \mathbb{R}$ . In the case we had another uniform (finite) upper bound, say  $l(a) < M, \forall a \in \mathbb{R}$ , we could have looked (in case  $2\pi < M$ ) in the scaled function,  $2\pi f(z)/M$ , and get instead of the inequality of Theorem 32 the following inequality:

$$\int_{-\pi}^{\pi} \Phi \left( \frac{2\pi}{M} \Re\{f(re^{i\theta})\} \right) d\theta \leq \int_{-\pi}^{\pi} \Phi \left( \frac{2\pi}{M} \Re\{F(re^{i\theta})\} \right) d\theta.$$

What if there was no uniform upper bound on the  $l(a)$ 's, i.e.  $\sup_{a \in \mathbb{R}} l(a) = +\infty$ ? We would not like to conformally map  $f(U)$  into a bounded subset of  $\mathbb{C}$ , say by an inversion:

$$\left( \frac{\alpha}{z - \beta} \right).$$

The reason is that there does not seem to be a simple relation between the Steiner symmetrization of the original function  $f(z)$  and the Steiner symmetrization of the transformed mapping:

$$\left( \frac{\alpha}{f(z) - \beta} \right).$$

What we can do is the following: Pick a number  $r_0, 0 < r_0 < 1$  and consider the function  $f(r_0 z)$ . We denote by  $F(r_0, z)$  the Steiner symmetrization of  $f(r_0 z)$ . Since the image:  $\{f(r_0 z) \mid |z| < 1\}$  is a bounded set, our theorems give us comparison between integrals that involve  $f(r_0 z)$  and those that involve  $F(r_0, z)$ . We now have to estimate the relations between integrals that involve  $F(r_0, z)$  and those that involve the Steiner symmetrization  $F(re^{i\theta})$ , when  $r_0 \rightarrow 1^-$ . For example, is the following limit claim holds true?  $\lim_{r_0 \rightarrow 1^-} F(r_0, z) = F(z), |z| < 1$ . In what sense? (uniform, uniform on compacta, other). This might not be easy, for we are treating the case in which  $f(U)$  is unbounded vertically, i.e.  $\sup_{a \in \mathbb{R}} l(a) = +\infty$  while  $f(r_0 U)$  is a bounded set for  $0 < r_0 < 1$ . The mappings  $F(z) : U \rightarrow f(U)^*, F(r_0, z) : U \rightarrow f(r_0 U)^*$  are conformal and onto and satisfy the conditions:  $F(0) = f(0) = f(r_0 \cdot 0) = F(r_0, 0)$ .

We need a kind of a continuity claim on families of conformal mappings  $U \rightarrow D_{r_0}$  such that  $D_{r_0} \rightarrow D$  when  $r_0 \rightarrow 1^-$  in some sense (What do we mean by  $D_{r_0} \rightarrow D$ ?). Once again, in our model the domains  $D_{r_0} (0 < r_0 < 1)$  are bounded, while  $D$  is unbounded.

## 7 Steiner symmetrization and zero sets of bounded holomorphic functions in $U$

In this section we will point at some facts related to Problem number 2, on the list of problems in section 5. We will use the following definition: for  $2 \leq p \leq \infty, 0 < \alpha < \infty$  we define  $S(p, \alpha) = \{f \in H(U) \mid f \text{ is univalent in } U, f(0) = 0, 1 \leq f'(0), \alpha \leq \|f\|_p\}$ . We note that our normalization in the defining equation of  $S(p, \alpha)$  included the inequality,  $1 \leq |f'(0)|$  while now we gave up the absolute value and use instead  $1 \leq f'(0)$ . Problem 2 asks the following: Let  $f \in S(2, \alpha)$  and let  $\{c_{\phi_n}\}_{n=1}^{\infty}$  be a sequence of shrinking factors that deform  $f(U)$  into the minimal disk. What is the value of the following convergent infinite product  $\prod_{n=1}^{\infty} c_{\phi_n}$ ?

