The structure of rationally factorized Lax type flows and their analytical integrability

M. VOVK,
Lviv Polytechnic
National University
Dept. Applied Math.
Bandery Street 12,
79000 Lviv
UKRAINE
mira.i.kopych@gmail.com

P. PUKACH
Lviv Polytechnic
National University
Dept. Applied Math.
Bandery Street 12,
79000 Lviv
UKRAINE
ppukach@gmail.com

O. HENTOSH
IAPMM NASU
Naukova str., 3b
79060 Lviv
UKRAINE
ohen@ukr.net

Ya. PRYKARPATSKY
University of Agriculture
Dept. Applied Math.
Kraków
POLAND
yarpry@gmail.com

Abstract: In the article we construct a wide class of differential-functional dynamical systems, whose rich algebraic structure makes their integrability analytically effective. In particular, there is analyzed in detail the operator Lax type equations for factorized seed elements, there is proved an important theorem about their operator factorization and the related analytical solution scheme to the corresponding nonlinear differential-functional dynamical systems.

Keywords: associative algebras, automorphisms, compatibility condition, factorized flows, central extension, Casimir invariants

1 The basic associative algebra case

There is considered an associative functional algebra $\mathcal{A} \subset C^{\infty}(\mathbb{S}^1;\mathbb{C})$, admitting the automorphism

$$T \circ a(x) := a(x + \delta i) \qquad (1)$$

for any $a \in \mathcal{A}$, being a simple shift on $i\delta \in i\mathbb{R}\backslash\{0\} \subset \mathbb{C}$, $i^2 = -1$, $2\pi/\delta \notin \mathbb{Z}_+$, along the complexified loop parameter $x \in \mathbb{S}^1 \subset \mathbb{C}$. The linear and invariant trace-functional $\tau : \mathcal{A} \to \mathbb{C}$ is defined for any $a \in \mathcal{A}$ by the natural expression :

$$\tau(a) := \int_{\mathbb{S}^1} a(x)dx. \qquad (2)$$

Having constructed the basic Lie algebra $\mathcal{G}$ of homomorphisms $A(T) \in Hom\mathcal{A}$, where

$$A(T) \sim \sum^{j \ll \infty} a_j(x)T^j \qquad (3)$$

for $a_j(y) \in \mathcal{A}, j \ll \infty$. As the related Lax type integrable dynamical systems are generated [3, 5, 4, 1, 13, 11] by the Casimir invariants $\gamma \in I(\mathcal{G}^*)$ of the basic Lie algebra $\mathcal{G}$, satisfying the determining equation

$$[\nabla\gamma(l), l] = 0, \qquad (4)$$

we will be interested in a seed Lax element $l \in \mathcal{G}^*$, chosen in the following rationally factorized form:

$$l := F_n(T)^{-1} \circ Q_{n+p}(T), \qquad (5)$$

where by definition, the elements

$$F_n(T) := \sum_{j=\overline{0,n}} f_j(x)T^j,$$
$$Q_{n+p}(T) := \sum_{j=\overline{0,n+p}} q_j(x)T^j \qquad (6)$$

are some polynomial homomorphisms of $\mathcal{A}$ for fixed n and $p \in \mathbb{Z}_+$.

The following **problem** [2, 1, 6, 8] arises: *construct the corresponding dynamical systems on the elements $F_n(T), Q_{n+p}(T) \in Hom\mathcal{A}$, which will possess an infinite hierarchy of functional invariants and will be analytically integrable.*

It is natural to consider the general Lax type flow

$$dl/dt = [l, \nabla\gamma(l)_+], \tag{7}$$

for the rational element (5), generated by a Casimir functional $\gamma \in I(\mathcal{G}^*)$ and determined by the expression (4). Now let us observe that $\gamma := tr(\gamma(l) = tr(\gamma(\tilde{l}))$ for any analytical mapping $\gamma(l) \in \mathcal{G}$, where we have introduced, by definition, the factorized element $\tilde{l} := Q_{n+p}F_n^{-1} \in \mathcal{G}^*$. In addition, the element $\tilde{l} = Q_{n+p}F_n^{-1} \in \mathcal{G}^*$ satisfies the similar to (7) evolution equation

$$d\tilde{l}/dt = [\tilde{l}, \nabla\gamma(\tilde{l})_+] \tag{8}$$

for the same Casimir functional $\gamma \in I(\mathcal{G}^*)$, whose gradient, similarly to (4), is determined from the algebraic relationship

$$[\tilde{l}, \nabla\gamma(\tilde{l})] = 0. \tag{9}$$

Taking now into account these two compatible equations (7) and (8) one easily derives the following factorization theorem.

Theorem 1 *The operator evolution equations*

$$dF_n/dt = F_n\nabla\gamma(l)_+ - \nabla\gamma(\tilde{l})_+F_n,$$
$$dQ_{n+p}/dt = Q_{n+p}\nabla\gamma(l)_+ - \nabla\gamma(\tilde{l})_+Q_{n+p} \tag{10}$$

factorize the Lax type flows (7) and (8) for all $t \in \mathbb{R}$ with elements $l = F_n^{-1}Q_{n+p} \in \mathcal{G}^*$ and $\tilde{l} = Q_{n+p}F_n^{-1} \in \mathcal{G}^*$, respectively, where the corresponding Casimir invariants $\gamma \in I(\mathcal{G}^*)$ satisfy the relationship $\gamma|_{l=F_n^{-1}Q_{n+p}} = \gamma|_{\tilde{l}=Q_{n+p}F_n^{-1}}$ for any F_n and $Q_{n+p} \in G_+$.

As a simple consequence from Theorem 1 one can derive the following proposition.

