On the length spectrum for compact locally symmetric spaces of real rank one

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Abstract: - We obtain improved asymptotic estimate for the function enumerating prime geodesics over compact locally symmetric space of real rank one.

Key-Words: - length spectrum, zeta functions, logarithmic derivative, entire and meromorphic functions, functional equations, admissible lifts

1 Introduction

Let Y be a compact n-dimensional locally symmetric Riemannian manifold with strictly negative sectional curvature.

As it is well known, Y is isometric to a double coset space $\Gamma\backslash G/K = \Gamma\backslash X$ where, $G;K;\Gamma$ and X denote: a connected, rank-one, semi-simple Lie group; a maximal compact subgroup of G; a discrete, torsion-free, co-compact subgroup of G and the universal Riemannian covering space of Y, respectively.

For the sake of simplicity, we shall consider Y as $\Gamma\backslash G/K$ with Γ acting as isometries on X.

According to [8, pp. 134-135], prime geodesic theorem states that

$$\#\left\{C_\gamma \middle| l(\gamma)\le x\right\} \overset{def}{=} \pi_\Gamma(x) = \int_1^{\log x} \frac{e^{\alpha t}}{t}dt + O\left(x^\eta\right) \qquad (1)$$

as $x \to +\infty$, where C_γ denotes a prime geodesic of the length $l(\gamma)$ over Y, η is a constant (depending on Γ) such that $\left(1-\frac{1}{2n}\right)\alpha \le \eta < \alpha$ and α is defined by

$$\alpha = \begin{cases} (n-1)\left(-\underline{\xi}\right)^{\frac{1}{2}}, & \overline{\xi} = \underline{\xi}, \\ \dfrac{4n(n-1)\left(-\overline{\xi}\right)+S}{6n\left(-\overline{\xi}\right)^{\frac{1}{2}}}, & \overline{\xi} \ne \underline{\xi} \end{cases},$$

where $\overline{\xi}\left(\underline{\xi}\right)$ and S denote *sup(inf)* of the sectional curvatures of Y and the scalar curvature of Y, respectively.

Note that the elements of the set introduced by (1) are labeled by γ since every prime geodesic over Y corresponds to a conjugacy class of a primitive hyperbolic element $\gamma \in \Gamma$.

It is also convenient to define a norm $N(\gamma) = e^{l(\gamma)}$ for hyperbolic $\gamma \in \Gamma$ (see also, Section 3).

Suppose that the Riemannian metric over Y induced from the Killing form is normalized so that the sectional curvature of Y varies between -4 and -1.

Now, (see, [8, p. 136]), $\alpha = n+q-1$, where $q = 0,1,3,7$ depending on whether X is a real, a complex or a quaternionic hyperbolic space or the Cayley hyperbolic plane.

If we calculate the integral on the right side of (1) by parts, we obtain a weaker from of the prime geodesic theorem i.e., $\lim_{n\to+\infty} \pi_\Gamma(x) \frac{\alpha \log x}{x^\alpha} = 1$. The same resut in the same form was also proved in [13] as well as in [15], where Y is not compact space but has a finite volume. The result of [15] that corresponds to the real hyperbolic manifolds with cusps case was later refined in [20]. There, the author applied a variant of the techniques [17], [18]. He also applied the Ruelle zeta function instead of the Selberg zeta. Finally, in [1], we adapted the approach [21] to the setting [20] and thus further improved the result [20] so to coincide with the best known estimate in the case of compact Riemann surface Y, i.e., with the estimate (see, [21], [4])

$$\pi_\Gamma(x) = \sum_{s_k \in \left(\frac{3}{4}, 1\right]} li\left(x^{s_k}\right) + O\left(x^{\frac{3}{4}} (\log x)^{-1}\right) \qquad (2)$$

as $x \to +\infty$, where s_k denotes a zero of the Selberg zeta function associated to Y.

In this paper we improve the estimate (1) so to be in accordance with (2).

Our main tool will be the zeta functions of Selberg and Ruelle [7]. In particular, we shall utilize the fact that these functions are meromorphic functions of order not larger than n. This was proved in [2], [3] when n is even. An analogous result for odd n will be derived in Section 4.

The structure of the paper is as follows: Section 2 provides some necessary background and preliminary material. In Section 3 we introduce the zeta functions and assemble those theorems and facts we will need. In Section 4 we prove a number of auxiliary results related to the analytic properties of the zeta functions introduced in Section 3. In Section 5 we state and prove the main result. Section 6 is devoted to a derivation of the functional equations of the zeta functions in the form [22] not presented in [7]. In Section 7 we give concluding remarks.

2 Preliminaries

In the sequel, we follow the notation of [7].

Assume that G is a linear group.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$ and M the centralizer of $\mathfrak{a}$ in K with the Lie algebra $\mathfrak{m}$.

Let $\Phi(\mathfrak{g}, \mathfrak{a})$ be the root system and $\Phi^+(\mathfrak{g}, \mathfrak{a}) \subset \Phi(\mathfrak{g}, \mathfrak{a})$ a system of positive roots. By W we denote the Wely group of $\Phi(\mathfrak{g}, \mathfrak{a})$. Let

$$\mathfrak{n} = \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{n}_\alpha$$

be the sum of the root spaces. Then, the Iwasawa decomposition $G = KAN$ corresponds to the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \oplus \mathfrak{n}$. Define

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})} \dim(\mathfrak{n}_\alpha) \alpha .$$

Let $\mathfrak{a}^+$ be the half line in $\mathfrak{a}$ on the which the positive roots take positive values. Put $A^+ = \exp(\mathfrak{a}^+) \subset A$.

Let $\sigma \in \hat{M}$.

Following [7, p. 27], we distinguish between two cases.

Case (a): σ is invariant under the action of the Weyl group W.

Note that this case may occur if n is either even or an odd number. Moreover, this case includes all σ if n is even.

By Proposition 1.1 (n odd) and Proposition 1.2. (n even) in [7, pp. 20-23], there is an element $\gamma \in R(K)$ such that $i^*(\gamma) = \sigma$. According to [7, p. 27], γ is uniquely determined by this condition if n is odd. Here, $i^*: R(K) \to R(M)$ is the restriction map induced by the embedding $i: M \hookrightarrow K$, where $R(K)$ and $R(M)$ are the representation rings over $\mathbb{Z}$ of K and M, respectively.

In [7, p. 28], the authors introduced the operators $A_d(\gamma, \sigma)$ and $A_{Y,\chi}(\gamma, \sigma)$. These operators correspond to spaces X_d and Y, respectively. Here, χ is a finite-dimensional unitary representation of Γ and X_d denotes a compact dual space of the symmetric space X.

Case (b): σ is not invariant under the action of the Weyl group W.

In this case n is odd and X is the real hyperbolic space $H\mathbb{R}^n$.

By Proposotion 1.1 in [7, p. 20], there is a unique element $\gamma' \in \widehat{Spin(n)}$ and a splitting $s \otimes \gamma' = \gamma^+ \oplus \gamma^-$, where s is the spin representation of $Spin(n)$ and $\gamma^\pm$ are reperentations of K, such that for the non-trivial element $w \in W$

$$\sigma - w\sigma = sign(v_k)\left(s^+ - s^-\right) i^*(\gamma')$$

and

$$\sigma + w\sigma = i^*\left(\gamma^+ - \gamma^-\right),$$

where v_k is the last coordinate of the highest weight of σ (see, [7, Section 1.1.2]) and $s^\pm$ are the half-spin representations of $Spin(n-1)$.

Define $\gamma = \gamma^+ - \gamma^- \in R(K)$ and $\gamma^s = \gamma^+ - \gamma^- \in R(K)$. Now, we define the operators $A_{Y,\chi}(\gamma, \sigma)$ and $A_{Y,\chi}(\gamma^s, \sigma)$ in the same way as in the case (a). The

operator $A_{Y,\chi}\left(\gamma^{s},\sigma\right)$ corresponds to a Dirac operator. We make the Dirac operator $D_{Y,\chi}(\sigma)$ unique by reasoning exactly as in [7, p. 29].
Being self-adjoint, the operator $D_{Y,\chi}(\sigma)$ satisfies $A_{Y,\chi}\left(\gamma^{s},\sigma\right)=\left|D_{Y,\chi}(\sigma)\right|$.
Let $E_A(.)$ be the family of spectral projections of a normal operator A. Put

$$m_{\chi}(s,\gamma,\sigma)=TrE_{A_{Y,\chi}(\gamma,\sigma)}(\{s\}),$$

$$m_d(s,\gamma,\sigma)=TrE_{A_d(\gamma,\sigma)}(\{s\})$$

and

$$m_{\chi}^{s}(s,\sigma)=Tr\left(E_{D_{Y,\chi}(\sigma)}(\{s\})\right)-Tr\left(E_{D_{Y,\chi}(\sigma)}(\{-s\})\right),$$

for $s\in\mathbb{C}$.

Definition 1. *[7, p. 49, Def. 1.17] Let n be even and $\sigma\in\hat{M}$. Then, $\gamma\in R(K)$ is called σ-admissible if $i^{*}(\gamma)=\sigma$ and $m_d(s,\gamma,\sigma)=P_{\sigma}(s)$ for all $0\le s\in L(\sigma)$.*

Here, $P_{\sigma}(s)$ resp. $L(\sigma)$ denote the polynomial resp. the lattice given by [7, Definition 1.13, p. 47; see also p. 40]. In particular, $L(\sigma)=T(\epsilon_{\sigma}+\mathbb{Z})$, where T and $\epsilon_{\sigma}\in\left\{0,\frac{1}{2}\right\}$ and given by the same definition.
By [7, p. 49, Lemma 1.18], there exists a σ-admissible $\gamma\in R(K)$ for every $\gamma\in\hat{M}$ when n is even.
Moreover, if n is odd, then the unique element in $R(K)$ corresponding to $\sigma\in\hat{M}$ is a priori admissible in some sense (see, [7, p. 54, Prop. 1,22]).