For  $f \in S(2, \alpha)$  and  $\phi \in \mathbb{R}$ , we denoted by  $g_{\phi}$  the Steiner symmetrization of  $e^{i\phi} f(z)$ . It has the following four properties:

1.  $g_\phi \in H(U)$  is one-to-one in  $U$ , and  $g_\phi(0) = 0$ .
2.  $1 \leq |e^{i\phi} f'(0)| \leq g'_\phi(0)$ . Note that  $|e^{i\phi} f'(0)| = f'(0) \geq 1$ .
3.  $\alpha \leq \|e^{i\phi} f\|_2 \leq \|g_\phi\|_2$ . Note that  $\|e^{i\phi} f\|_2 = \|f\|_2 \geq \alpha$ .
4.  $\int_{-\pi}^{\pi} |g'_\phi(e^{i\theta})| d\theta \leq \int_{-\pi}^{\pi} |e^{i\phi} f'(e^{i\theta})| d\theta$ . Note that  $\int_{-\pi}^{\pi} |e^{i\phi} f'(e^{i\theta})| d\theta = \int_{-\pi}^{\pi} |f'(e^{i\theta})| d\theta$ .

The shrinking factor that corresponds to  $f$  and to  $\phi$  is the following number:

$$c_\phi = \max \left\{ \frac{1}{g'_\phi(0)}, \frac{\alpha}{\|g_\phi\|_2} \right\}.$$

Thus by properties 2 and 3 we have  $0 < c_\phi \leq 1$ . By Theorem 3.4, if  $0 < \alpha < \infty$ , and if  $f \in S(2, \alpha)$  and  $\{\phi_n\}_{n=1}^\infty$  is any sequence of real numbers, then  $0 < \prod_{n=1}^\infty c_{\phi_n} \leq 1$ . This theorem was the source of Problem 2.

**Proposition 38**

$$\sum_{n=1}^\infty \log \left( \frac{1}{c_{\phi_n}} \right) < \infty. \tag{7}$$

$$\sum_{n=1}^\infty (1 - c_{\phi_n}) < \infty. \tag{8}$$

**Proof.**

We prove equation (7): By Theorem 21 we have:  $\log \prod_{n=1}^\infty c_{\phi_n} = \sum_{n=1}^\infty \log c_{\phi_n} > -\infty$ . Hence  $\sum_{n=1}^\infty \log(1/c_{\phi_n}) < \infty$ .

Now we prove equation (8): We write  $c_{\phi_n} = 1 - b_{\phi_n}$ . Then  $0 \leq b_{\phi_n} < 1$ . Also we have  $\prod_{n=1}^\infty c_{\phi_n} = \prod_{n=1}^\infty (1 - b_{\phi_n})$ . For  $0 \leq b < 1$  we have  $e^{-b} = 1 - b + b^2/2! - \dots \geq 1 - b$ , because this MaClaurin expansion is a Leibniz series. This also implies that:  $0 \leq e^{-b} - (1 - b) \leq b^2/2$ . So  $\prod_{n=1}^\infty e^{b_{\phi_n}} \geq \prod_{n=1}^\infty (1 - b_{\phi_n}) > 0$ . Thus:  $\exp(-\sum_{n=1}^\infty b_{\phi_n}) > 0$  and so  $\sum_{n=1}^\infty b_{\phi_n} < \infty$ . But  $b_{\phi_n} = 1 - c_{\phi_n}$  and we conclude that  $\sum_{n=1}^\infty (1 - c_{\phi_n}) < \infty$ .  $\diamond$

The inequality  $\sum_{n=1}^\infty (1 - c_{\phi_n}) < \infty$  says that the sequence  $\{c_{\phi_n}\}_{n=1}^\infty$  satisfies the Blaschke condition. Hence this sequence is precisely the zero set of a bounded analytic function in  $U$ . In fact the corresponding Blaschke product converges in  $U$ :

$$\prod_{n=1}^\infty \left( \frac{z - c_{\phi_n}}{1 - c_{\phi_n} z} \right).$$

**Proposition 39** If  $0 < \alpha < \infty$ , and if  $f \in S(2, \alpha)$ , and if  $\{\phi_n\}_{n=1}^\infty$  is any sequence of real numbers, then the infinite product:

$$B_{\{\phi_n\}}(z) = \prod_{n=1}^\infty \left( \frac{z - c_{\phi_n}}{1 - c_{\phi_n} z} \right),$$

is a Blaschke product, i.e. it is uniformly convergent on compact subsets of  $U$ , and  $\{c_{\phi_n}\}_{n=1}^\infty$  is the zero set of the resulting bounded (by 1) analytic function,  $B_{\{\phi_n\}}(z)$ .

This naturally leads to the question: is the converse of Proposition 39 holds true?