Proposition 2 *There exist such smooth mappings $\Phi, \Psi : \mathbb{R} \to G$ to the formal operator subgroup $G \simeq \exp\mathcal{G}$ satisfying the linear evolution equations*

$$d\Phi/dt + \nabla\gamma(\tilde{l})_+\Phi = 0,$$
$$d\Psi/dt + \nabla\gamma(l)_+\Psi = 0 \tag{11}$$

$\Phi|_{t=0} = \bar{\Phi}, B|_{t=0} = \bar{B} \in G$, generated, respectively, by the Lie algebra elements $\nabla\gamma(l)_+$ and $\nabla\gamma(\tilde{l})_+ \in \mathcal{G}_+$, that

$$F_n := \Psi\ \bar{F}_n\Phi^{-1}, Q_{n+p} := \Psi\ \bar{Q}_{n+p}\Phi^{-1}, \tag{12}$$

where, by definition, the elements $\bar{F}_n$ and $\bar{Q}_{n+p} \in G$ are constant with respect to the evolution parameter $t \in \mathbb{R}$.

Proof: It is enough to check, using (11) that the group elements (12) really satisfy the factorized evolution equations (10). □

Now based on Proposition 3 we can take into account, with no loss of generality, that the group elements $A, B \in G$ for all $t \in \mathbb{R}$ can be represented as operator series

$$\Phi(x;t) \sim I + \sum_{j\in\mathbb{Z}_+} a_j(x;t)T^{-j},$$
$$\Psi(x;t) \sim I + \sum_{j\in\mathbb{Z}_+} b_j(x;t)T^{-j}, \tag{13}$$

whose coefficients can be found recurrently from the expressions (12), rewritten in the following useful for calculations form:

$$\begin{aligned}(I + \textstyle\sum_{j\in\mathbb{Z}_+} b_j(x;t))T^{-j}) \circ \bar{F}_n = \\ = F_n \circ (I + \textstyle\sum_{j\in\mathbb{Z}_+} a_j(x;t)T^{-j}), \\ (I + \textstyle\sum_{j\in\mathbb{Z}_+} b_j(x;t)T^{-j}) \circ \bar{Q}_{n+p} = \\ = Q_{n+p} \circ (I + \textstyle\sum_{j\in\mathbb{Z}_+\setminus\{0\}} a_j(x;t)T^{-j}),\end{aligned} \tag{14}$$

where the group elements $\bar{F}_n$ and $\bar{Q}_{n+p} \in G$ are considered to be given *a priori* constant in the following, motivated by the expression (6), operator series form:

$$\bar{F}_n(T) \sim \sum_{j\in\mathbb{Z}_+} \bar{f}_jT^{n-j},$$
$$\bar{Q}_{n+p}(T) \sim \sum_{j\ \in\mathbb{Z}_+} \bar{q}_jT^{n+p-j}\ , \tag{15}$$

where $d\bar{f}_j/dt = 0 = d\bar{q}_j/dt, j \in \mathbb{Z}_+$, for all $t \in \mathbb{R}$. The results obtained above mean, in particular, that the expressions (14) can be effectively used for finding exact analytical solutions to the resulting differential-functional equations naturally following from the operator evolution equations (10), generated by a suitably chosen Casimir functional $\gamma \in I(\mathcal{G}^*)$. This and other related aspects of this important problem of finding exact analytical solutions will be analyzed in detail in other work under preparation.

2 The centrally extended basic associative algebra case

Consider now the case when the basic associative functional algebra $\mathcal{A} \subset C^\infty(\mathbb{S}^1;\mathbb{C})$ is extended as the `loop` algebra $C_{S^1}(\mathcal{A})$ of smooth mappings $\{\mathbb{S}^1 \to \mathcal{A}\}$. The corresponding Lie algebra $C_{S^1}(\mathcal{G})$ of linear homomorphisms $C_{S^1}(\mathcal{A})$, naturally generated by the complexified homomorphic shifts

(1) along the cyclic variable $x \in \mathbb{S}^1$, can be centrally extended to the Lie algebra $C_{S^1}(\widehat{\mathcal{G}})$ via the standard [13, 1, 9, 11] Maurer-Cartan cocycle

$$\omega_2(A(T), B(T)) := \int_{\mathbb{S}^1} dy\, \langle A(T), dB(T)/dy \rangle, \tag{16}$$

which also admits the natural splitting subject to the positive and negative degrees of the basic homomorphism (1) into two subalgebras

$$C_{\mathbb{S}^1}(\widehat{\mathcal{G}}) = C_{\mathbb{S}^1}(\widehat{\mathcal{G}})_+ + C_{\mathbb{S}^1}(\widehat{\mathcal{G}})_-. \tag{17}$$

The latter makes it possible to construct the adjoint splitting

$$C_{\mathbb{S}^1}(\widehat{\mathcal{G}}^*) = C_{\mathbb{S}^1}(\widehat{\mathcal{G}}^*)_+ + C_{\mathbb{S}^1}(\widehat{\mathcal{G}}^*)_- \tag{18}$$

and define for any factorized element $(l, 1) \in C_{\mathbb{S}^1}(\widehat{\mathcal{G}}^*)$ the following integrable Hamiltonian flows

$$dl/dt = [l - d/dy, \nabla\gamma(l)_+], \tag{19}$$

where the Casimir functionals $\gamma \in I(C_{\mathbb{S}^1}(\widehat{\mathcal{G}}^*))$ satisfy the gauge type differential-functional equation

$$[l, \nabla\gamma(l)] = d\nabla\gamma(l)/dy \tag{20}$$

for all $y \in \mathbb{S}^1$. Here as above we will consider the case when a rationally factorized element $l(T) \in C_{\mathbb{S}^1}(\mathcal{G}^*)$ is given in the form

$$l(T) := F_n\,(T)^{-1} \circ Q_{n+p}(T), \tag{21}$$

where, by definition, the elements

$$F_n(T) := \sum_{j=\overline{0,n}} f_j(x; y) T^j,$$
$$Q_{n+p}(T) := \sum_{j=\overline{0,n}+p} q_j(x; y) T^j \tag{22}$$

belong to the formal operator subgroup $C_{\mathbb{S}^1}(G_+) := \exp(C_{\mathbb{S}^1}(\mathcal{G}_+)) \simeq I + C_{\mathbb{S}^1}(\mathcal{G}_+)$.