3 Zeta functions

Since $\Gamma\subset G$ is co-compact and torsion-free, there are only two types of conjugacy classes: the class of the identity $e\in\Gamma$ and classes of hyperbolic elements.
Let Γ_{h} resp. $\mathrm{P}\Gamma_{\mathrm{h}}$ denote the set of the Γ-conjugacy classes of hyperbolic resp. primitive hyperbolic elements in Γ.
It is well known that every hyperbolic element $g\in G$ is conjugated to some element $a_g m_g\in A^{+}M$ (see, e.g., [13]-[15]). Following [7, p. 59], we put $l(g)=\left|log\left(a_g\right)\right|$.
For $s\in\mathbb{C}, Re(s)>2\rho$, the Ruelle zeta function is defined by the infinite product (see, [7, p. 96])

$$Z_{R,\chi}(s,\sigma)=\prod_{\gamma_0\in \mathrm{P}\Gamma_{\mathrm{h}}}\det\left(1-\left(\sigma\left(m_{\gamma_0}\right)\otimes\chi\left(\gamma_0\right)\right)e^{-sl(\gamma_0)}\right)^{(-1)^{n-1}},$$

where σ and χ are finite-dimensional unitary representations of M and Γ, respectively.
For $s\in\mathbb{C}, Re(s)>\rho$, the Selberg zeta function is defined by the infinite product (see, [7, p. 97])

$$Z_{S,\chi}(s,\sigma)=$$
$$\prod_{\gamma_0\in \mathrm{P}\Gamma_{\mathrm{h}}}\prod_{k=0}^{+\infty}\det\left(1-\left(\sigma\left(m_{\gamma_0}\right)\otimes\chi(\gamma_0)\otimes S^{k}\left(Ad\left(m_{\gamma_0}a_{\gamma_0}\right)_{\bar{\mathfrak{n}}}\right)\right)e^{-(s+\rho)l(\gamma_0)}\right),$$

where S^{k} is the k-th symmetric power of an endomorphism, $\bar{\mathfrak{n}}=\theta\mathfrak{n}$ and θ is the Cartan involution of $\mathfrak{g}$.
In the case (b) we also define

$$S_{\chi}(s,\sigma)=Z_{S,\chi}(s,\sigma)Z_{S,\chi}(s,w\sigma),$$

the super zeta function

$$S_{\chi}^{s}(s,\sigma)=\frac{Z_{S,\chi}(s,\sigma)}{Z_{S,\chi}(s,w\sigma)}$$

and the super Ruelle zeta function

$$Z_{R,\chi}^{s}(s,\sigma)=\frac{Z_{R,\chi}(s,\sigma)}{Z_{R,\chi}(s,w\sigma)}, \quad (3)$$

where $w\in W$ is non-trivial element.
As known, the Ruelle zeta function can be expressed in terms of Selberg zeta functions (see, e.g., [10]-[12]). By [7, pp. 99-100], there exist sets $I_p=\left\{(\tau,\lambda)\middle|\tau\in\hat{M},\lambda\in\mathbb{R}\right\}$ such that

$$Z_{R,\chi}(s,\sigma)=\prod_{p=0}^{n-1}\prod_{(\tau,\lambda)\in I_p}Z_{S,\chi}(s+\rho-\lambda,\tau\otimes\sigma)^{(-1)^{p}}. \quad (4)$$

Let Λ rep. Υ denote the set of all elements λ resp. τ that appear in (4).
The following theorem holds true (see, [7, p. 113, Th. 3.15]).

Theorem A. *The zeta functions* $Z_{S,\chi}(s,\sigma)$, $S_\chi(s,\sigma)$ *and* $S_\chi^s(s,\sigma)$ *have meromorphic continuation to all of* $\mathbb{C}$.
If n *is even and* γ *is* σ*-admissible, then the singularitities of* $Z_{S,\chi}(s,\sigma)$ *are the following ones:*

- *at* $\pm \mathrm{i}s$ *of order* $m_\chi(s,\gamma,\sigma)$ *if* $s\neq 0$ *is an eigenvalue of* $A_{Y,\chi}(\gamma,\sigma)$,
- *at* $s=0$ *of order* $2m_\chi(0,\gamma,\sigma)$ *if* 0 *is an eigenvalue of* $A_{Y,\chi}(\gamma,\sigma)$,
- *at* $-s$, $s\in T(\mathbb{N}-\epsilon_\sigma)$ *of order* $2\frac{d_Y \dim(\chi)\mathrm{vol}(Y)}{\mathrm{vol}(X_d)} m_d(s,\gamma,\sigma)$. *Then* $s>0$ *is an eigenvalue of* $A_d(\gamma,\sigma)$.

If two such points coincide, then the orders add up.
If n *is odd, then the singulatities of* $Z_{S,\chi}(s,\sigma)$ *(case* (a)*) and of* $S_\chi(s,\sigma)$ *(case(*b*)) are:*

- *at* $\pm \mathrm{i}s$ *of order* $m_\chi(s,\gamma,\sigma)$ *if* $s\neq 0$ *is an eigenvalue of* $A_{Y,\chi}(\gamma,\sigma)$,
- *at* $s=0$ *of order* $2m_\chi(0,\gamma,\sigma)$ *if* 0 *is an eigenvalue of* $A_{Y,\chi}(\gamma,\sigma)$.
-

In case (b) *(*n *odd) the singularities of* $S_\chi^s(s,\sigma)$ *are at* $\mathrm{i}s$ *and have order* $m_\chi^s(s,\sigma)$ *if* $s\in\mathbb{R}$ *is an eigenvalue of* $D_{Y,\chi}(\sigma)$.
For odd n *in case* (b) *the zeta function* $Z_{S,\chi}(s,\sigma)$ *has singularities at* $\mathrm{i}s, \pm s\in \mathrm{spec}\left(A_{Y,\chi}\left(\gamma^s,\sigma\right)\right)$ *of order* $\frac{1}{2}\left(m_\chi(|s|,\gamma,\sigma)+m_\chi^s(s,\sigma)\right)$ if $s\neq 0$ and $m_\chi(0,\gamma,\sigma)$ if $s=0$.

Here, $d_Y=-(-1)^{\frac{n}{2}}$ if n is even and $d=1$ otherwise.
We have proved the following theorem.

Theorem B. *[2, p. 528, Th. 4.1] If* n *is even and* γ *is* σ*-admissible, then there exist entire functions* $Z_1(s), Z_2(s)$ *of order at most* n *such that*

$$Z_{S,\chi}(s,\sigma)=\frac{Z_1(s)}{Z_2(s)},$$

where the zeros of $Z_1(s)$ *correspond to the zeros of* $Z_{S,\chi}(s,\sigma)$ *and the zeros of* $Z_2(s)$ *correspond to the poles of* $Z_{S,\chi}(s,\sigma)$. *The orders of the zeros of* $Z_1(s)$ *resp.* $Z_2(s)$ *equal the orders of the corresponding zeros resp. poles of* $Z_{S,\chi}(s,\sigma)$.

4 Auxiliary results

Theorem 2. *Let* n *be an odd number.*
If $f(s)\in\left\{Z_{S,\chi}(s,\sigma), S_\chi(s,\sigma), S_\chi^s(s,\sigma)\right\}$, *then there exist entire functions* $Z_1(s), Z_2(s)$ *of order at most* n *such that*

$$f(s)=\frac{Z_1(s)}{Z_2(s)},$$

where the zeros of $Z_1(s)$ correspond to the zeros of $f(s)$ and the zeros of $Z_2(s)$ correspond to the poles of $f(s)$. The orders of the zeros of $Z_1(s)$ resp. $Z_2(s)$ equal the orders of the corresponding zeros resp. poles of $f(s)$.

Proof. Denote by S the set of singularities of $f(s)$.
If $f(s)=Z_{S,\chi}(s,\sigma)$ (case (a)) or $f(s)=S_\chi(s,\sigma)$ (case (b)), then, reasoning in the same way as in [2, pp. 529-530], we obtain that

$$\sum_{s\in S\setminus\{0\}} |s|^{-(n+\varepsilon)}=O(1)$$

for any $\varepsilon>0$.
Compared to the even-dimensional case, this case is somewhat simpler. Namely, the singularities that correspond to $A_d(\gamma,\sigma)$ are missing.
Now, proceeding in exactly the same way as in [2, p. 530] (see also, [5, p. 14]), we obtain the claim.
Let $f(s)=S_\chi^s(s,\sigma)$ (case(a)).
Put $n(r)=\#\left\{s\in \mathrm{spec} D_{Y,\chi}(\sigma) \,\middle|\, |s|\leq r\right\}$.
Since $D_{Y,\chi}^2(\sigma)=A_{Y,\chi}^2\left(\gamma^s,\sigma\right)$ and $A_{Y,\chi}^2\left(\gamma^s,\sigma\right)$ is an elliptic operator of the second order, we have the estimate

$$n(r)\sim Cr^n,\ r\to+\infty.$$

Now,

$$\sum_{s\in S\setminus\{0\}} |s|^{-(n+\varepsilon)} = \sum_{\substack{s\in S\setminus\{0\}\\ 0<|s|<1}} |s|^{-(n+\varepsilon)} + \sum_{\substack{s\in S\setminus\{0\}\\ |s|\geq 1}} |s|^{-(n+\varepsilon)} =$$
$$= O(1) + \sum_{\substack{s\in specD_{Y,\chi}(\sigma)\\ |s|\geq 1}} \left| m_{\chi}^{s}(s,\sigma) \right| |s|^{-(n+\varepsilon)}$$
$$= O\left(\int_{1}^{+\infty} t^{-(n+\varepsilon)} dn(t) \right) = O(1)$$

for any $\varepsilon > 0$.
Hence, by the same argumentation as in [2, p. 530], the assertion follows.
Finally, if $f(s) = Z_{S,\chi}(s,\sigma)$ (case(a)), the theorem follows from the fact that $Z_{S,\chi}^{2}(s,\sigma) = S_{\chi}(s,\sigma) S_{\chi}^{s}(s,\sigma)$. This completes the proof. □

Corollary 3. *Let n be an odd number. If $f(s) \in \left\{ Z_{R,\chi}(s,\sigma), Z_{R,\chi}^{s}(s,\sigma) \right\}$, then*

$$f(s) = \frac{Z_1(s)}{Z_2(s)},$$

where $Z_1(s), Z_2(s)$ are entire functions of order at most n over $\mathbb{C}$.

Proof. By (3), it is enough to prove the claim for $f(s) = Z_{R,\chi}(s,\sigma)$. However, if $f(s) = Z_{R,\chi}(s,\sigma)$, then the claim is an immediate consequence of the formula (4) and Theorem 2. □

Lemma 4. *If n is even and γ is σ-admissible, then*

$$P_{\sigma}(w) = \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} w^{n-2k-1},$$

where

$$p_{n-2k-1} = \frac{2T}{\left(\frac{n}{2} - k - 1 \right)!} c_{-\left(\frac{n}{2} - k \right)}, \; k = 0,1,\ldots,\frac{n}{2} - 1,$$

$$c_{-\frac{n}{2}} = \frac{\left(\frac{n}{2} - 1 \right)!}{2T}$$

and the numbers c_k are defined by the asymptotic expression

$$\operatorname{Tr} e^{-tA_d(\gamma,\sigma)^2} \overset{t\to 0}{\sim} \sum_{k=-\frac{n}{2}}^{\infty} c_k t^k .$$

Proof. By [7, pp. 47-48], $P_{\sigma}(0) = 0$, $P_{\sigma}(w) = -P_{\sigma}(w)$ and $P_{\sigma}(w) = w \cdot Q_{\sigma}(w)$, here Q_{σ} is an even polynomial. Hence, P_{σ} is an odd polynomial. Moreover, P_{σ} is a monic polynomial of degree $n-1$ (see, e.g., [6, pp. 17-19], [23, pp. 240-243].
Put

$$P_{\sigma}(w) = \sum_{k=0}^{\frac{n}{2}-1} p_{n-2k-1} w^{n-2k-1}, \; p_{n-1} = 1.$$

By [7, p. 118], $Q_{\sigma}(w) = \sum_{k=0}^{\frac{n}{2}-1} q_{n-2k-2} w^{n-2k-2}$, where $q_{2i} = \frac{2T}{i!} c_{-(i+1)}$, $i = 0,1,\ldots,\frac{n}{2} - 1$. In other words,

$$p_{n-2k-1} = q_{n-2k-2} = \frac{2T}{\left(\frac{n}{2} - k - 1 \right)!} c_{-\left(\frac{n}{2} - k \right)},$$