Problem 2'. Let:

$$\prod_{n=1}^\infty \left( \frac{z - c_{\alpha_n}}{1 - c_{\alpha_n} z} \right)$$

be a Blaschke product all of whose zeros  $\{\alpha_n\}_{n=1}^\infty$  are positive numbers,  $0 < \alpha_n \leq 1$ . Is there a number  $0 < \alpha < \infty$  and a function  $f \in S(2, \alpha)$ , and a sequence of real numbers  $\{\phi_n\}_{n=1}^\infty$  such that  $\forall n \in \mathbb{Z}^+$ ,  $c_{\phi_n} = \alpha_n$  the corresponding shrinking factors?

An explanation. We are given the data  $\{\alpha_n\}_{n=1}^\infty$  and should come up with an  $0 < \alpha < \infty$ ,  $f \in S(2, \alpha)$  and real numbers  $\{\phi_n\}_{n=1}^\infty$  such that:

1. If  $\bar{g}_{\phi_1}$  is the Steiner symmetrization of  $e^{i\phi_1} f$  then,

$$c_{\phi_1} = \max \left\{ \frac{1}{\bar{g}'_{\phi_1}(0)}, \frac{\alpha}{\|\bar{g}_{\phi_1}\|_2} \right\} = \alpha_1, \text{ and,}$$

$$g_{\phi_1} = c_{\phi_1} \bar{g}_{\phi_1}.$$

2. If  $(\overline{g_{\phi_1}})_{\phi_2}$  is the Steiner symmetrization of  $e^{i\phi_2} g_{\phi_1}$  then,

$$c_{\phi_2} = \max \left\{ \frac{1}{\overline{g_{\phi_1}}'_{\phi_2}(0)}, \frac{\alpha}{\|(\overline{g_{\phi_1}})_{\phi_2}\|_2} \right\} = \alpha_2, \text{ and}$$

$$(g_{\phi_1})_{\phi_2} = c_{\phi_2} (\overline{g_{\phi_1}})_{\phi_2}.$$

3. If  $(\overline{(g_{\phi_1})_{\phi_2}})_{\phi_3}$  is the Steiner symmetrization of  $e^{i\phi_3} (g_{\phi_1})_{\phi_2}$  then,

$$c_{\phi_3} = \max \left\{ \frac{1}{(\overline{(g_{\phi_1})_{\phi_2}})'_{\phi_3}(0)}, \frac{\alpha}{\|(\overline{(g_{\phi_1})_{\phi_2}})_{\phi_3}\|_2} \right\} = \alpha_3,$$

$$\text{and } ((g_{\phi_1})_{\phi_2})_{\phi_3} = c_{\phi_3} (\overline{(\overline{g_{\phi_1}})_{\phi_2}})_{\phi_3}, \text{ etc.} \dots$$

An idea.

We show that for each  $N \in \mathbb{Z}^+$  we can construct

a number  $0 < \alpha < \infty$ , a function  $f_N \in S(2, \alpha)$  and real numbers  $\{\phi_n^{(N)}\}_{n=1}^\infty$  such that this solves the finite problem, i.e.  $\forall n \in \mathbb{Z}^+, 1 \leq n \leq N, c_{\phi_n^{(N)}} = \alpha_n$  the corresponding shrinking factors. This gives a sequence of functions  $\{f_N\}_{N=1}^\infty \subseteq S(2, \alpha)$ , and a sequence of sequences of shrinking factors  $\{c_{\phi_n^{(N)}}\}_{n=1}^N = \{\alpha_1, \dots, \alpha_N\}$  on the numbers  $\{\phi_n^{(N)}\}_{n=1}^\infty$ . We then might try to prove that the Blaschke condition  $\sum_{n=1}^\infty (1 - \alpha_n) < \infty$  implies that the limit  $f(z) = \lim_{N \rightarrow \infty} f_N(z)$  exists uniformly on compact subsets of  $U$ . We also might hope to prove that the infinite set of limits:  $\lim_{N \rightarrow \infty} \phi_n^{(N)} = \phi_n^{(\infty)}$  exist and finally that the data  $f(z) \in S(2, \alpha)$  and  $\{\phi_n^{(\infty)}\}_{n=1}^\infty$  solves Problem 2', namely that we have  $c_{\phi_n^{(\infty)}} = \alpha_n, \forall n \in \mathbb{Z}^+$ . So we first want to solve:

**Problem 2 (finite).** Given  $N$  numbers  $\alpha_n, 0 < \alpha_1 < \alpha_2 < \dots < \alpha_N < 1$ , find a number  $0 < \alpha < \infty$  and a function  $f_N \in S(2, \alpha)$  and a sequence of real numbers  $\{\phi_n\}_{n=1}^N$  such that  $c_{\phi_n} = \alpha_n, n = 1, 2, \dots, N$ .