In this case we also can not make use of the expansions (13), thus forcing us to apply the Lie-algebraic scheme of [2, 1, 10]. Namely, we will formulate the following similar statements without proof.

Lemma 3 *For any factorized in the rational form (21) element $l \in C_{\mathbb{S}^1}(\mathcal{G}^*)$ there exists, as $2\pi/\delta \notin \mathbb{Z}_+$, an invertible mapping $\Phi(T) \in C_{\mathbb{S}^1}(G_-)$, $\Phi(T)|_{x=0} = I$, and such an element $\bar{l} \in C_{\mathbb{S}^1}(\mathcal{G}^*)$ that the following functional operator relationship*

$$(\partial/\partial y - l(T)) \circ \Phi(T) = \Phi(T) \circ (\partial/\partial y - \bar{l}(T)\,) \tag{23}$$

holds, where $\partial\bar{l}(T)/\partial t = 0 = \partial\bar{l}(T)/\partial x$, that is the element $\bar{l}(T) \in C_{\mathbb{S}^1}(\mathcal{G}^*)$ is constant both with respect to the evolution parameter $t \in \mathbb{R}$ and the functional algebra $\mathcal{A}$ parameter $x \in \mathbb{S}^1$.

Proof: *Sketch of a proof.* Taking into account that $l(T) := F_n\,(T)^{-1} \circ Q_{n+p}(T) \in C_{\mathbb{S}^1}(\mathcal{G}^*)$, the operator relationship (23) can be equivalently rewritten as

$$\begin{aligned}(F_n\,(T)\partial/\partial y - Q_{n+p}(T)) \circ \Phi(T) = \\ = F_n\,(T) \circ \Phi(T) \circ (\partial/\partial y - \bar{l}(T)),\end{aligned} \tag{24}$$

where

$$\bar{l}(T) \sim \sum_{j \in \mathbb{Z}_+} l_j(y) T^{p-j} \tag{25}$$

is constant, it allows to determine recurrently all coefficients of the corresponding invertible operator expansion

$$\Phi(T) \sim I + \sum_{j \in \mathbb{Z}_+} \varphi_j(x; y) T^{-j} \tag{26}$$

for all $(x; y) \in \mathbb{S}^1 \times \mathbb{S}^1$. The latter proves the lemma.

Theorem 4 *The following functionals*

$$\begin{aligned}\gamma_j = Tr(T^j \bar{l}(T)) = \int_{\mathbb{S}^1} tr(\bar{l}_j(y))dy = \\ = \int_{\mathbb{S}^1} \tau(\bar{l}_j(y))dy,\end{aligned} \tag{27}$$

where, by definition,

$$\bar{l} \sim \sum_{j \in \mathbb{Z}_+} \bar{l}_j(y) T^{p-j}, \tag{28}$$

are for all $j \in \mathbb{Z}_+$ the Casimir invariants for the centrally extended loop Lie algebra $C_{\mathbb{S}^1}(\widehat{\mathcal{G}})$.

Based on Theorem 4 one can find that the corresponding gradients

$$\nabla\gamma_j(l) = \Phi(T) T^j \Phi(T)^{-1} \tag{29}$$

, $j \in \mathbb{Z}_+$, for the countable hierarchy of Casimir functionals (27) satisfy the determining relationship (20). In addition, from (23) one ensues that the following operator expression for the case $l = F_n^{-1} Q_{n+p} \in C_{\mathbb{S}^1}(\mathcal{G}^*)$:

$$\bar{l} = \Phi(T)^{-1}(l - \partial/\partial y)\Phi(T), \tag{30}$$

holds for all $y \in \mathbb{S}^1$, where the invertible mapping $\Phi(T) \in C_{\mathbb{S}^1}(\hat{G})$ satisfies the evolution equation

$$d\Phi(T)/dt + \nabla\gamma(l)_+ \Phi(T) = 0 \tag{31}$$

for all $t \in \mathbb{R}$. Similarly one can state that there exists a suitably chosen mapping $\Psi(T) \in C_{\mathbb{S}^1}(G)$ for the case $\tilde{l} = Q_{n+p}F_n^{-1} \in C_{\mathbb{S}^1}(\mathcal{G}^*)$, such that

$$\bar{\tilde{l}} = \Psi(T)^{-1}(\tilde{l} - \partial/\partial y)\Psi(T) \tag{32}$$

holds for some constant element $\bar{\tilde{l}} \in C_{\mathbb{S}^1}(\mathcal{G}^*)$ with respect to both the evolution variable $t \in \mathbb{R}$ and the functional parameter $x \in \mathbb{S}^1$, where the invertible mapping $\Psi(T) \in C_{\mathbb{S}^1}(G)$ satisfies the evolution equation

$$d\Psi(T)/dt + \nabla\gamma(l)_+\Psi(T) = 0. \tag{33}$$

Moreover, the element $\tilde{l} \in C_{\mathbb{S}^1}(\mathcal{G}^*)$ satisfies the Lax type evolution equation

$$d\tilde{l}/dt = [\tilde{l} - d/dy, \nabla\gamma(\tilde{l})_+] \tag{34}$$

for all $t \in \mathbb{R}$. Taking into account that the expression (32) can be equivalently rewritten as

$$\begin{aligned}(F_n\ (T)\partial/\partial y - Q_{n+p}(T)\ ) \circ \Psi(T) = \\ = F_n\ (T) \circ \Psi(T) \circ (\partial/\partial y - \bar{\tilde{l}}(T)),\end{aligned} \tag{35}$$

from (35), (24) and evolution equations (31), (33) one can derive the corresponding factorized evolution equations

$$\begin{aligned}d\mathrm{F}_n/dt = \mathrm{F}_n\nabla\gamma(l)_+ - \nabla\gamma(\tilde{l})_+\mathrm{F}_n, \\ \mathrm{Q}_n/dt = \mathrm{Q}_n\nabla\gamma(l)_+ - \nabla\gamma(\tilde{l})_+\mathrm{Q}_n,\end{aligned} \tag{36}$$