$k = 0,1,\ldots,\frac{n}{2} - 1$. This completes the proof. □

Lemma 5. *Let n be an odd number. Put $r = 1$ in the case (a) and $r = 2$ in the casae (b). Then,*

$$P_{\sigma}(w) = \sum_{k=0}^{\frac{n-1}{2}} p_{n-2k-1} w^{n-2k-1},$$

where

$$p_{n-2k-1} = \frac{(n-2k) T \operatorname{vol}(X_d)}{r\pi \dim(\chi) \operatorname{vol}(Y)} c_{k-\frac{n}{2}} \Gamma\left(k - \frac{n}{2} \right),$$

$$k = 0,1,\ldots,\frac{n-1}{2},$$

$$c_{-\frac{n}{2}} = \frac{r\pi \dim(\chi) \operatorname{vol}(Y)}{nT \operatorname{vol}(X_d) \Gamma\left(-\frac{n}{2} \right)}$$

and the numbers $c_{k-\frac{n}{2}}$ are defined by the asymptotic expression

$$\operatorname{Tr} e^{-tA_{Y,\chi}(\gamma,\sigma)^2} \overset{t\to 0}{\sim} \sum_{k=0}^{\infty} c_{k-\frac{n}{2}} t^{k-\frac{n}{2}} .$$

Proof. By [7, p. 48], $P_{\sigma}(-w) = P_{\sigma}(w)$. Hence, P_{σ} is an even polynomial. Reasoning as in the proof of Lemma 4, we put

$$P_\sigma(w)=\sum_{k=0}^{\frac{n-1}{2}} p_{n-2k-1}w^{n-2k-1},\ p_{n-1}=1.$$

By [7, p. 125],

$$\frac{r\pi\dim(\chi)\operatorname{vol}(Y)}{T\operatorname{vol}(X_d)}\int_0^w P_\sigma(t)dt=\sum_{k=0}^{\frac{n-1}{2}} c_{k-\frac{n}{2}}\Gamma\left(k-\frac{n}{2}\right)w^{n-2k}.$$

Hence,

$$\frac{r\pi\dim(\chi)\operatorname{vol}(Y)}{T\operatorname{vol}(X_d)}P_\sigma(w)=\sum_{k=0}^{\frac{n-1}{2}}(n-2k)c_{k-\frac{n}{2}}\Gamma\left(k-\frac{n}{2}\right)w^{n-2k-1}.$$

This completes the proof. □

Lemma 6. *Let n be an even number. Suppose that H is a half-plane of the form $\operatorname{Re}(s)<-(2\rho+\varepsilon)$, $\varepsilon>0$, minus the union of a set of congruent disks about the points $-s$, $s\in T(\mathbb{N}-\epsilon_{\tau\otimes\sigma})+\rho-\lambda$, $\lambda\in\Lambda$, $\tau\in\Upsilon$. Then there exists a constant C_R such that*

$$\left|\frac{Z'_{R,\chi}(s,\sigma)}{Z_{R,\chi}(s,\sigma)}\right|\le C_R|s|^{n-1}$$

for $s\in H$.

Proof. The identity (4) implies

$$\frac{Z'_{R,\chi}(s,\sigma)}{Z_{R,\chi}(s,\sigma)}=\sum_{p=0}^{n-1}(-1)^p\sum_{(\tau,\lambda)\in I_p}\frac{Z'_{S,\chi}(s+\rho-\lambda,\tau\otimes\sigma)}{Z_{S,\chi}(s+\rho-\lambda,\tau\otimes\sigma)}.\quad(5)$$

Since n is even, we know that there exists a $\tau\otimes\sigma$-admissible $\gamma_{\tau\otimes\sigma}\in R(K)$ for every $\tau\in\Upsilon$.

Recall Theorem A. Now, it is enough to prove that if K is a half-plane of the form $\operatorname{Re}(s)<-(\rho+\varepsilon)$, $\varepsilon>0$, minus the union of a set of congruent disks about the points $-s$, $s\in T(\mathbb{N}-\epsilon_{\tau\otimes\sigma})$, $\tau\in\Upsilon$, then there exists a constant C_S such that for all $\tau\in\Upsilon$

$$\left|\frac{Z'_{S,\chi}(s,\tau\otimes\sigma)}{Z_{S,\chi}(s,\tau\otimes\sigma)}\right|\le C_S|s|^{n-1}$$

for $s\in K$.

The proof is independent of the choice of τ. We simplify our notation by omitting the latter.

By [7, p. 118, Th. 3.19], $Z_{S,\chi}(s,\sigma)$ has the representation

$$Z_{S,\chi}(s,\sigma)=\det\left(A_{Y,\chi}(\gamma_\sigma,\sigma)^2+s^2\right)\det\left(A_d(\gamma_\sigma,\sigma)+s\right)^{-\frac{2\dim(\chi)\chi(Y)}{\chi(X_d)}}\cdot$$

$$\exp\left(\frac{\dim(\chi)\chi(Y)}{\chi(X_d)}\sum_{m=1}^{\frac{n}{2}}c_{-m}\frac{s^{2m}}{m!}\left(\sum_{r=1}^{m-1}\frac{1}{r}-2\sum_{r=1}^{2m-1}\frac{1}{r}\right)\right).$$

Hence, (see, [7, pp. 120-122])

$$Z_{S,\chi}(-s,\sigma)=Z_{S,\chi}(s,\sigma)\cdot\left(\frac{\det\left(A_d(\gamma_\sigma,\sigma)-s\right)}{\det\left(A_d(\gamma_\sigma,\sigma)+s\right)}\right)^{-\frac{2\dim(\chi)\chi(Y)}{\chi(X_d)}}$$

$$=Z_{S,\chi}(s,\sigma)\cdot\left(\frac{D^+(s)}{D^-(s)}\right)^{-\frac{2\dim(\chi)\chi(Y)}{\chi(X_d)}}\quad(6)$$

$$=Z_{S,\chi}(s,\sigma)\cdot$$

$$\exp\left(-\frac{\pi}{T}\int_0^s P_\sigma(w)\begin{Bmatrix}\tan\left(\frac{\pi w}{T}\right), & \epsilon_\sigma=\frac{1}{2}\\ -\cot\left(\frac{\pi w}{T}\right), & \epsilon_\sigma=0\end{Bmatrix}dw\right)^{-\frac{2\dim(\chi)\chi(Y)}{\chi(X_d)}}$$

$$=Z_{S,\chi}(s,\sigma)\cdot e^{K\int_0^s P_\sigma(w)\begin{Bmatrix}\tan\left(\frac{\pi w}{T}\right), & \epsilon_\sigma=\frac{1}{2}\\ -\cot\left(\frac{\pi w}{T}\right), & \epsilon_\sigma=0\end{Bmatrix}dw},$$

where $K=\dfrac{2\pi\dim(\chi)\chi(Y)}{\chi(X_d)T}$.

Consider the case $\epsilon_\sigma=\frac{1}{2}$. The case $\epsilon_\sigma=0$ is discussed similarly.

The identity (6) implies

$$-\frac{Z'_{S,\chi}(-s,\sigma)}{Z_{S,\chi}(-s,\sigma)}=\frac{Z'_{S,\chi}(s,\sigma)}{Z_{S,\chi}(s,\sigma)}+KP_\sigma(s)\tan\left(\frac{\pi s}{T}\right).$$

Since $\dfrac{Z'_{S,\chi}(s,\sigma)}{Z_{S,\chi}(s,\sigma)}$ is bounded on every half-plane $\operatorname{Re}(s)>\rho+\varepsilon,\ \varepsilon>0$, we conclude that $\dfrac{Z'_{S,\chi}(-s,\sigma)}{Z_{S,\chi}(-s,\sigma)}$ is bounded on K. Morever, $\tan\left(\dfrac{\pi s}{T}\right)$ is bounded on the complement of the union of congruent disks about the points $T\left(k+\frac{1}{2}\right)=T\left(k+\epsilon_{\sigma}\right)$, $k\in\mathbb{Z}$. Hence (see, Lemma 4), the assertion follows. □

Lemma 7. *Let n be an odd number. Suppose that H is a half-plane of the form $\mathrm{Re}(s)<-(2\rho+\varepsilon)$, $\varepsilon>0$. Then there exists a constant C_R such that*

$$\left|\frac{Z'_{R,\chi}(s,\sigma)}{Z_{R,\chi}(s,\sigma)}\right|\le C_R|s|^{n-1}$$

for $s\in H$.

Proof. The identity (5) holds true.

Hence, by Theorem A, it is enough to prove that if K is a half-plane of the form $\mathrm{Re}(s)<-(\rho+\varepsilon)$, $\varepsilon>0$, then there exists a constant C_S such that for all $\tau\in\Upsilon$

$$\left|\frac{Z'_{S,\chi}(s,\tau\otimes\sigma)}{Z_{S,\chi}(s,\tau\otimes\sigma)}\right|\le C_S|s|^{n-1}$$

for $s\in K$.

The proof does not depend on the choice of τ. Hence, we simplify our notation by omitting it.

Suppose that the case (*a*) holds true, i.e., that $\sigma\in\hat{M}$ is Weyl-invariant.

By [7, p. 116, Th. 3.17],

$$Z_{S,\chi}(s,\sigma)=e^{\frac{2\pi\dim(\chi)\mathrm{vol}(Y)}{T\,\mathrm{vol}(X_d)}\int_0^s P_\sigma(w)dw}Z_{S,\chi}(-s,\sigma).$$

Hence,

$$\frac{Z'_{S,\chi}(s,\sigma)}{Z_{S,\chi}(s,\sigma)}=\frac{2\pi\dim(\chi)\mathrm{vol}(Y)}{T\,\mathrm{vol}(X_d)}P_{\sigma(s)}-\frac{Z'_{S,\chi}(-s,\sigma)}{Z_{S,\chi}(-s,\sigma)}.$$

Here, the polyunomial P_σ corresponds to the case (*a*) $(r=1)$ of Lemma 5.

Now, by the same reasoning as in the proof of Lemma 6, the theorem follows.

Suppose that the case (*b*) holds true. Now, $\sigma\in\hat{M}$ is not Weyl-invariant. Therefore, by [7, p. 116, Th. 3.18],

$$Z_{S,\chi}(s,\sigma)=e^{\pi i\eta(D_{Y,\chi}(\sigma))+\frac{2\pi\dim(\chi)\mathrm{vol}(Y)}{T\,\mathrm{vol}(X_d)}\int_0^s P_\sigma(w)dw}Z_{S,\chi}(-s,w\sigma)$$

for non-trivial $w\in W$, where $\eta\left(D_{Y,\chi}(\sigma)\right)$ is the eta invariant of $D_{Y,\chi}(\sigma)$.

We obtain

$$\frac{Z'_{S,\chi}(s,\sigma)}{Z_{S,\chi}(s,\sigma)}=\frac{2\pi\dim(\chi)\mathrm{vol}(Y)}{T\,\mathrm{vol}(X_d)}P_\sigma(s)-\frac{Z'_{S,\chi}(-s,w\sigma)}{Z_{S,\chi}(-s,w\sigma)}.$$

Here, the polynomial P_σ corresponds to the case (*b*) $(r=2)$ of Lemma 5. Namely, this can be easily seen from the derivation of the functional equation (3.21) in [7, p. 117].