**Remark 40** We recall that we have defined the shrinking factor as follows:

$$c_\phi = \max \left\{ \frac{1}{g'_\phi(0)}, \frac{\alpha}{\|g_\phi\|_2} \right\}, \text{ so } 0 < c_\phi \leq 1,$$

and since  $c_\phi = 1 - b_\phi$ , also  $b_\phi = 1 - c_\phi$ . Thus we have the following identities:

$$b_\phi = \min \left\{ 1 - \frac{1}{g'_\phi(0)}, 1 - \frac{\alpha}{\|g_\phi\|_2} \right\},$$

and

$$\frac{1}{c_\phi} = \min \left\{ g'_\phi(0), \frac{1}{\alpha} \cdot \|g_\phi\|_2 \right\}.$$

Thus we can restate Proposition 38 as follows:

**Proposition 41**

$$\sum_{n=1}^\infty \log \left( \min \left\{ g'_{\phi_n}(0), \frac{1}{\alpha} \cdot \|g_{\phi_n}\|_2 \right\} \right) < \infty. \quad (9)$$

$$\sum_{n=1}^\infty \min \left\{ 1 - \frac{1}{g'_{\phi_n}(0)}, 1 - \frac{\alpha}{\|g_{\phi_n}\|_2} \right\} < \infty. \quad (10)$$

We will need to use results on the convergence of sequences of conformal mappings. The next section surveys the results we will be using.

## 8 Convergence of a sequence of conformal mappings

We will use the following two references:

- [6], sections 120 through 124, pages 74-77.
- [11], section 5 pages 54-62.

The exposition in [11] is easier for us being more modern but the results were proven by Constantin Carathéodory in [6]. So we take the parts we need mostly from [11]. Section 5 in [11] is titled: "Convergence theorems on the conformal mapping of a sequence of domains":

Suppose we have a sequence of univalent domains  $B_1, B_2, \dots$ , in the  $z$ -plane, each including  $z = 0$ . If there exists a disk  $|z| < \rho$ , where  $\rho > 0$ , that belongs to all the domains in  $B_n$ , we define the kernel of this sequence of domains as the largest domain containing  $z = 0$  such that an arbitrary closed subset of it belongs to all the domains  $B_n$  from some  $n$  on. By "largest domain" is meant the domain containing any other domain possessing this property. If such a disk does not exist, the kernel of the sequence of domains  $B_1, B_2, \dots$  is defined to be the point  $z = 0$ . We shall say that the sequence of domains  $B_1, B_2, \dots$  converges to the kernel  $B$ , and we shall denote this by writing  $B_n \rightarrow B$ , if every subsequence of these domains has  $B$  as its kernel. In particular, if a sequence of simply connected domains,  $B_1, B_2, \dots, B_n, \dots$  that include  $z = 0$  converges to the limiting domain  $B$  (also including  $z = 0$ ) in the sense that all boundary points of the domains  $B_n$  from some  $n$  on are arbitrary close to the boundary of the domain  $B$ , and all points of the boundary of the domain  $B$  are arbitrary close to the boundaries of the domains  $B_n$ , then this sequence has the domain  $B$  as its kernel and it converges to that kernel.

**In our application later on the domains in the sequence of domains are Steiner symmetric and will turn out to satisfy exactly the assumptions of the previous paragraph.**

Convergence to the kernel is guaranteed also for a sequence of domains  $B_n$  that include  $z = 0$  and satisfy the condition  $B_1 \subseteq B_2 \subseteq B_3 \subseteq \dots$ , or for a sequence of domains  $B_n$  that contains a neighborhood of the point  $z = 0$  and satisfy the condition  $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$ .