for the elements $\mathrm{F}_n := F_n \in C_{\mathbb{S}^1}(G)$ and $\mathrm{Q}_{n+p} := Q_{n+p} - F_n\partial/\partial y \in C_{\mathbb{S}^1}(\hat{G})$, which allow the following natural representations

$$\begin{aligned}\mathrm{F}_n := \Psi(T)\ \bar{\mathrm{F}}_n\Phi(T)^{-1}, \\ \mathrm{Q}_{n+p} := \Psi(T)\ \bar{\mathrm{Q}}_{n+p}\Phi(T)^{-1}\end{aligned} \tag{37}$$

with $\bar{\mathrm{F}}_n := \bar{F}_n \in C_{\mathbb{S}^1}(G)$ and $\bar{\mathrm{Q}}_n := \bar{Q}_n - \bar{F}_n\partial/\partial y \in C_{\mathbb{S}^1}(G)$ being constants with respect to the evolution variables $t \in \mathbb{R}$ and $x \in \mathbb{S}^1$. Taking now into account the above expressions (37) and (31) one easily obtains from (36) the following evolutions equations

$$dF_n/dt = F_n\nabla\gamma(l)_+ - \nabla\gamma(\tilde{l})_+F_n, \tag{38}$$

$$\begin{aligned}dQ_{n+p}/dt = Q_{n+p}\nabla\gamma(l)_+ - \nabla\gamma(\tilde{l})_+Q_{n+p} - \\ F_n\partial\nabla\gamma(l)_+/\partial y\end{aligned}$$

on the basic operator factors F_n and $Q_{n+p} \in C_{\mathbb{S}^1}(G)$. The corresponding invertible mapping Φ and $\Psi \in C_{\mathbb{S}^1}(G)$, satisfying, respectively, the expressions (37), can be recurrently constructed from the algebraic relationships

$$F_n(I + \sum_{j\in\mathbb{Z}_+} a_j(x;y)T^{-j}) = (I + \sum_{j\in\mathbb{Z}_+} b_j(x;y)T^{-j})\bar{F}_n,$$

$$\begin{aligned}(Q_{n+p} - F_n\partial/\partial y)(I + \sum_{j\in\mathbb{Z}_+} a_j(x;y)T^{-j}) = \\ = (I + \sum_{j\in\mathbb{Z}_+} b_j(x;y)T^{-j})(\bar{Q}_n - \bar{F}_n\partial/\partial y)\end{aligned} \tag{39}$$

in the series expansion form:

$$\begin{aligned}\Phi(T) \sim I + \sum_{j\in\mathbb{Z}_+} a_j(x;y)T^{-j}, \\ \Psi(T) \sim I + \sum_{j\in\mathbb{Z}_+} b_j(x;y)T^{-j}.\end{aligned} \tag{40}$$

The statements above we can formulate as the next factorization theorem.

Theorem 5 *The operator evolution equations*

$$\begin{aligned}dF_n/dt = F_n\nabla\gamma(l)_+ - \nabla\gamma(\tilde{l})_+F_n, \\ dQ_{n+p}/dt = Q_{n+p}\nabla\gamma(l)_+ - \nabla\gamma(\tilde{l})_+ = \\ = Q_{n+p} - F_n\partial\nabla\gamma(l)_+/\partial y\end{aligned} \tag{41}$$

factorize the Lax type flows (19) and (34) with elements $l = F_n^{-1}Q_{n+p} \in C_{\mathbb{S}^1}(\mathcal{G}^*)$ and $\tilde{l} = Q_{n+p}F_n^{-1} \in C_{\mathbb{S}^1}(\mathcal{G}^*)$, respectively, where the corresponding Casimir invariants $\gamma \in I(C_{\mathbb{S}^1}(\widehat{\mathcal{G}}^*))$ satisfy the relationship $\gamma|_{l=F_n^{-1}Q_{n+p}} = \gamma|_{\tilde{l}=Q_{n+p}F_n^{-1}}$ for any F_n and $Q_{n+p} \in C_{\mathbb{S}^1}(G_+)$.

2.1 Example

As an example of a rationally factorized operator $l \in C_{\mathbb{S}^1}(\mathcal{G}^*)$ one can consider the following simple expression

$$l := T^{-1}(T^2 + Tv + uI), \tag{42}$$

where functions $u, v \in C(\mathbb{S}^1 \times \mathbb{R}; \mathbb{R})$. The corresponding elements $F_1 := T$, $Q_2 := T^2 + Tv + uI) \in C_{\mathbb{S}^1}(G_+)$ generate the factorized evolution equations (10), where gradients of the corresponding Casimir functionals $\gamma \in I(C_{\mathbb{S}^1}(\widehat{\mathcal{G}}^*))$ can be found recurrently from the relationships (29) jointly with the relationships (24), (25) and (26). From the corresponding calculations one ensues the system of integrable evolution functional equations

$$u_t = u(Tv - v), v_t = v(T^{-1}u - u) \tag{43}$$

on the elements $u, v \in C(\mathbb{S}^1; \mathbb{R})$.