Now, reasoning as the previous case, the theorem follows. □

Lemma 8. *Let $c,d\in\mathbb{R}, c<d$. There exists a sequence $\{y_j\}$, $y_j\to+\infty$ as $j\to+\infty$, such that*

$$\frac{Z'_{R,\chi}(x+\mathrm{i}y,\sigma)}{Z_{R,\chi}(x+\mathrm{i}y,\sigma)}=O\left(y_j^{2n}\right)$$

for $x\in(c,d)$.

Proof. Consider the identity (5).

It is enough to prove that there exists a sequence $\{y_j\}$, $y_j\to+\infty$ as $j\to+\infty$, such that for all $\tau\in\Upsilon$

$$\left|\frac{Z'_{S,\chi}(x+\mathrm{i}y,\tau\otimes\sigma)}{Z_{S,\chi}(x+\mathrm{i}y,\tau\otimes\sigma)}\right|=O\left(y_j^{2n}\right)$$

for $x\in(a,b)$, where $a=c-\rho, b=d+\rho$.

We consider the interval I_1 given by $\mathrm{i}t$, $t_0-1<t\le t_0+1$, where $t_0>2\rho$ is fixed.

It suffices to prove that there existy $y\in(t_0-1,t_0+1]$ such that for all $\tau\in\Upsilon$

$$\left|\frac{Z'_{S,\chi}(x+\mathrm{i}y,\tau\otimes\sigma)}{Z_{S,\chi}(x+\mathrm{i}y,\tau\otimes\sigma)}\right|=O\left(y^{2n}\right) \quad (7)$$

for $x\in(a,b)$.

Let S_R be the set of all singularities of all zeta functions $Z_{S,\chi}(s,\tau\otimes\sigma)$, $\tau\in\Upsilon$. Let $N_R(t)$ be the number of elements in S_R on the interval $\mathrm{i}x$, $0<x\le t$.

Let $N(t)$ be the number of singularities of $Z_{S,\chi}(s,\sigma)$ on the same interval. By Theorem A, these singularities (for even n) are given in terms of eigenvalues of $A_{Y,\chi}(\gamma_\sigma,\sigma)$, where $\gamma_\sigma\in R(K)$ is some admissible lift of σ. If n is odd, the singularities in the case (*a*) resp. the case (*b*) are given in terms of eigenvalues of $A_{Y,\chi}(\gamma,\sigma)$ resp.

$A_{\Upsilon,\chi}\left(\gamma^s,\sigma\right)$. Hence, according to [9, p. 89, Th. 9.1.], $N(t)=D_1t^n+O\left(t^{n-1}(\log t)^{-1}\right)$ for some explicitly known constant D_1. However, the O - term does not improve our result. For the sake of simplicity, we take $N(t)=O\left(t^n\right)$. Consequently, $N_R(t)=O\left(t^n\right)$.

It follows immediately that the number of singularities of $Z_{S,\chi}(s,\sigma)$ on I_1 is $O\left(t_0^n\right)$.

Similarly, the number of elements in S_R on I_1 is $O\left(t_0^n\right)$, i.e., it is at most $\lfloor C_1t_0^n\rfloor$ for some constant C_1.

Denote by I_2 the interval $\mathrm{i}t$, $t_0-\frac{3}{4}<t\le t_0+\frac{3}{4}$.

Since $I_2\subset I_1$, the number of elements in S_R on I_2 is at most $\lfloor C_1t_0^n\rfloor$.

Let us divide the interval I_2 into $1+\lfloor C_1t_0^n\rfloor$ equal intervals. By the Dirichlet principle, one of them does not contain any element from S_R. Let $\mathrm{i}y$ by the midpoint of such an interval. We shall prove that y satisfies (7) for $x\in(a,b)$ and all $\tau\in\Upsilon$. The proof does not depend on the choice of $\tau\in\Upsilon$. We simplify our notation by omitting it, i.e., we prove that

$$\frac{Z'_{S,\chi}(x+\mathrm{i}y,\sigma)}{Z_{S,\chi}(x+\mathrm{i}y,\sigma)}=O\left(y^{2n}\right)$$

for $x\in(a,b)$.

By Theorem B and Theorem 2, $Z_1(s)$ and $Z_2(s)$ are entire functions of order at most n. Hence, there are canonical product expressions for $Z_1(s)$ and $Z_2(s)$ of the form (see, e.g., [10, p. 509])

$$Z_i(s)=s^{n_i}e^{g_i(s)}\prod_{\alpha\in R_i\setminus\{0\}}\left(1-\frac{s}{\alpha}\right)\exp\left(\frac{s}{\alpha}+\frac{s^2}{2\alpha^2}+...+\frac{s^n}{n\alpha^n}\right),$$

$i=1,2$, here R_i is the set of zeros of $Z_i(s)$, n_i is the order of the zero of $Z_i(s)$ at $s=0$, $g_i(s)$ is a polynomial of degree at most n.

Therefore,

$$\frac{Z'_{S,\chi}(s,\sigma)}{Z_{S,\chi}(s,\sigma)}=\frac{1}{s}(n_1-n_2)+g'_1(s)-g'_2(s)$$
$$+\sum_{i=1,2}(-1)^{i-1}\sum_{\alpha\in R_i\setminus\{0\}}\left(\frac{s}{\alpha}\right)^n\frac{1}{s-\alpha}.$$

We have

$$|\mathrm{i}y-\alpha|\ge\frac{1}{2}\cdot\frac{\frac{3}{2}}{1+\lfloor C_1t_0^n\rfloor}\ge\frac{3}{4}\cdot\frac{1}{1+C_1t_0^n}>\frac{3}{4}\cdot\frac{1}{1+C_1\left(y+\frac{3}{4}\right)^n}\ge\frac{C_2}{y^n}$$

for some constant C_2 and all $\alpha\in R_i$, $i=1,2$.

Now, for a small fixed $\varepsilon>0$ and the choice $s_x=x+\mathrm{i}y$, $x\in(a,b)$, we have

$$\frac{Z'_{S,\chi}(s_x,\sigma)}{Z_{S,\chi}(s_x,\sigma)}=\frac{1}{s_x}(n_1-n_2)+g'_1(s_x)-g'_2(s_x)$$
$$+\sum_{k=1}^{8}\sum_{\beta\in A_k}\left(\frac{s_x}{\beta}\right)^n\frac{1}{s_x-\beta},$$

where β denotes a singularity of $Z_{S,\chi}(s,\sigma)$ and

$$A_1=\left\{\beta\,\middle|\,\beta<0,|\beta|>\rho+\varepsilon\right\},$$
$$A_2=\left\{\beta\,\middle|\,0<|\beta|\le\rho+\varepsilon\right\},$$
$$A_3=\left\{\beta\,\middle|\,\beta=\mathrm{i}t,\rho+\varepsilon<t\le t_0-1\right\},$$
$$A_4=\left\{\beta\,\middle|\,\beta\in I_1\right\},$$
$$A_5=\left\{\beta\,\middle|\,\beta=\mathrm{i}t,t>t_0+1\right\},$$
$$A_6=\left\{\beta\,\middle|\,\beta=\mathrm{i}t,\rho+\varepsilon<t\le t_0-1\right\},$$
$$A_7=\left\{\beta\,\middle|\,-\beta\in I_1\right\},$$
$$A_8=\left\{\beta\,\middle|\,\beta=\mathrm{i}t,t>t_0+1\right\}.$$

Note that $A_1=\emptyset$ for odd n.

Since $\sum_{\beta\in A_1\neq\emptyset}\frac{1}{|\beta|^n}$ converges and $|s_x-\beta|\ge y$ for $\beta\in A_1\neq\emptyset$, we get

$$\sum_{\beta\in A_1\neq\emptyset}\left(\frac{s_x}{\beta}\right)^n\frac{1}{s_x-\beta}=O\left(y^n\sum_{\beta\in A_1\neq\emptyset}\frac{1}{|\beta|^n}\frac{1}{|s_x-\beta|}\right)=O\left(y^{n-1}\right).$$

Furthermore, A_2 is a finite set. Hence,

$$\sum_{\beta\in A_2}\left(\frac{s_x}{\beta}\right)^n\frac{1}{s_x-\beta}=O\left(y^n\sum_{\beta\in A_2}\frac{1}{|\beta|^n}\frac{1}{|s_x-\beta|}\right)=O\left(y^{n-1}\right)$$

since $|s_x-\beta|\ge y-\rho-\varepsilon>C_3y$ for some constant C_3 and all $\beta\in A_2$.

Similarly, $|s_x-\beta|\ge y-t_0+1>\frac{1}{4}$ and $|\beta|>\rho+\varepsilon$ for $\beta\in A_3$. Hence,

$$\sum_{\beta\in A_3}\left(\frac{s_x}{\beta}\right)^n\frac{1}{s_x-\beta}=O\left(y^n\sum_{\beta\in A_3}\frac{1}{|\beta|^n}\frac{1}{|s_x-\beta|}\right)=O\left(y^n\sum_{\beta\in A_3}1\right)$$
$$=O\left(y^n(t_0-1)^n\right)=O\left(y^{2n}\right).$$

If $\beta\in A_4$, then $|s_x-\beta|\geq|\mathrm{i}y-\beta|>\frac{C_2}{y^n}$ and $|\beta|>y-\frac{7}{4}>C_4y$ for some constant C_4. Therefore,

$$\sum_{\beta\in A_4}\left(\frac{s_x}{\beta}\right)^n\frac{1}{s_x-\beta}=O\left(y^n\sum_{\beta\in A_4}\frac{1}{|\beta|^n}\frac{1}{|s_x-\beta|}\right)=O\left(y^n\sum_{\beta\in A_4}1\right)$$
$$=O\left(y^nt_0^n\right)=O\left(y^n\left(y+\frac{3}{4}\right)^n\right)=O\left(y^{2n}\right).$$

Similarly, $|s_x-\beta|\geq t-y>C_5t$ for some constant C_5 and $\beta=\mathrm{i}t\in A_5$. One has

$$\sum_{\beta\in A_5}\left(\frac{s_x}{\beta}\right)^n\frac{1}{s_x-\beta}=O\left(y^n\sum_{\beta\in A_5}\frac{1}{|\beta|^n}\frac{1}{|s_x-\beta|}\right)$$
$$=O\left(y^n\int_{t_0+1}^{+\infty}\frac{1}{t^{n+1}}dN(t)\right)=O\left(y^n\int_{t_0+1}^{+\infty}t^{-2}dt\right)$$
$$=O\left(y^n(t_0+1)^{-1}\right)=O\left(y^{n-1}\right).$$

If $\beta\in A_6$, then $|s_x-\beta|>y+\rho+\varepsilon>y$ and $|\beta|>\rho+\varepsilon$. Hence,

$$\sum_{\beta\in A_6}\left(\frac{s_x}{\beta}\right)^n\frac{1}{s_x-\beta}=O\left(y^n\sum_{\beta\in A_6}\frac{1}{|\beta|^n}\frac{1}{|s_x-\beta|}\right)$$
$$=O\left(y^{n-1}\sum_{\beta\in A_6}1\right)=O\left(y^{2n-1}\right).$$

Similarly, $|s_x-\beta|>y+t_0-1>y$ and $|\beta|>t_0-1>y-\frac{7}{4}>C_4y$ for $\beta\in A_7$. We have

$$\sum_{\beta\in A_7}\left(\frac{s_x}{\beta}\right)^n\frac{1}{s_x-\beta}=O\left(y^n\sum_{\beta\in A_7}\frac{1}{|\beta|^n}\frac{1}{|s_x-\beta|}\right)$$
$$=O\left(y^{-1}\sum_{\beta\in A_7}1\right)=O\left(y^{n-1}\right).$$

If $\beta\in A_8$, then $|s_x-\beta|\geq y+t>t$ for $\beta=-\mathrm{i}t\in A_8$. Therefore,

$$\sum_{\beta\in A_8}\left(\frac{s_x}{\beta}\right)^n\frac{1}{s_x-\beta}=O\left(y^n\sum_{\beta\in A_8}\frac{1}{|\beta|^n}\frac{1}{|s_x-\beta|}\right)$$
$$=O\left(y^n\int_{t_0+1}^{+\infty}\frac{1}{t^{n+1}}dN(t)\right)=O\left(y^{n-1}\right).$$

Finally, $\frac{1}{s_x}(n_1-n_2)=O\left(y^{-1}\right)$ and $g_1'(s_x)-g_2'(s_x)=O\left(y^{n-1}\right)$.