**Theorem A (Carathéodory, [7]).** Suppose that we have a sequence of functions  $z = f_n(\xi)$ , where  $n = 1, 2, \dots$ , that are regular in the disk  $|\xi| < 1$ . Suppose  $f_n(0) = 0$  and  $f'_n(0) > 0$  for  $n = 1, 2, \dots$ . Suppose that, for each  $n$ , the function  $f_n(\xi)$  maps the disk  $|\xi| < 1$  onto a domain  $B_n$ . For the sequence

$\{f_n(\xi)\}$  to converge in  $|\xi| < 1$  to a finite function, it is necessary and sufficient that the sequence  $\{B_n\}$  converge to the kernel  $B$ , which is either the point  $z = 0$  or a domain having more than one boundary point. When convergence exists, it is uniform inside the disk  $|\xi| < 1$ . If the limit function  $f(\xi) \neq \text{const.}$ , it maps  $|\xi| < 1$  onto the kernel  $B$ , and the sequence  $\{\phi_n(z)\}$  of inverse functions  $\phi_n(z)$  converges uniformly inside  $B$  to the function  $\phi(z)$  inverse to  $f(\xi)$ . (Thus it is assumed that the functions  $f_n(\xi)$  are conformal).

**Remark 42** It was proved in [4] and in [16] that the conditions of Theorem A are also necessary and sufficient for convergence in mean of  $\{f'_n(\xi)\}$  to  $f'(\xi)$ , that is, necessary and sufficient for:

$$\lim_{n \rightarrow \infty} \int_B |f'_n(\xi) - f'(\xi)|^2 d\sigma = 0,$$

where  $f'_n(\xi)$  is taken equal to 0 outside the domain  $B_n$

Theorem A gives the conditions for convergence of univalent functions only in the open disk  $|\xi| < 1$ . For the convergence of univalent functions in the closed disk  $|\xi| \leq 1$ , we give the following theorem, confining ourselves to domains of the Jordan type.

**Theorem B (Radó, [17]).** Let  $\{B_n\}$ ,  $n = 1, 2, \dots$  denote a sequence of simply connected domains each including the point  $z = 0$  and each bounded by a Jordan curve. Denote the boundary of  $B_n$  by  $C_n$ . Suppose that the sequence  $\{B_n\}$  converges to a domain  $B$  (its kernel) bounded by a Jordan curve  $C$ . Let  $\{f_n(\xi)\}$  denote a sequence of functions  $f_n(\xi)$  such that, for each  $n$ ,  $f_n(0) = 0$ ,  $f'_n(0) > 0$  and  $f_n(\xi)$  maps the unit disk  $|\xi| < 1$  onto the domain  $B_n$ . For the sequence  $\{f_n(\xi)\}$  to converge uniformly on the closed disk  $|\xi| \leq 1$  to a function  $z = f(\xi)$  that vanishes at 0, has positive first derivative at 0, and maps the open disk  $|\xi| < 1$  onto the domain  $B$ , it is necessary and sufficient that for every  $\epsilon > 0$ , there exists a number  $N > 0$  such that, for  $n > N$ , there exists a continuous one-to-one correspondence between the points of the curves  $C_n$  and  $C$  such that the distance between any point of  $C_n$  and the corresponding point of  $C$  will be less than  $\epsilon$ .

**Remark 43** In the case of domains with arbitrary boundaries, not in particular of Jordan type, the question of convergence in the closed disk, has been thoroughly investigated by Markušević, [16].

We are now ready to tackle Problem number 2 that appear on the list of problems in section 5.

## 9 The product of infinitely many shrinking factors

Let  $f(z) \in S(2, \alpha)$ , and let  $\{\phi_n\}_{n=1}^\infty$  be any sequence of real numbers. We defined recursively:  $\bar{g}_{\phi_1}$  is the Steiner symmetrization of  $e^{i\phi_1} f(z)$ , and  $g_{\phi_1} = c_{\phi_1} \bar{g}_{\phi_1}$  where  $c_{\phi_1}$  is the shrinking factor given by,

$$c_{\phi_1} = \max \left\{ \frac{1}{\bar{g}'_{\phi_1}(0)}, \frac{\alpha}{\|\bar{g}_{\phi_1}\|_2} \right\}.$$

Next,  $(\bar{g}_{\phi_1})_{\phi_2}$  is the Steiner symmetrization of  $e^{i\phi_2} g_{\phi_1}(z)$ , and  $(g_{\phi_1})_{\phi_2} = c_{\phi_2} (\bar{g}_{\phi_1})_{\phi_2}$ , where  $c_{\phi_2}$  is the shrinking factor given by,

$$c_{\phi_2} = \max \left\{ \frac{1}{(\bar{g}_{\phi_1})'_{\phi_2}(0)}, \frac{\alpha}{\|(\bar{g}_{\phi_1})_{\phi_2}\|_2} \right\}.$$

The process proceeds indefinitely.