3 Special functional-algebraic realizations

The algebraic scheme devised in Section 2 makes it possible to be effectively modified for the case when the associative functional algebra $\mathcal{A}$ is chosen to be the algebra of smooth pseudo-differential operators $\mathrm{PDO}(\mathbb{S}^1)$, acting on the functional space $C^\infty(\mathbb{S}^1;\mathbb{R})$and endowed with the natural commutator Lie structure. The resulting Lie algebra $\mathcal{G} := (\mathrm{PDO}(\mathbb{S}^1); [\cdot,\cdot])$ is split into direct sum of two subalgebras, $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$:

$$\mathcal{G}_+ := \Big\{ \sum_{j\in\mathbb{Z}_+} a_j(x)\partial^j : \forall j\ a_j(x) \in C^\infty(\mathbb{S}^1;\mathbb{R})\Big\},$$
$$\mathcal{G}_- := \Big\{ \sum_{j\in\mathbb{Z}_+} b_j(x)\partial^{-j} : \forall j\ b_j(x) \in C^\infty(\mathbb{S}^1;\mathbb{R})\Big\},$$

where, by definition, $\partial := \partial/\partial x$ and $\partial \cdot \partial^{-1} = 1$ for $x \in \mathbb{S}^1$. Moreover, the Lie algebra $\mathcal{G}$ is metrized by means of the invariant trace form

$$(a, b) := Tr(a \cdot b),\ Tr(c) := \int_{\mathbb{S}^1} (\mathrm{res}_\partial\ c)\ dx \quad (44)$$

for any a, b and $c \in \mathcal{G}$, allowing to identify the adjoint space $\mathcal{G}^* \simeq \mathcal{G}$.

Taking into account these preliminaries a similar to that, posed in Section 2, problem arises: *construct the corresponding operator dynamical systems on the elements $F_n(\partial), Q_{n+p}(\partial) \in \mathcal{G}$, which will possess an infinite hierarchy of functional invariants and will be analytically integrable.*

As above we consider the general Lax type flow

$$dl/dt = [l, \nabla\gamma(l)_+], \quad (45)$$

for the rational element

$$l(\partial) := F_n(\partial)^{-1}Q_{n+p}(\partial), \quad (46)$$

generated by a Casimir functional $\gamma \in I(\mathcal{G}^*)$ and determined by the expression (4). One observes that $\gamma := tr(\gamma(l) = tr(\gamma(\tilde{l}))$ for any analytical mapping $\gamma(l) \in \mathcal{G}$, where we have introduced, by definition, the factorized element $\tilde{l} := Q_{n+p}F_n^{-1} \in \mathcal{G}^*$. Also the element $\tilde{l} = Q_{n+p}F_n^{-1} \in \mathcal{G}^*$ satisfies the similar to (7) evolution equation

$$d\tilde{l}/dt = [\tilde{l}, \nabla\gamma(\tilde{l})_+] \quad (47)$$

for the same Casimir functional $\gamma \in I(\mathcal{G}^*)$, whose gradient, similarly to (4), is determined from the algebraic relationship

$$[\tilde{l}, \nabla\gamma(\tilde{l})] = 0. \quad (48)$$

Taking now into account these two compatible equations (45) and (47) one easily derives the following factorization theorem.

Theorem 6 *The differential operator evolution equations*

$$dF_n/dt = F_n\nabla\gamma(l)_+ - \nabla\gamma(\tilde{l})_+F_n, \quad (49)$$
$$dQ_{n+p}/dt = Q_{n+p}\nabla\gamma(l)_+ - \nabla\gamma(\tilde{l})_+Q_{n+p}$$

factorize the Lax type flows (45) and (47) for all $t \in \mathbb{R}$ with elements $l = F_n^{-1}Q_{n+p} \in \mathcal{G}^*$ and $\tilde{l} = Q_{n+p}F_n^{-1} \in \mathcal{G}^*$, respectively, where the corresponding Casimir invariants $\gamma \in I(\mathcal{G}^*)$ satisfy the relationship $\gamma|_{l=F_n^{-1}Q_{n+p}} = \gamma|_{\tilde{l}=Q_{n+p}F_n^{-1}}$ for any F_n and $Q_{n+p} \in G_+$.

From Theorem 6 one easily ensues the following proposition.

Proposition 7 *There exist such smooth mappings $\Phi, \Psi : \mathbb{R} \to G$ to the formal operator subgroup $G \simeq \exp\mathcal{G}$ satisfying the linear evolution equations*

$$\frac{d\Phi}{dt} + \nabla\gamma(\tilde{l})_+\Phi = 0,\ \ \frac{d\Psi}{dt} + \nabla\gamma(l)_+\Psi = 0 \quad (50)$$

$\Phi|_{t=0} = \bar{\Phi}, B|_{t=0} = \bar{B} \in G$, generated, respectively, by the pseudo-differential Lie algebra elements $\nabla\gamma(l)_+$ and $\nabla\gamma(\tilde{l})_+ \in \mathcal{G}_+$, that

$$F_n := \Psi\,\bar{F}_n\Phi^{-1},\ \ Q_{n+p} := \Psi\,\bar{Q}_{n+p}\Phi^{-1}, \quad (51)$$

where, by definition, the elements $\bar{F}_n$ and $\bar{Q}_{n+p} \in G$ are some constant expressions with respect to the evolution parameter $t \in \mathbb{R}$.

4 The Poisson structures and Hamiltonian analysis on the extended phase space

Let us consider equation (7), the first equation of (11) and its adjoint expression:

$$d\hat{l}/dt = [\hat{l}, \nabla\gamma(\hat{l})_+],\ \ d\hat{f}/dt + \nabla\gamma(\hat{l})_+\hat{f} = 0,$$
$$d\hat{f}^*/dt - \nabla\gamma(\hat{l})^*_+\hat{f} = 0 \quad (52)$$

for vector elements $\hat{f} \in W$ and $\hat{f}^* \in W^*$, respectively, where W denotes a representation space for the group G and W^* is its natural conjugation with respect to the natural bilinear form $< \cdot, \cdot >$, realizing the standard paring between spaces W^* and W. Put also by

$$\nabla\gamma(\hat{l}, \hat{f}, \hat{f}^*) := (\delta\gamma/\delta\hat{l},\, \delta\gamma/\delta\hat{f},\, \delta\gamma/\delta\hat{f}^*)$$

an extended gradient vector at a point $(\hat{l}; \tilde{f}, \tilde{f}^*) \in \mathcal{G}^* \oplus W \oplus W^*$ for any smooth functional $\gamma \in \mathcal{D}(\mathcal{G}^* \oplus W \oplus W^*)$.