We obtain

$$\frac{Z'_{S,\chi}(s_x,\sigma)}{Z_{S,\chi}(s_x,\sigma)}=O\left(y^{2n}\right).$$

This completes the proof. □

5 Main result

Theorem 9. *Let Y be a compact, n-dimensional, locally symmetric Riemannian manifold with strictly negative sectional curvature. Then,*

$$\pi_\Gamma(x)=\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\mathrm{li}\left(x^{s^{p,\tau,\lambda}}\right)$$
$$+O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}(\log)^{-1}\right)$$

as $x\to+\infty$, where $s^{p,\tau,\lambda}$ is a singularity of the Selberg zeta function $Z_S(s+\rho-\lambda,\tau)$.

Proof. Fix some finite-dimensional unitary representations $\chi\in\hat{\Gamma}$ and $\sigma\in\hat{M}$.

We simplify our notation by omitting χ and σ in the sequel.

For $g\in\Gamma$, let $n_\Gamma(g)=\#\left(\Gamma_g/\langle g\rangle\right)$, where Γ_g is the centralizer of g in Γ and $\langle g\rangle$ is the group generated by g.

If $\gamma\in\Gamma_{\mathrm{h}}$ then $\gamma=\gamma_0^{n_\Gamma(\gamma)}$ for some $\gamma_0\in P\Gamma_{\mathrm{h}}$.

For $\gamma\in\Gamma_{\mathrm{h}}$ we introduce $\Lambda_0(\gamma)=\Lambda_0\left(\gamma_0^{n_\Gamma(\gamma)}\right)=\log N(\gamma_0)$.

By [7, pp. 96-97, (3.4)],

$$\frac{Z'_R(s)}{Z_R(s)}=(-1)^{n+1}\sum_{\gamma\in\Gamma_{\mathrm{h}}}\Lambda_0(\gamma)N(\gamma)^{-s},\mathrm{Re}(s)>2\rho. \quad (8)$$

We define

$$\psi_j(x)=\int_0^x \psi_{j-1}(t)\,dt,\ j=1,2,\ldots,$$

where

$$\psi_0(x)=\sum_{\gamma\in\Gamma_h, N(\gamma)\le x}\Lambda_0(\gamma).$$

Let $k\ge 2n$ be an integer and $x>1$, $c>2\rho$.
By [19, p. 31, Th. B.] and (8)

$$\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{Z_R'(s)}{Z_R(s)}\frac{x^s}{s(s+1)\ldots(s+k)}ds$$

$$=(-1)^{n+1}\sum_{\gamma\in\Gamma_h}\Lambda_0(\gamma)\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left(\frac{x}{N(\gamma)}\right)^s\frac{ds}{s(s+1)\ldots(s+k)}$$

$$=(-1)^{n+1}\sum_{\gamma\in\Gamma_h,\frac{x}{N(\gamma)}\ge 1}\Lambda_0(\gamma)\frac{1}{k!}\left(1-\frac{1}{\frac{x}{N(\gamma)}}\right)^k$$

$$=(-1)^{n+1}\frac{1}{k!}\sum_{\gamma\in\Gamma_h,N(\gamma)\le x}\Lambda_0(\gamma)\left(1-\frac{N(\gamma)}{x}\right)^k.$$

On the other hand, by [19, p. 18, Th. A.]

$$\psi_k(x)=\frac{1}{k!}\sum_{\gamma\in\Gamma_h,N(\gamma)\le x}\Lambda_0(x-N(\gamma))^k.$$

Hence,

$$\psi_k(x)=\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}(-1)^{n+1}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{s(s+1)\ldots(s+k)}ds.$$

Assume that $c'\ll -2\rho$ is not a pole of the integrand of $\psi_k(x)$. Note that the identity (4) and Theorem A yield that no $c'<-k$ is a pole of the integrand of $\psi_k(x)$ if n is odd.
By Lemma 6 resp. Lemma 7, if n is even resp. odd $\frac{Z_R'(s)}{Z_R(s)}=O\left(|s|^{n-1}\right)$ on the line $\mathrm{Re}(s)=c'$. Furthermore, by Lemma 8, there exists a sequence $\{y_j\}$, $y_j\to+\infty$ as $j\to+\infty$, such that

$$\frac{Z_R'(t+\mathrm{i}y_j)}{Z_R(t+\mathrm{i}y_j)}=O\left(y_j^{2n}\right)$$

for $t\in[c',c]$.
Fix some $y_j\gg 1$.

By construction of $\{y_j\}$, we know that no pole of $\frac{Z_R'(s)}{Z_R(s)}$ occurs on the line $\mathrm{Im}(s)=y_j$.
Applying the Cauchy residue theorem to the integrand of $\psi_k(x)$ over the rectangle $R(c',y_j)$ given by vertices $c-\mathrm{i}y_j$, $c+\mathrm{i}y_j$, $c'+\mathrm{i}y_j$, $c'-\mathrm{i}y_j$, we obtain

$$\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}y_j}^{c+\mathrm{i}y_j}(-1)^{n+1}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{s(s+1)\ldots(s+k)}ds$$

$$=\sum_{z\in R(c',y_j)}\mathrm{Res}_{s=z}\left((-1)^{n+1}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{s(s+1)\ldots(s+k)}\right) \qquad (9)$$

$$+\frac{1}{2\pi \mathrm{i}}\int_{c'-\mathrm{i}}^{c'+\mathrm{i}}+\frac{1}{2\pi \mathrm{i}}\int_{c'-\mathrm{i}y_j}^{c'-\mathrm{i}}+\frac{1}{2\pi \mathrm{i}}\int_{c'+\mathrm{i}}^{c'+\mathrm{i}y_j}+\frac{1}{2\pi \mathrm{i}}\int_{c'+\mathrm{i}y_j}^{c+\mathrm{i}y_j}+\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}y_j}^{c'-\mathrm{i}y_j}.$$

We have

$$\frac{1}{2\pi \mathrm{i}}\int_{c'-\mathrm{i}}^{c'+\mathrm{i}}\left((-1)^{n+1}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{s(s+1)\ldots(s+k)}\right)ds$$

$$=O\left(x^{c'+k}\int_{c'-\mathrm{i}}^{c'+\mathrm{i}}|ds|\right)=O\left(x^{c'+k}\int_{-1}^{1}dv\right)=O\left(x^{c'+k}\right),$$

$$\frac{1}{2\pi \mathrm{i}}\int_{c'+\mathrm{i}}^{c'+\mathrm{i}y_j}\left((-1)^{n+1}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{s(s+1)\ldots(s+k)}\right)ds$$

$$=O\left(x^{c'+k}\int_{c'+\mathrm{i}}^{c'+\mathrm{i}y_j}\frac{|ds|}{|s|^{k-n-2}}\right)=O\left(x^{c'+k}\int_1^{y_j}\frac{dv}{v^{k-n+2}}\right)=O\left(x^{c'+k}\right),$$

$$\frac{1}{2\pi \mathrm{i}}\int_{c'+\mathrm{i}y_j}^{c+\mathrm{i}y_j}\left((-1)^{n+1}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{s(s+1)\ldots(s+k)}\right)ds=O\left(\frac{x^{c+k}}{y_j^{k+1-2n}}\right).$$

Symilarly,

$$\frac{1}{2\pi \mathrm{i}}\int_{c'-\mathrm{i}y_j}^{c'-\mathrm{i}}\left((-1)^{n+1}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{s(s+1)\ldots(s+k)}\right)ds=O\left(x^{c'+k}\right),$$

$$\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}y_j}^{c'-\mathrm{i}y_j}\left((-1)^{n+1}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{s(s+1)\ldots(s+k)}\right)ds=O\left(\frac{x^{c+k}}{y_j^{k+1-2n}}\right).$$

Hence, by (9) and (5)

$$\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}y_j}^{c+\mathrm{i}y_j}\left((-1)^{n+1}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{s(s+1)...(s+k)}\right)ds$$
$$=\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in R(c',y_j)}c_z(p,\tau,\lambda,k) \qquad (10)$$
$$+O\left(x^{c'+k}\right)+O\left(\frac{x^{c+k}}{y_j^{k+1-2n}}\right),$$

where

$$c_z(p,\tau,\lambda,k)=\mathrm{Res}_{s=z}\left(\frac{Z_S'(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)}\frac{x^{s+k}}{s(s+1)...(s+k)}\right).$$

Letting $j\to+\infty, c'\to-\infty$ in (10), we get

$$\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left((-1)^{n+1}\frac{Z_R'(s)}{Z_R(s)}\frac{x^{s+k}}{s(s+1)...(s+k)}\right)ds$$
$$=\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in A_k^{p,\tau,\lambda}}c_z(p,\tau,\lambda,k),$$

i.e.,

$$\psi_k(x)=\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in A_k^{p,\tau,\lambda}}c_z(p,\tau,\lambda,k), \quad (11)$$

where $A_k^{p,\tau,\lambda}$ denotes the set of poles of

$$\frac{Z_S'(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)}\frac{x^{s+k}}{s(s+1)...(s+k)}.$$

Take $k=2n$. By (11),

$$\psi_{2n}(x)=\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in A_{2n}^{p,\tau,\lambda}}c_z(p,\tau,\lambda,2n)$$
$$=\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in A^{p,\tau,\lambda}}c_z(p,\tau,\lambda), \qquad (12)$$

where, for the sake of simplicity, we denote by $A^{p,\tau,\lambda}$ the set of poles of $\frac{Z_S'(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)}\frac{x^{s+2n}}{s(s+1)...(s+2n)}$ and by $c_z(p,\tau,\lambda)$ the residue at $s=z$.

$\frac{Z_S'(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)}\frac{x^{s+2n}}{s(s+1)...(s+2n)}$ corresponds to some $(\tau,\lambda)\in I_p$ for some $p\in\{0,1,2,...,n-1\}$.

By Theorem A, the singularities of $Z_S(s+\rho-\lambda,\tau)$, for even n are: at $\pm\mathrm{i}s-\rho+\lambda$ of order $m(s,\gamma_\tau,\tau)$ if $s\neq0$ is an eigenvalue of $A_Y(\gamma_\tau,\tau)$, at $-\rho+\lambda$ of order $2m(0,\gamma_\tau,\tau)$ if 0 is an eigenvalue of $A_Y(\gamma_\tau,\tau)$, at $-s-\rho+\lambda$, $s\in T(\mathbb{N}-\epsilon_\tau)$ of order $-2(-1)^{\frac{n}{2}}\frac{\mathrm{vol}(Y)}{\mathrm{vol}(X_d)}m_d(s,\gamma_\tau,\tau)$ (in this case $s>0$ is an eigenvalue of $A_d(\gamma_\tau,\tau)$). Here, γ_τ is some τ - admissible element in $R(K)$.