**Theorem 44** Let  $f(z) \in S(2, \alpha)$ , then for any sequence  $\{\phi_n\}_{n=1}^\infty$  of real numbers the limit function  $F = \lim_{n \rightarrow \infty} (\dots((g_{\phi_1})_{\phi_2}) \dots)_{\phi_n}$  exists and the convergence is uniform on compact subsets of  $U$ . The image  $F(U)$  is a Steiner symmetric domain that includes  $z = 0$  and  $F \in S(2, \alpha)$ .

### Proof.

Let us define  $B_n = (\dots((g_{\phi_1})_{\phi_2}) \dots)_{\phi_n}(U)$ ,  $n = 1, 2, 3, \dots$ . Then by the definition of the recursive process, since  $(\dots((g_{\phi_1})_{\phi_2}) \dots)_{\phi_n}$  is the Steiner symmetrization of  $e^{i\phi_n} (\dots((g_{\phi_1})_{\phi_2}) \dots)_{\phi_{n-1}}$ , it follows that the domain  $\bar{B}_n = (\dots((g_{\phi_1})_{\phi_2}) \dots)_{\phi_n}(U)$  is Steiner symmetric, but  $B_n = c_{\phi_n} \bar{B}_n$ , a multiple by a number  $0 < c_{\phi_n} \leq 1$  of a Steiner symmetric domain. Hence  $B_n$  itself is Steiner symmetric, and in particular the sequence  $\{B_n\}_{n=1}^\infty$  is a sequence of simply connected domains, each of which contains the point  $z = 0$ , and by the definition of the Steiner symmetrization all of the domains  $B_n$  contain the disk  $|z| < \text{dist}(0, \partial f(U))$ . Hence, the kernel of  $\{B_n\}_{n=1}^\infty$ , say  $B$ , exists and  $\partial B$  is the faithful limit of  $\partial B_n$  as  $n \rightarrow \infty$ . Hence  $B_n \rightarrow B$  and by Theorem A (Carathéodory) the limit  $F = \lim_{n \rightarrow \infty} (\dots((g_{\phi_1})_{\phi_2}) \dots)_{\phi_n}$  exists and is uniform on compact subsets of  $U$ . Thus  $B = F(U)$ , and  $F : U \rightarrow F(U)$  is conformal and satisfies the following normalization  $F(0) = 0$ ,  $F'(0) > 1$  and  $\|F\|_2 \geq \alpha$ . This proves that  $F \in S(2, \alpha)$  which is consistent with the fact that  $S(p, \alpha)$  is a compact family (recall the proof of Proposition 3). Moreover, the limit  $F(U)$  of the Steiner symmetric domains  $\{(\dots((g_{\phi_1})_{\phi_2}) \dots)_{\phi_n}(U)\}_{n=1}^\infty$  is Steiner symmetric. This is consistent with fact that  $F(U)$  is a simply

connected domain being the conformal image of  $U$ .  $\diamond$

We can now give a sharp lower bound for the infinite product  $\prod_{n=1}^{\infty} c_{\phi_n}$  of the shrinking factors, that appears in the Problem 2 on the list of problems in section 5.

**Theorem 45** 1. *Let  $f \in S(2, \alpha)$  and let  $\{\phi_n\}_{n=1}^{\infty}$  be any sequence of real numbers. Let  $\{c_{\phi_n}\}_{n=1}^{\infty}$  be the corresponding sequence of the shrinking factors. Then we have the following estimate:*

$$\max\{1, \alpha\} \cdot \left\{ \frac{1}{\pi} \int_0^{2\pi} \Re\{f(e^{i\theta})\} \Re\{e^{i\theta} f'(e^{i\theta})\} d\theta \right\}^{-1/2} \leq \prod_{n=1}^{\infty} c_{\phi_n} \leq 1,$$

and these bounds on  $\prod_{n=1}^{\infty} c_{\phi_n}$  are sharp bounds.