On the space $\mathcal{G}^*$ there exists the canonical Poisson structure

$$\delta\gamma/\delta\hat{l} :\xrightarrow{\tilde{\theta}} [\hat{l}, (\delta\gamma/\delta\hat{l})_+] - [\hat{l}, \delta\gamma/\delta\hat{l}]_+ \;, \tag{53}$$

where $\tilde{\theta} : T^*(\mathcal{G}^*) \to T(\mathcal{G}^*) \simeq \mathcal{G}$ is a Poisson operator at a point $\hat{l} \in \mathcal{G}^*$. Similarly on the space $W \oplus W^*$ there exists the canonical Poisson structure

$$(\delta\gamma/\delta\hat{f},\, \delta\gamma/\delta\hat{f}^*) :\xrightarrow{\tilde{J}} (-\delta\gamma/\delta\hat{f}^*,\, \delta\gamma/\delta\hat{f}), \tag{54}$$

where $\tilde{J} : T^*(W \oplus W^*) \to T(W \oplus W^*)$ is the Poisson operator corresponding to the symplectic form $\omega^{(2)} =< d\hat{f}^*, \wedge d\hat{f} >$ at a point $(\hat{f}, \hat{f}^*) \in W \oplus W^*$. It should be noted here that the Poisson structure (53) generates equations (7) and (8) for any Casimir functional $\gamma \in \mathrm{I}(\mathcal{G}^*)$.

Thus, on the extended phase space $\mathcal{G}^* \oplus W \oplus W^*$ one can obtain a new Poisson structure as the tensor product $\tilde{\Theta} := \tilde{\theta} \otimes \tilde{J}$ of the structures (53) and (54).

Consider now the following Backlund [1] transformation:

$$(\hat{l}; \hat{f}, \hat{f}^*) :\xrightarrow{B} (l = \; l(\hat{l}; \hat{f}, \hat{f}^*), \;\; f = \hat{f}, \;\; f^* = \hat{f}^*), \tag{55}$$

generating on $\mathcal{G}^* \oplus W \oplus W^*$ some Poisson structure $\Theta : T^*(\mathcal{G}^* \oplus W \oplus W^*) \to T(\mathcal{G}^* \oplus W \oplus W^*)$. The *main condition imposed on the mapping (55) is the coincidence of the resulting dynamical system*

$$(dl/dt;\, df/dt,\, df^*/dt) := -\Theta\nabla\bar{\gamma}(\, l; f, f^*) \tag{56}$$

with the evolution equations

$$\begin{aligned} &dl/dt = [l, \nabla\gamma(l)_+], \quad df/dt = \nabla\gamma\;(l)_+ f, \\ &df^*/dt = -\nabla\gamma\;(l)_+ f^* \end{aligned} \tag{57}$$

in the case when $\bar{\gamma} := \gamma \in \mathrm{I}(\mathcal{G}^*)$, being not dependent on the variables $(f, f^*) \in W \oplus W^*$.

To satisfy that condition we will find variation of the functional $\bar{\gamma} := \gamma|_{l=l(\hat{l},\hat{f},\hat{f}^*)} \in \mathcal{D}(\mathcal{G}^* \times W \oplus W^*)$, generated by a Casimir functional $\gamma \in \mathrm{I}(\mathcal{G}^*)$, under the constraint $\delta\tilde{l} = 0$, taking into account the evolutions (52) and the Backlund transformation (55) definition. One easily obtains that

$$\begin{aligned} &\delta\bar{\gamma}(\hat{l}; \hat{f}, \hat{f}^*)\Big|_{\delta\hat{l}=0} = \langle \delta\bar{\gamma}/\delta\hat{f}, \delta\hat{f}\rangle + \langle \delta\bar{\gamma}/\delta\hat{f}^*, \delta\hat{f}^*\rangle \\ &= \langle -d\hat{f}^*/dt, \delta\hat{f}\rangle + \langle d\hat{f}/dt, \delta\hat{f}^*\rangle\Big|_{\hat{f}=f,\, \hat{f}^*=f^*} = \\ &= \langle (\delta\gamma/\delta\; l)_+^* \hat{f}^*, \delta\hat{f}\rangle + \langle (\delta\gamma/\delta\; l)_+ \hat{f}, \delta\hat{f}^*\rangle = \\ &= \langle \hat{f}^*, (\delta\gamma/\delta\; l)_+ \delta\hat{f}\rangle + \langle (\delta\gamma/\delta\; l)_+ \hat{f}, \delta\hat{f}^*\rangle = \\ &= (\delta\gamma/\delta\; l, (\delta\hat{f})\xi^{-1} \otimes \hat{f}^*) + (\delta\gamma/\delta\; l, \hat{f}\xi^{-1} \otimes \delta\hat{f}^*) = \\ &= \Big(\delta\gamma/\delta\; l, \delta(\hat{f}\xi^{-1} \otimes \hat{f}^*)\Big) := (\delta\gamma/\delta\; l, \delta\; l)\;, \end{aligned} \tag{58}$$

giving rise to the relationship

$$\delta\; l|_{\delta\tilde{l}=0} = \delta(\hat{f}\xi^{-1} \otimes \hat{f}^*) := \;\delta(\hat{f}\xi^{-1} \otimes \hat{f}^*). \tag{59}$$

Having assumed now the linear dependence of l on $\tilde{l} \in \hat{\mathcal{G}}^*$ one gets right away from (59) that

$$l = \tilde{l} + \hat{f}\xi^{-1} \otimes \hat{f}^*. \tag{60}$$

Thus, the Backlund transformation (55) can be rewritten as

$$(\hat{l}; \hat{f}, \hat{f}^*) :\xrightarrow{B} (\; l = \hat{l} + \hat{f}\;\xi^{-1} \otimes \hat{f}^*; f = \hat{f}\;, f^* = \hat{f}^*). \tag{61}$$