Note that the singularities of $Z_S(s+\rho-\lambda,\tau)$ at $-s-\rho+\lambda$, $s\in T(\mathbb{N}-\epsilon_\tau)$ are all less than $-\rho+\lambda$. Furthermore, the singularities of $Z_S(s+\rho-\lambda,\tau)$ than correspond to $A_Y(\gamma_\tau,\tau)$ are contained in the union of the interval $[-2\rho+\lambda,\lambda]$ with the line $-\rho+\lambda+\mathrm{i}\mathbb{R}$. An overlap between these two kinds of singularities may occur inside $[-2\rho+\lambda,-\rho+\lambda)$ (see, [7, pp. 114-115]).

If n is odd (case (*a*)), the singularities of $Z_S(s+\rho-\lambda,\tau)$ are: at $\pm\mathrm{i}s-\rho+\lambda$ of order $m(s,\gamma,\tau)$ if $s\neq0$ is an eigenvalue of $A_Y(\gamma,\tau)$, at $-\rho+\lambda$ of order $2m(0,\gamma,\tau)$ if 0 is an eigenvalue of $A_Y(\gamma,\tau)$.

If n is odd (case (*b*)), the singularities of $Z_S(s+\rho-\lambda,\tau)$ are at $\mathrm{i}s-\rho+\lambda, \pm s\in spec\left(A_Y\left(\gamma^s,\tau\right)\right)$ of order $\frac{1}{2}m(|s|,\gamma,\tau)+\frac{1}{2}m^s(s,\tau)$ if $s\neq0$ and $m(0,\gamma,\tau)$ if $s=0$.

Therefore, if n is odd, the singularities of $Z_S(s+\rho-\lambda,\tau)$ are contained in the union of the interval $[-2\rho+\lambda,\lambda]$ with the line $-\rho+\lambda+\mathrm{i}\mathbb{R}$.

The integers $0,-1,...,-2n$ are simple poles of $\frac{x^{s+2n}}{s(s+1)...(s+2n)}$. These integers may also appear as simple poles of $\frac{Z_S'(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)}$, i.e., as singularities of $Z_S(s+\rho-\lambda,\tau)$. Denote by $I_{p,\tau,\lambda}$ the set of such integers. Put $I'_{p,\tau,\lambda}$ to be the difference $\{0,-1,...,-2n\}\setminus I_{p,\tau,\lambda}$. The set of the remaining singular-rities $s^{p,\tau,\lambda}$ of $Z_S(s+\rho-\lambda,\tau)$ will be denoted by $S^{p,\tau,\lambda}$.

Reasoning as in [17, pp. 88-89], we write

$$\frac{Z_S'(s+\rho-\lambda,\tau)}{Z_S(s+\rho-\lambda,\tau)}=\frac{o_z^{p,\tau,\lambda}}{s-z}\left(1+\sum_{i=1}^{+\infty}a_{i,z}^{p,\tau,\lambda}(s-z)^i\right),$$

where z is a singularity of $Z_S(s+\rho-\lambda,\tau)$ and $o_z^{p,\tau,\lambda}$ is the order of z.

Now, for $s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}$,

$$
\begin{aligned}
&c_{s^{p,\tau,\lambda}}(p,\tau,\lambda)\\
&=\lim_{s\to s^{p,\tau,\lambda}}\left(s-s^{p,\tau,\lambda}\right)\frac{Z_s'(s+\rho-\lambda,\tau)}{Z_s(s+\rho-\lambda,\tau)}\frac{x^{s+2n}}{s(s+1)\ldots(s+2n)}\\
&=\lim_{s\to s^{p,\tau,\lambda}}\left(s-s^{p,\tau,\lambda}\right)\frac{o^{p,\tau,\lambda}_{s^{p,\tau,\lambda}}}{s-s^{p,\tau,\lambda}}\cdot\\
&\left(1+\sum_{i=1}^{+\infty}a^{p,\tau,\lambda}_{i,s^{p,\tau,\lambda}}\left(s-s^{p,\tau,\lambda}\right)^i\right)\frac{x^{s+2n}}{s(s+1)\ldots(s+2n)}\\
&=o^{p,\tau,\lambda}_{s^{p,\tau,\lambda}}\frac{x^{s^{p,\tau,\lambda}+2n}}{s^{p,\tau,\lambda}\left(s^{p,\tau,\lambda}+1\right)\ldots\left(s^{p,\tau,\lambda}+2n\right)}.
\end{aligned}
\tag{13}
$$

Let $-j\in I_{p,\tau,\lambda}$. We have

$$c_{-j}(p,\tau,\lambda)=\lim_{s\to -j}\frac{d}{ds}\left((s+j)^2\frac{Z_s'(s+\rho-\lambda,\tau)}{Z_s(s+\rho-\lambda,\tau)}\frac{x^{s+2n}}{s(s+1)\ldots(s+2n)}\right).$$

Since

$$
\begin{aligned}
&(s+j)^2\frac{Z_s'(s+\rho-\lambda,\tau)}{Z_s(s+\rho-\lambda,\tau)}\frac{x^{s+2n}}{s(s+1)\ldots(s+2n)}\\
&=o^{p,\tau,\lambda}_{-j}\left(1+\sum_{i=1}^{+\infty}a^{p,\tau,\lambda}_{i,-j}(s+j)^i\right)\frac{x^{s+2n}}{\prod_{\substack{l=0\\ l\neq j}}^{2n}(s+l)}\\
&=o^{p,\tau,\lambda}_{-j}\frac{x^{s+2n}}{\prod_{\substack{l=0\\ l\neq j}}^{2n}(s+l)}+o^{p,\tau,\lambda}_{-j}a^{p,\tau,\lambda}_{1,-j}(s+j)\frac{x^{s+2n}}{\prod_{\substack{l=0\\ l\neq j}}^{2n}(s+l)}+\ldots
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{d}{ds}\left((s+j)^2\frac{Z_s'(s+\rho-\lambda,\tau)}{Z_s(s+\rho-\lambda,\tau)}\frac{x^{s+2n}}{s(s+1)\ldots(s+2n)}\right)\\
&=\frac{o^{p,\tau,\lambda}_{-j}}{\prod_{\substack{l=0\\ l\neq j}}^{2n}(s+l)}x^{s+2n}\log x-\frac{o^{p,\tau,\lambda}_{-j}}{\prod_{\substack{l=0\\ l\neq j}}^{2n}(s+l)}\sum_{\substack{l=0\\ l\neq j}}^{2n}\frac{1}{s+l}x^{s+2n}\\
&+\frac{o^{p,\tau,\lambda}_{-j}}{\prod_{\substack{l=0\\ l\neq j}}^{2n}(s+l)}a^{p,\tau,\lambda}_{1,-j}x^{s+2n}+o^{p,\tau,\lambda}_{-j}a^{p,\tau,\lambda}_{1,-j}(s+j)\frac{d}{ds}\left(\frac{x^{s+2n}}{\prod_{\substack{l=0\\ l\neq j}}^{2n}(s+l)}\right)+\ldots,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
c_{-j}(p,\tau,\lambda)&=\frac{o^{p,\tau,\lambda}_{-j}}{\prod_{\substack{l=0\\ l\neq j}}^{2n}(-j+l)}x^{-j+2n}\log x\\
&+\frac{o^{p,\tau,\lambda}_{-j}}{\prod_{\substack{l=0\\ l\neq j}}^{2n}(-j+l)}\left(-\sum_{\substack{l=0\\ l\neq j}}^{2n}\frac{1}{-j+l}+a^{p,\tau,\lambda}_{1,-j}\right)x^{-j+2n}.
\end{aligned}
\tag{14}
$$

Finally, let $-j\in I'_{p,\tau,\lambda}$. Now,

$$
\begin{aligned}
&c_{-j}(p,\tau,\lambda)\\
&=\lim_{s\to -j}\left((s+j)\frac{Z_s'(s+\rho-\lambda,\tau)}{Z_s(s+\rho-\lambda,\tau)}\frac{x^{s+2n}}{s(s+1)\ldots(s+2n)}\right)\\
&=\frac{Z_s'(-j+\rho-\lambda,\tau)}{Z_s(-j+\rho-\lambda,\tau)}\frac{x^{-j+2n}}{\prod_{\substack{l=0\\ l\neq j}}^{2n}(-j+l)}.
\end{aligned}
\tag{15}
$$

We denote:

$I_{-2n}=\{0,-1,..,-2n\}$,

$B_{p,\tau,\lambda}=\left\{-j\in I_{-2n}\,\middle|\,c_{-j}(p,\tau,\lambda)=O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right)\right\}$,

$B'_{p,\tau,\lambda}=I_{-2n}\backslash B_{p,\tau,\lambda}$,

$S^{p,\tau,\lambda}_{\mathbb{R}}=S^{p,\tau,\lambda}\cap\mathbb{R}$,

$S^{p,\tau,\lambda}_{-\rho+\lambda}=S^{p,\tau,\lambda}\backslash S^{p,\tau,\lambda}_{\mathbb{R}}$,

$C^1_{p,\tau,\lambda}=\left\{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{\mathbb{R}}\,\middle|\,s^{p,\tau,\lambda}\leq -2n-1\right\}$,

$C^2_{p,\tau,\lambda}=\left\{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{\mathbb{R}}\,\middle|\,-2n-1<s^{p,\tau,\lambda}\leq -2n+2\rho\frac{n+\rho-1}{n+2\rho-1}\right\}$,

$C^3_{p,\tau,\lambda}=\left\{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{\mathbb{R}}\,\middle|\,-2n+2\rho\frac{n+\rho-1}{n+2\rho-1}<s^{p,\tau,\lambda}\leq 2\rho\frac{n+\rho-1}{n+2\rho-1}\right\}$,

$C^4_{p,\tau,\lambda}=\left\{s^{p,\tau,\lambda}\in S^{p,\tau,\lambda}_{\mathbb{R}}\,\middle|\,2\rho\frac{n+\rho-1}{n+2\rho-1}<s^{p,\tau,\lambda}\leq 2\rho\right\}$.