2. *Let us define recursively the following sequence of mappings:  $f_{\phi_1}$  is the Steiner symmetrization of  $e^{i\phi_1} f$ . For  $n \in \mathbb{Z}^+$ , let  $(\dots((f_{\phi_1})_{\phi_2}) \dots)_{\phi_{n+1}}$  be the Steiner symmetrization of  $e^{i\phi_{n+1}}(\dots((f_{\phi_1})_{\phi_2}) \dots)_{\phi_n}$ . Then the limit function  $G(z) = \lim_{n \rightarrow \infty} (\dots((f_{\phi_1})_{\phi_2}) \dots)_{\phi_n}(z)$  exists and is uniform on compact subsets of  $U$ . Moreover, we have the following identity:*

$$\prod_{n=1}^{\infty} c_{\phi_n} = \max\left\{ \frac{1}{|G'(0)|}, \frac{\alpha}{\|G\|_2} \right\}.$$

**Remark 46** *Theorem 45 gives some kind of solution to Problem 2 on the list of problems that appear in section 5.*

**A proof of Theorem 45.**

1. We will use the recursive sequence  $\{f_{\phi_n}\}_{n=1}^{\infty}$ , that was defined in part 2 of Theorem 9.2. Then as in the proof of Theorem 44 that dealt with the sequence  $\{(\dots((g_{\phi_1})_{\phi_2}) \dots)_{\phi_n}\}_{n=1}^{\infty}$ , based on Theorem A (Carathéodory), the limit  $G = \lim_{n \rightarrow \infty} (\dots((f_{\phi_1})_{\phi_2}) \dots)_{\phi_n}$  exists and is uniform on compact subsets of  $U$ . We note that the recursive process outlined by the newer sequence  $\{(\dots((f_{\phi_1})_{\phi_2}) \dots)_{\phi_n}\}_{n=1}^{\infty}$  is simpler than the original recursive process that was described on section 3, in that we do not multiply by the shrinking factors  $c_{\phi_n}$  after each symmetrization was done. The purpose of those multiplications was to optimize, i.e. make as small as possible, each element of the sequence of functions produced. That without leaving the family  $S(2, \alpha)$ . We will soon see that if our goal was to optimize the limiting function and not the each element of the sequence, then this can be accomplished by a

single multiplication by just one shrinking factor. In fact this is the key idea for the current proof.

The first step is to note that each element of the old sequence, the  $g$ -sequence, is a multiple by a constant of the corresponding element of the new system, the  $f$ -sequence. The constant, though is not a single shrinking factor. Clearly by the definitions of  $g_{\phi_1}$  and of  $f_{\phi_1}$  we have the formula  $g_{\phi_1} = c_{\phi_1} f_{\phi_1}$ . Next,  $(f_{\phi_1})_{\phi_2}$  is the conformal mapping  $U \rightarrow (e^{i\phi_2} f_{\phi_1}(U))^*$  where  $A^*$  denotes the Steiner symmetrization of the domain  $A$ . This conformal mapping is normalized as follows:  $(f_{\phi_1})_{\phi_2}(0) = 0$  and  $(f_{\phi_1})'_{\phi_2}(0) > 0$ . That definition of  $(f_{\phi_1})_{\phi_2}$  should be compared with the definition of  $(g_{\phi_1})_{\phi_2}$  which equals the shrinking factor,  $c_{\phi_2}$  multiplying the conformal mapping  $U \rightarrow (e^{i\phi_2} g_{\phi_1}(U))^*$ . But we already have the formula  $e^{i\phi_2} g_{\phi_1}(U) = e^{i\phi_2}(c_{\phi_1} f_{\phi_1}(U)) = c_{\phi_1}(e^{i\phi_2} f_{\phi_1}(U))$ . Thus we deduce that the Steiner symmetrization of  $e^{i\phi_2} g_{\phi_1}(U)$  equals to  $c_{\phi_1}$  times the Steiner symmetrization of  $e^{i\phi_2} f_{\phi_1}(U)$ . In other words the relation between the image of the first conformal mapping to the image of the second conformal mapping is multiplication by the shrinking factor  $c_{\phi_1}$ , where we recall that  $0 < c_{\phi_1} \leq 1$ . Hence by composition of conformal mappings we get the following (second) formula  $(g_{\phi_1})_{\phi_2} = c_{\phi_2} c_{\phi_1} (f_{\phi_1})_{\phi_2}$ . Similarly, the general case follows by an inductive argument. We obtain the general formula:

$$(\dots((g_{\phi_1})_{\phi_2}) \dots)_{\phi_n} = \left( \prod_{k=1}^n c_{\phi_k} \right) \cdot (\dots((f_{\phi_1})_{\phi_2}) \dots)_{\phi_n},$$

$$\forall n \in \mathbb{Z}^+.$$

Passing to the limit  $n \rightarrow \infty$  gives us:

$$\lim_{n \rightarrow \infty} (\dots((g_{\phi_1})_{\phi_2}) \dots)_{\phi_n} = \left( \prod_{k=1}^{\infty} c_{\phi_k} \right) \cdot (\dots((f_{\phi_1})_{\phi_2}) \dots)_{\phi_n},$$

or simply (using our notations for the limits):

$$F = \left( \prod_{k=1}^{\infty} c_{\phi_k} \right) \cdot G. \tag{11}$$

If the sequence  $\{\phi_n\}_{n=1}^{\infty}$  deforms  $f(U)$  to a disk  $D(0, R)$  in the newer process, then this disk has an area which equals the area of  $f(U)$ . We obtain the following equation with the unknown  $R$ :

$$\pi R^2 = \int_0^{2\pi} \Re\{f(e^{i\theta})\} \Re\{e^{i\theta} f'(e^{i\theta})\} d\theta.$$

Hence:

$$R = \left\{ \frac{1}{\pi} \int_0^{2\pi} \Re\{f(e^{i\theta})\} \Re\{e^{i\theta} f'(e^{i\theta})\} d\theta \right\}^{1/2}.$$



This means that for this particular sequence  $\{\phi_n\}_{n=1}^\infty$ , We obtain the very simple formula for the conformal mapping  $G$ , namely that  $G : U \rightarrow D(0, R)$ , given by  $G(z) = R \cdot z$ . In particular we get  $G'(0) = \|G\|_2 = R$  and hence the corresponding shrinking factor, which optimize  $G$  is given by:

$$\max \left\{ \frac{1}{R}, \frac{\alpha}{R} \right\} = \max \{1, \alpha\} \cdot \frac{1}{R} = \\ = \max \{1, \alpha\} \left\{ \frac{1}{\pi} \int_0^{2\pi} \Re\{f(e^{i\theta})\} \Re\{e^{i\theta} f'(e^{i\theta})\} d\theta \right\}^{-1/2}.$$

This concludes the proof of the inequality of part 1, including its sharpness.

2. In the general case the limiting function for an arbitrary sequence of real numbers  $\{\phi_n\}_{n=1}^\infty$  is the conformal mapping  $G : U \rightarrow G(U)$ , where  $G(0) = 0$  and  $G'(0) > 0$ . Now the general shrinking factor (of  $G$ ) is given by:

$$c = \max \left\{ \frac{1}{G'(0)}, \frac{\alpha}{\|G\|_2} \right\}.$$

On the other hand, by equation 11 this shrinking factor is given by the infinite product:

$$c = \prod_{n=1}^\infty c_{\phi_n}.$$

This concludes the proof of part 2 of our theorem.  $\diamond$

**Corollary 47** Let  $f \in S(2, \alpha)$  and let  $\{\phi_n\}_{n=1}^\infty$  be any sequence of real numbers. Let  $G$  be the limiting function of the newer recursive process, i.e.  $G = \lim_{n \rightarrow \infty} (\dots ((f_{\phi_1})_{\phi_2}) \dots)_{\phi_n}$ . Then we have the sharp estimate:

$$\max \{1, \alpha\} \cdot \left\{ \frac{1}{\pi} \int_0^{2\pi} \Re\{f(e^{i\theta})\} \Re\{e^{i\theta} f'(e^{i\theta})\} d\theta \right\}^{-1/2} \leq \\ \leq \max \left\{ \frac{1}{G'(0)}, \frac{\alpha}{\|G\|_2} \right\} \leq 1.$$

In particular, if  $\alpha$  is taken to be small enough, then:

$$G'(0) \leq \left\{ \frac{1}{\pi} \int_0^{2\pi} \Re\{f(e^{i\theta})\} \Re\{e^{i\theta} f'(e^{i\theta})\} d\theta \right\}^{1/2}.$$

This last upper bound is sharp.

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