Now by means of simple calculations via [1] the isomorphism formula

$$\Theta = B^{'}\tilde{\Theta}B^{'*}\;,$$

where $B^{'} : T(\mathcal{G}^* \oplus W \oplus W^*) \to T(\mathcal{G}^* \oplus W \oplus W^*)$ is a Frechet derivative of (61), one finds easily the following form of the Backlund transformed Poisson structure Θ on $\mathcal{G}^* \oplus W \oplus W^*$:

$$\Theta : \nabla\gamma(\; l; f, f^*) \to \begin{pmatrix} [\; l, (\delta\gamma/\delta\; l)_+] - [\; l, \delta\gamma/\delta\; l]_+ \\ +(f\xi^{-1} \otimes (\delta\gamma/\delta f) - (\delta\gamma/\delta f^*)\xi^{-1} \otimes f^*); \\ -\delta\gamma/\delta f^* - (\delta\gamma/\delta\; l)_+ f \\ \delta\gamma/\delta f + (\delta\gamma/\delta\; l)_+^* f^* \end{pmatrix}, \tag{62}$$

where $\gamma \in D(\mathcal{G}^* \oplus W \oplus W^*)$ is an arbitrary smooth functional. The obtained Backlund transformation (61) makes it possible to formulate the following theorem.

Theorem 8 *The set of differential-operator dynamical systems (57) on $\mathcal{G}^* \oplus W \oplus W^*$ is Hamiltonian with respect to the Poisson structure (62) and has the form (56) for $\gamma := \bar{\gamma} \in \mathrm{I}(\mathcal{G}^*)$, being chosen Casimir functionals on $\mathcal{G}^*$.*

Based on the expression (56) one can construct a new hierarchy of Hamiltonian evolution equations describing commutative flows generated by involutive with respect to the natural Poisson bracket (54) Casimir invariants $\gamma \in \mathrm{I}(\hat{\mathcal{G}}^*)$, extended on the space $\mathcal{G}^* \oplus W \oplus W^*$.

Proceed now to considering flows (7) and (8) as Hamiltonian systems on $\mathcal{G}^* \times \mathcal{G}^*$ subject to the following tensor doubled standard Poisson structure:

$$\vartheta : \nabla\gamma(l) \longrightarrow \begin{pmatrix} [\nabla\gamma(l)_+, l] - [\nabla\gamma(l), l]_+ \\ \left[\nabla\tilde{\gamma}_+(\tilde{l}), \tilde{l}\right] - \left[\nabla\tilde{\gamma}(\tilde{l}), \tilde{l}\right]_+ \end{pmatrix}, \tag{63}$$

where $\gamma(l) = \gamma(\tilde{l})$ and $\gamma \in \mathcal{D}(\mathcal{G}^* \times \mathcal{G}^*)$ is an arbitrary smooth functional on $\mathcal{G}^* \times \mathcal{G}^*$. Concerning the transformation

$$\Phi(Q, F; \tilde{l}, l) = 0 \Leftrightarrow \tilde{l} - QF^{-1} = 0, \quad l - F^{-1}Q = 0, \tag{64}$$

which can be evidently considered as a usual Backlund [1] transformation, we can construct a new Poisson structure $\eta : T^*(\mathcal{G}_+^* \times \mathcal{G}_+^*) \longrightarrow T(\mathcal{G}_+^* \times \mathcal{G}_+^*)$ on the space $\mathcal{G}_+^* \times \mathcal{G}_+^*$ with respect to the phase variables $(F, Q) \in \mathcal{G}_+^* \times \mathcal{G}_+^*$. Thereby one finds [1] the corresponding to (63) and (64) transformed Poisson structure $\eta : T^*(\mathcal{G}_+^* \times \mathcal{G}_+^*) \longrightarrow T(\mathcal{G}_+^* \times \mathcal{G}_+^*)$ at $(F, Q) \in \mathcal{G}_+^* \times \mathcal{G}_+^*$, where

$$\eta = T\vartheta T^*, \quad T = \Phi'_{(\tilde{l},l)} \Phi'^{-1}_{(Q,F)}. \tag{65}$$

Making use of the expressions

$$\Phi'_{(Q,F)} = \begin{pmatrix} -(.)F^{-1}\tilde{l}(.)F^{-1} \\ -F^{-1}(.) & F^{-1}(.)l \end{pmatrix}, \Phi'_{(l,\tilde{l})} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\Phi'^{-1}_{(Q,F)} = \begin{pmatrix} -(1 - \tilde{l} \otimes l^{-1})^{-1}(.)F & (1 - \tilde{l} \otimes l^{-1})^{-1}\tilde{l}F(.)l^{-1} \\ -(1 - \tilde{l} \otimes l^{-1})^{-1}(.)F & (1 - \tilde{l} \otimes l^{-1})^{-1}F(.), \end{pmatrix}$$

$$(\Phi'^*_{(Q,F)})^{-1} = \begin{pmatrix} -F(.)(1 - \tilde{l}^{-1} \otimes \tilde{l})^{-1} & -F(.)(1 - l^{-1} \otimes \tilde{l})^{-1} \\ l^{-1}(.)F\tilde{l}(1 - l^{-1} \otimes \tilde{l})^{-1} & (.)F(1 - l^{-1} \otimes \tilde{l})^{-1} \end{pmatrix}$$

jointly with the ϑ- structure (63), one gets from (65) that

$$\eta = \begin{pmatrix} -(1 - \tilde{l} \otimes l^{-1})^{-1}(.)F & (1 - \tilde{l} \otimes l^{-1})^{-1}\tilde{l}F(.)l^{-1} \\ -(1 - \tilde{l} \otimes l^{-1})^{-1}(.)F & (1 - \tilde{l} \otimes l^{-1})^{-1}F(.) \end{pmatrix} \times$$