Note that $C^k_{p,\tau,\lambda}=\emptyset$ for $k\in\{1,2\}$ when n is odd. Now, we can write

$$
\begin{aligned}
&\sum_{z\in A^{p,\tau,\lambda}}c_z(p,\tau,\lambda)\\
&=\sum_{z\in B_{p,\tau,\lambda}}c_z(p,\tau,\lambda)+\sum_{z\in B'_{p,\tau,\lambda}}c_z(p,\tau,\lambda)\\
&+\sum_{k=1}^{4}\sum_{z\in C^k_{p,\tau,\lambda}}c_z(p,\tau,\lambda)+\sum_{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}}c_z(p,\tau,\lambda).
\end{aligned}
\tag{16}
$$

Cosider the sum over $C^1_{p,\tau,\lambda}\neq\emptyset$ in (16).
Since $C^1_{p,\tau,\lambda}\subset S^{p,\tau,\lambda}_{\mathbb{R}}\subset S^{p,\tau,\lambda}$ and $z\leq-2n-1<-2\rho+\lambda$ for $z\in C^1_{p,\tau,\lambda}$, it follows from (13) than

$$\sum_{z\in C^1_{p,\tau,\lambda}\neq\emptyset} c_z(p,\tau,\lambda)$$
$$=\sum_{z\in C^1_{p,\tau,\lambda}\neq\emptyset} o_z^{p,\tau,\lambda}\frac{x^{z+2n}}{z(z+1)...(z+2n)}$$
$$=-2(-1)^{\frac{n}{2}}\frac{\mathrm{vol}(Y)}{\mathrm{vol}(X_d)}\sum_{k\geq\frac{1}{T}(2n+1-\rho+\lambda)+\epsilon_\tau} m_d\left(T(k-\epsilon_\tau),\gamma_\tau,\tau\right)$$
$$\cdot\frac{x^{-T(k-\epsilon_\tau)-\rho+\lambda+2n}}{\prod_{l=0}^{2n}\left(-T(k-\epsilon_\tau)-\rho+\lambda+l\right)}.$$

The fact that γ_τ is τ- admissible element yields $m_d(s,\gamma_\tau,\tau)=P_\tau(s)$ for all $0\leq s\in L(\tau)=T(\epsilon_\tau+\mathbb{Z})$. In particular, $m_d\left(T(k-\epsilon_\tau),\gamma_\tau,\tau\right)=P_\tau\left(T(k-\epsilon_\tau)\right)$ for $k\geq\frac{1}{T}(2n+1-\rho+\lambda)+\epsilon_\tau$. We obtain

$$\sum_{z\in C^1_{p,\tau,\lambda}\neq\emptyset} c_z(p,\tau,\lambda)$$
$$=O\left(x^{-1}\sum_{k\geq\frac{1}{T}(2n+1-\rho+\lambda)+\epsilon_\tau}\frac{\left|P_\tau\left(T(k-\epsilon_\tau)\right)\right|}{\left(T(k-\epsilon_\tau)+\rho-\lambda-2n\right)^{2n+1}}\right)$$
$$=O\left(x^{-1}\sum_{k\geq\frac{1}{T}(2n+1-\rho+\lambda)+\epsilon_\tau}\frac{\left(2n+1-\rho+\lambda+T\epsilon_\tau\right)^{2n+1}\left|P_\tau\left(T(k-\epsilon_\tau)\right)\right|}{T^{2n+1}k^{2n+1}}\right).$$

Hence, by Lemma 4,

$$\sum_{z\in C^1_{p,\tau,\lambda}} c_z(p,\tau,\lambda)=O\left(x^{-1}\sum_{k\geq\frac{1}{T}(2n+1-\rho+\lambda)+\epsilon_\tau}\frac{1}{k^{n+2}}\right)=O\left(x^{-1}\right). \tag{17}$$

The sum over $B_{p,\tau,\lambda}$ in (16) is a finite one. Therefore, by the definition of $B_{p,\tau,\lambda}$,

$$\sum_{z\in B_{p,\tau,\lambda}} c_z(p,\tau,\lambda)=O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right). \tag{18}$$

The sum over $C^2_{p,\tau,\lambda}\neq\emptyset$ is a finite one as well. Hence, by (13),

$$\sum_{z\in C^2_{p,\tau,\lambda}\neq\emptyset} c_z(p,\tau,\lambda)=\sum_{z\in C^2_{p,\tau,\lambda}\neq\emptyset} o_z^{p,\tau,\lambda}\frac{x^{z+2n}}{z(z+1)...(z+2n)}=O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right). \tag{19}$$

Combining (12) and (16)-(19), we obtain

$$\begin{aligned}\psi_{2n}(x)&=\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in B'_{p,\tau,\lambda}} c_z(p,\tau,\lambda)\\&+\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in C^3_{p,\tau,\lambda}} c_z(p,\tau,\lambda)\\&+\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in C^4_{p,\tau,\lambda}} c_z(p,\tau,\lambda)\\&+\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}} c_z(p,\tau,\lambda)\\&+O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right).\end{aligned} \tag{20}$$

Suppose $1<h\leq\frac{x}{2}$.
We introduce the operator

$$\Delta^+_{2n}f(x)=\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}f\left(x+(2n-i)h\right). \tag{21}$$

If f is at least $2n$ times differentiable function, then

$$\Delta^+_{2n}f(x)=\int_x^{x+h}\int_{t_{2n}}^{t_{2n}+h}...\int_{t_2}^{t_2+h}f^{(2n)}(t_1)dt_1...dt_{2n}. \tag{22}$$

The mean value theorem applied to (22) yields

$$\Delta^+_{2n}f(x)=h^{2n}f^{(2n)}(\tilde{x}), \tag{23}$$

where $\tilde{x}\in[x,x+2nh]$.
Since ψ_0 is nondecreasing, we obtain

$$\psi_0(x)\leq h^{-2n}\Delta^+_{2n}\psi_{2n}(x)\leq\psi_0(x+2nh). \tag{24}$$

Now, (20), (21) and the fact that $h\le\frac{x}{2}$, imply

$$\begin{aligned}&h^{-2n}\Delta_{2n}^{+}\psi_{2n}(x)\\&=\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in B'_{p,\tau,\lambda}}h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)\\&+\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in C^3_{p,\tau,\lambda}}h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)\\&+\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in C^4_{p,\tau,\lambda}}h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)\\&+\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{z\in S^{p,\tau,\lambda}_{-\rho+\lambda}}h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)\\&+O\left(h^{-2n}x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right).\end{aligned}\qquad(25)$$

Consider the sum over $B'_{p,\tau,\lambda}$ on the riht hand side of (25).
Let $z\in B'_{p,\tau,\lambda}, z=0$.
Suppose that $0\in I_{p,\tau,\lambda}$. Then, (14), (23) and the facts: $\left(x^n\log x\right)^{(n)}=n!\log x+n!\sum_{l=1}^{n}\frac{1}{l}$, $\left(x^n\right)^{(n)}=n!$, yield

$$h^{-2n}\Delta_{2n}^{+}c_0(p,\tau,\lambda)=o_0^{p,\tau,\lambda}\log\tilde{x}_{p,\tau,\lambda,0}+o_0^{p,\tau,\lambda}a_{1,0}^{p,\tau,\lambda},\qquad(26)$$

where $\tilde{x}_{p,\tau,\lambda,0}\in[x,x+2nh]$.
If $0\in I'_{p,\tau,\lambda}$, then

$$h^{-2n}\Delta_{2n}^{+}c_0(p,\tau,\lambda)=\frac{Z'_S(\rho-\lambda,\tau)}{Z_S(\rho-\lambda,\tau)}\qquad(27)$$

by (15). Let $z\in B'_{p,\tau,\lambda}, z=-j\le -1$.
Suppose that $-j\in I_{p,\tau,\lambda}$.
Since $\left(x^k\log x\right)^{(n)}=k!(-1)^{n-k-1}\frac{(n-k-1)!}{x^{n-k}}$ and $\left(x^k\right)^{(n)}=0$ for $0\le k<n, k\in\mathbb{N}$, we get

$$h^{-2n}\Delta_{2n}^{+}c_{-j}(p,\tau,\lambda)=o_{-j}^{p,\tau,\lambda}\frac{\tilde{x}_{p,\tau,\lambda,-j}^{-j}}{-j},\qquad(28)$$

where $\tilde{x}_{p,\tau,\lambda,-j}\in[x,x+2nh]$.
If $-j\in I_{p,\tau,\lambda}$, then

$$h^{-2n}\Delta_{2n}^{+}c_{-j}(p,\tau,\lambda)=0.\qquad(29)$$

Now, (26)-(29) and the fact that $h\le\frac{x}{2}$, imply

$$\sum_{z\in B'_{p,\tau,\lambda}}h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)=O(\log x).\qquad(30)$$

Consider the sum over $C^3_{p,\tau,\lambda}$ on the right hand side of (25). Let $z\in C^3_{p,\tau,\lambda}$.
By (13) and (23)

$$\begin{aligned}\left|h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)\right|&=\left|o_z^{p,\tau,\lambda}\frac{\tilde{x}_{p,\tau,\lambda,z}}{z}\right|\\&=\frac{\left|o_z^{p,\tau,\lambda}\right|}{|z|}\tilde{x}^z_{p,\tau,\lambda,z}\le\frac{\left|o_z^{p,\tau,\lambda}\right|}{|z|}\tilde{x}_{p,\tau,\lambda,z}^{2\rho\frac{n+\rho-1}{n+2\rho-1}},\end{aligned}$$

where $\tilde{x}_{p,\tau,\lambda,z}\in[x,x+2nh]$. Hence, $h\le\frac{x}{2}$ and the fact that $C^3_{p,\tau,\lambda}$ is a finite set, yield

$$\sum_{z\in C^3_{p,\tau,\lambda}}h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)=O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right).\qquad(31)$$

Similarly, the sum over $C^4_{p,\tau,\lambda}$ on the right hand side of (25) is a finite one. We have

$$h^{-2n}\Delta_{2n}^{+}c_{s^{p,\tau,\lambda}}(p,\tau,\lambda)=o_{S^{p,\tau,\lambda}}^{p,\tau,\lambda}\frac{\tilde{x}_{s^{p,\tau,\lambda}}^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}}$$

for $s^{p,\tau,\lambda}\in C^4_{p,\tau,\lambda}$, where $\tilde{x}_{s^{p,\tau,\lambda}}\in[x,x+2nh]$. Hence, reasoning as in [21, p. 246] and [20, p. 101], we obtain

$$\begin{aligned}&\sum_{z\in C^4_{p,\tau,\lambda}}h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)\\&=\sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}}+O\left(h^{2\rho}\right),\end{aligned}\qquad(32)$$

where $s^{p,\tau,\lambda}$ is counted $o^{p,\tau,\lambda}_{s^{p,\tau,\lambda}}$ times in the last sum.
Finally, we estimate the sum over $S^{p,\tau,\lambda}_{-\rho+\lambda}$ in (25). Let $z\in S^{p,\tau,\lambda}_{-\rho+\lambda}$. By (13),

$$c_z(p,\tau,\lambda)=o_z^{p,\tau,\lambda}\frac{x^{z+2n}}{z(z+1)\dots(z+2n)}.$$

We derive two estimates for $h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)$. Firstly, by (21),

$$h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda) = h^{-2n}\frac{o_z^{p,\tau,\lambda}}{z(z+1)\dots(z+2n)}\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}\left(x+(2n-i)h\right)^{z+2n}.$$

Since $h\le\frac{x}{2}$, we obtain

$$h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)=O\left(h^{-2n}|z|^{-2n-1}x^{-\rho+\lambda+2n}\right). \quad (33)$$

Secondly, by (22)

$$\left|h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)\right| = \left|h^{-2n}\frac{o_z^{p,\tau,\lambda}}{z}\int_x^{x+h}\int_{t_{2n}}^{t_{2n}+h}\dots\int_{t_2}^{t_2+h}t_1^z\,dt_1\dots dt_{2n}\right| \le h^{-2n}\left|o_z^{p,\tau,\lambda}\right||z|^{-1}\int_x^{x+h}\int_{t_{2n}}^{t_{2n}+h}\dots\int_{t_2}^{t_2+h}t_1^{-\rho+\lambda}dt_1\dots dt_{2n}.$$