$$\times \begin{pmatrix} \begin{array}{l} -\left[\tilde{l}, (1 - \tilde{l} \otimes l^{-1})^{-1}(.)F(1 - \tilde{l} \otimes l^{-1})^{-1}(.)\right]_+, \\ -\left[l^{-1}(.)F\tilde{l}(1 - l^{-1} \otimes \tilde{l})^{-1}, l\right] + \\ +\left[\left((.)F(1 - l^{-1} \otimes \tilde{l})^{-1}\right)_+, l\right], \end{array} \\ \begin{array}{l} -\left[\left(F(.)(1 - l^{-1} \otimes \tilde{l})^{-1}\right)_+, \tilde{l}\right] + \\ +\left[\left(F(.)(1 - l^{-1} \otimes \tilde{l})^{-1}\right), \tilde{l}\right]_+ \\ \left[\left((.)F(1 - l^{-1} \otimes \tilde{l})^{-1}\right)_+, l\right] - \\ -\left[(.)F(1 - l^{-1} \otimes \tilde{l})^{-1}, l\right]_+ \end{array} \end{pmatrix} \tag{66}$$

at $\tilde{l} = QF^{-1}$ and $l = F^{-1}Q \in \mathcal{G}^*$.

Let now take any Casimir functional $\gamma \in \mathrm{I}(\mathcal{G}^*)$. Then one construct from the Poisson bracket (66) the following Hamiltonian flow on $\mathcal{G}_+^* \times \mathcal{G}_+^*$:

$$\frac{d}{dt}(Q, F) = \eta\nabla\gamma(Q, F), \tag{67}$$

where $(Q, F) \in \mathcal{G}_+^* \times \mathcal{G}_+^*$ and $t \in \mathbb{R}$ is the temporal evolution parameter. The flow (67) is characterized by the following theorem.

Theorem 9 *The Hamiltonian vector field d/dt on $\mathcal{G}_+^* \times \mathcal{G}_+^*$, defined by (67), and the vector field d/dt on $\mathcal{G}_+^* \times \mathcal{G}_+^*$, defined by (10), coincide.*

Proof: Proof of this theorem consists in simple but a slightly tedious calculation of the expression (67).

Remark 10 *The theorem above solves completely a problem posed in [8] about Hamiltonian formulation of the factorized differential-operator equations (10).*

5 Examples

5.1 Example 1

We consider the following pseudo-differential factorized expression

$$l(\partial) = (\partial + u)^{-1}[(\partial + u)(\partial^2 + 2v) - 2w] \tag{68}$$

for $F_1 := \partial + u, Q_3 := (\partial + u)(\partial^2 + 2v) - 2w \in \mathcal{G}_+, (u, v, w) \in C^\infty(\mathbb{S}^1; \mathbb{R}^3)$. The respectively factorized differential operator evolution equations

(49) give rise to the following [6] interesting system

$$u_t = 2uu_x + 2v_x - u_{xx}, \qquad (69)$$
$$v_t = 2w_x, \quad w_t = w_{xx} + 2(wu)_x$$

of completely integrable evolution equations.

Remark 11 *It is worth to mention that the derived above system of integrable equations (69) allows the following degenerate purely differential-matrix linear spectral problem:*

$$\begin{pmatrix} (\partial^2 + 2v)(\partial - u) & -\lambda \\ \partial - u & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = 0 \qquad (70)$$

for $(f, g) \in L_2(\mathbb{S}^1; \mathbb{C}^2)$ and arbitrary spectral parameter $\lambda \in \mathbb{C}$.

5.2 Example 2

A next example is related with the pseudo-differential factorized expression

$$l(\partial) = [(\partial+w)(\partial+p)]^{-1}[(\partial+w)(\partial+p)\partial+(\partial+p)u+v] \qquad (71)$$

for $F_2 := (\partial + w)(\partial + p)$ and $Q_3 := (\partial + w)(\partial + p)\partial + (\partial+p)u + v \in \mathcal{G}_+, (u, p, v, w) \in C^\infty(\mathbb{S}^1; \mathbb{R}^4)$. From the factorized differential operator evolution equations (49) one easily ensues the system

$$u_t = u_{2x} + 2v_x + 2(uw)_x, \qquad (72)$$
$$v_t = v_{2x} + 2vw_x + 2(pw)_x,$$
$$w_t = -w_{xx} + 2u_x + 2ww_x,$$
$$p_t = -p_{xx} - 2w_{2x} + 2u_x + 2pp_x,$$

of completely integrable evolution flows on $C^\infty(\mathbb{S}^1; \mathbb{R}^4)$, considered also before in [12, 6] in the context of generating a new class of integrable dispersionless systems of hydrodynamic type equations.

5.3 Example 3

Let us put now the following pseudo-differential factorized expression

$$l(\partial) = \partial+(1/4-\alpha^2\partial^2)^{-1}(\gamma\partial^2+v/2 +\beta/4)+\gamma\alpha^{-2}, \qquad (73)$$

where α, β and $\gamma \in \mathbb{R}$ are constants, $v \in C^\infty(\mathbb{S}^1; \mathbb{R}), F_2 := 1/4 - \alpha^2\partial^2$ and $Q_3 := \gamma\partial^2 + v/2 + \beta/4 \in \mathcal{G}_+$. The related factorized differential operator evolution equations (49) are reduced for the gradient element $\nabla\gamma(l)_+ = \nabla\gamma(l) - \nabla\gamma(l)_- = \frac{1}{2}u_x - u\partial - (1/4-\alpha^2\partial^2)^{-1}(\gamma\partial^3+\partial\, v/2 + \beta\partial/4) \in \mathcal{G}_- \oplus \{\partial\}$, where and $u := (1 - \alpha^2\partial^2)^{-1}v \in C^\infty(\mathbb{S}^1; \mathbb{R})$ and the element $\partial \in \mathcal{G}$ is a character of the Lie algebra $\mathcal{G}$, that is $(\partial, [\mathcal{G}_\pm, \mathcal{G}_\pm]) = 0$, to the following evolution flow:

$$v_t + \beta u_x + uv_x + 2vu_x + \gamma u_{3x} = 0, \qquad (74)$$

describing simple wave motion [7] of the Euler equations for shallow water dynamics.

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