Hence, by the mean value theorem and the fact that $h\le\frac{x}{2}$,

$$h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)=O\left(|z|^{-1}x^{-\rho+\lambda}\right). \quad (34)$$

Let $M>2\rho$. Now, using (33) and (34), we deduce

$$\sum_{z\in S_{-\rho+\lambda}^{p,\tau,\lambda}}h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda) = \sum_{\substack{z\in S_{-\rho+\lambda}^{p,\tau,\lambda}\\ |-\rho+\lambda|<|z|\le M}}h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda)+\sum_{\substack{z\in S_{-\rho+\lambda}^{p,\tau,\lambda}\\ |z|>M}}h^{-2n}\Delta_{2n}^{+}c_z(p,\tau,\lambda) \quad (35)$$

$$=O\left(x^{-\rho+\lambda}\sum_{\substack{z\in S_{-\rho+\lambda}^{p,\tau,\lambda}\\ |-\rho+\lambda|<|z|\le M}}|z|^{-1}\right)+O\left(h^{-2n}x^{-\rho+\lambda+2n}\sum_{\substack{z\in S_{-\rho+\lambda}^{p,\tau,\lambda}\\ |z|>M}}|z|^{-2n-1}\right)$$

$$=O\left(x^{-\rho+\lambda}\int_{|-\rho+\lambda|}^{M}t^{-1}dN_{p,\tau,\lambda}(t)\right)+O\left(h^{-2n}x^{-\rho+\lambda+2n}\int_M^{+\infty}t^{-2n-1}dN_{p,\tau,\lambda}(t)\right)$$
$$=O\left(x^{-\rho+\lambda}M^{n-1}\right)+O\left(h^{-2n}x^{-\rho+\lambda+2n}M^{-n-1}\right),$$

where $N_{p,\tau,\lambda}(t)=O(t^n)$ denotes the number of singularities of $Z_S(s+\rho-\lambda,\tau)$ on the interval $-\rho+\lambda+\mathrm{i}x$, $0<x\le t$.

Combining (25), (30)-(32) and (35), we obtain

$$h^{-2n}\Delta_{2n}^{+}\psi_{2n}(x) = \sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}} +O\left(h^{2\rho}\right)+O\left(x^{\rho}M^{n-1}\right) +O\left(h^{-2n}x^{\rho+2n}M^{-n-1}\right)+O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right). \quad (36)$$

Substituting $h=x^{\frac{n+\rho-1}{n+2\rho-1}}$, $M=x^{\frac{\rho}{n+2\rho-1}}$ into (36) and taking into account (24), we get

$$\psi_0(x)\le\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}} +O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right). \quad (37)$$

Analogously, (see, e.g. [20, pp. 101-102]), one proves

$$\psi_0(x)\ge\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}} +O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right). \quad (38)$$

Combining (37) and (38), we conclude that

$$\psi_0(x)=\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}\frac{x^{s^{p,\tau,\lambda}}}{s^{p,\tau,\lambda}} +O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}\right). \quad (39)$$

Now, using (39) and following lines of [20, p. 102], we finally obtain

$$\pi_\Gamma(x)=\sum_{p=0}^{n-1}(-1)^{p+n+1}\sum_{(\tau,\lambda)\in I_p}\sum_{s^{p,\tau,\lambda}\in\left(2\rho\frac{n+\rho-1}{n+2\rho-1},2\rho\right]}li\left(x^{s^{p,\tau,\lambda}}\right) +O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}(\log x)^{-1}\right)$$

as $x\to+\infty$. This completes the proof. □

6 Functional equations

Theorem 10. *Let n be even. If γ is σ - admissibel, then there exists a bounded function $f(t)$ such that as $|t|\to\infty$,*

$$Z_{S,\chi}(\sigma_1+\mathrm{i}t,\sigma)=f(t)e^{g(t)}Z_{S,\chi}(-\sigma_1-\mathrm{i}t,\sigma),$$

where

$$g(t)=-\sum_{k=0}^{\frac{n}{2}-1}p_{n-2k-1}\frac{K}{n-2k}\sum_{l=1}^{\frac{n}{2}-k}\binom{n-2k}{2l-1}(-1)^l\sigma_1^{n-2k-2k+1}|t|^{2l-1}.$$

Proof. Reasoning as in the proof of Lemma 6, we obtain

$$Z_{S,\chi}(s,\sigma)=e^{-K\int_0^s P_\sigma(w)\left\{\begin{matrix}\tan\left(\frac{\pi w}{T}\right), & \epsilon_\sigma=\frac{1}{2}\\ -\cot\left(\frac{\pi w}{T}\right), & \epsilon_\sigma=0\end{matrix}\right\}dw}\cdot Z_{S,\chi}(-s,\sigma). \tag{40}$$

Suppose that $\epsilon_\sigma=\frac{1}{2}$.

It is known that as $|t|\to\infty$,

$$\tan\pi(\sigma_1+\mathrm{i}t)=\mathrm{i}\frac{t}{|t|}+O\left(e^{-2\pi|t|}\right).$$

This equation and Lemma 4 yield that at points on a vertical line away from the real axis one has

$$\begin{aligned}
&-K\int_0^s P_\sigma(w)\tan\left(\frac{\pi w}{T}\right)dw\\
&=-K\sum_{k=0}^{\frac{n}{2}-1}p_{n-2k-1}\int_0^s w^{n-2k-1}\tan\left(\frac{\pi w}{T}\right)dw\\
&=-K\sum_{k=0}^{\frac{n}{2}-1}p_{n-2k-1}\left(\frac{\sigma_1}{t}+\mathrm{i}\right)^{n-2k}\frac{t}{|t|}\mathrm{i}\frac{t^{n-2k}}{n-2k}\\
&-K\sum_{k=0}^{\frac{n}{2}-1}p_{n-2k-1}\left(\frac{\sigma_1}{t}+\mathrm{i}\right)^{n-2k}\int_0^t y^{n-2k-1}O\left(e^{-2\pi\frac{|y|}{T}}\right)dy\\
&=-\sum_{k=0}^{\frac{n}{2}-1}p_{n-2k-1}\frac{t}{|t|}\frac{K\mathrm{i}}{n-2k}\sum_{l=0}^{n-2k}\binom{n-2k}{l}\sigma_1^{n-k-l}t^l\mathrm{i}^l+O(1)\\
&=\sum_{k=0}^{\frac{n}{2}-1}p_{n-2k-1}\frac{t}{|t|}\frac{K\mathrm{i}}{n-2k}\sum_{l=0}^{n-2k}\binom{n-2k}{2l}\sigma_1^{n-2k-2l}t^{2l}\mathrm{i}^{2l}\\
&-\sum_{k=0}^{\frac{n}{2}-1}p_{n-2k-1}\frac{t}{|t|}\frac{K\mathrm{i}}{n-2k}\sum_{l=1}^{\frac{n}{2}-k}\binom{n-2k}{2l-1}\sigma_1^{n-2k-2l+1}t^{2l-1}\mathrm{i}^{2l-1}+O(1)\\
&=-\sum_{k=0}^{\frac{n}{2}-1}p_{n-2k-1}\frac{t}{|t|}\frac{K\mathrm{i}}{n-2k}\sum_{l=0}^{\frac{n}{2}-k}\binom{n-2k}{2l}(-1)^l\sigma_1^{n-2k-2l}t^{2l}\\
&-\sum_{k=0}^{\frac{n}{2}-1}p_{n-2k-1}\frac{K\mathrm{i}}{n-2k}\sum_{l=1}^{\frac{n}{2}-k}\binom{n-2k}{2l-1}(-1)^l\sigma_1^{n-2k-2l+1}|t|^{2l-1}+O(1),
\end{aligned} \tag{41}$$

where we are assuming that the integration is carried out along the line segment joining the origin to s.

Now, suppose that $\epsilon_\tau=0$.

It is not hard to verify that as $|t|\to\infty$,

$$\cot\pi(\sigma_1+\mathrm{i}t)=-\mathrm{i}\frac{t}{|t|}+O\left(e^{-2\pi|t|}\right).$$

Hence, reasoning as in the previous case, one obtains that at points on a vertical line away from the real axis

$$\begin{aligned}
&-K\int_0^s P_\sigma(w)\left(-\cot\left(\frac{\pi w}{T}\right)\right)dw\\
&=K\sum_{k=0}^{\frac{n}{2}-1}p_{n-2k-1}\int_0^s w^{n-2k-1}\cot\left(\frac{\pi w}{T}\right)dw\\
&=-K\sum_{k=0}^{\frac{n}{2}-1}p_{n-2k-1}\left(\frac{\sigma_1}{t}+\mathrm{i}\right)^{n-2k}\frac{t}{|t|}\mathrm{i}\frac{t^{n-2k}}{n-2k}\\
&+K\sum_{k=0}^{\frac{n}{2}-1}p_{n-2k-1}\left(\frac{\sigma_1}{t}+\mathrm{i}\right)^{n-2k}\int_0^t y^{n-2k-1}O\left(e^{-2\pi\frac{|y|}{T}}\right)dy\\
&=-\sum_{k=0}^{\frac{n}{2}-1}p_{n-2k-1}\frac{t}{|t|}\mathrm{i}\frac{K\mathrm{i}}{n-2k}\sum_{l=0}^{\frac{n}{2}-k}\binom{n-2k}{2l}(-1)^l\sigma_1^{n-2k-2l}t^{2l}\\
&-\sum_{k=0}^{\frac{n}{2}-1}p_{n-2k-1}\frac{K}{n-2k}\sum_{l=1}^{\frac{n}{2}-k}\binom{n-2k}{2l-1}(-1)^l\sigma_1^{n-2k-2l+1}|t|^{2l-1}+O(1).
\end{aligned} \tag{42}$$

Combininh (40) - (42), the assertion follows. □

Theorem 11. *Let n be odd. If $f(s)=Z_{S,\chi}(s,\sigma)$ (case (a)) or $f(s)=S_\chi(s,\sigma)$ (case (b)), then*

$$f(\sigma_1+\mathrm{i}t)=g(t)e^{h(t)}f(-\sigma_1-\mathrm{i}t),$$

where

$$g(t)=e^{2\mathrm{i}\sum_{k=0}^{\frac{n-1}{2}} c_{k-\frac{n}{2}}\Gamma\left(k-\frac{n}{2}\right)\sum_{l=1}^{\frac{n+1}{2}-k}\binom{n-2k}{2l-1}(-1)^{l-1}\sigma_1^{n-2k-2l+1}t^{2l-1}},$$

$$h(t)=2\sum_{k=0}^{\frac{n-1}{2}} c_{k-\frac{n}{2}}\Gamma\left(k-\frac{n}{2}\right)\sum_{l=0}^{\frac{n-1}{2}-k}\binom{n-2k}{2l}(-1)^{l}\sigma_1^{n-2k-2l}|t|^{2l}.$$

Proof. Let $r=1$ in the case (*a*) and $r=2$ in the case (*b*).
By [7, p. 116, Th. 3.17],

$$f(s)=e^{2\frac{r\pi\dim(\chi)\mathrm{vol}(Y)}{T\,\mathrm{vol}(X_d)}\int_0^s P_\sigma(w)dw} f(-s).$$

Reduce the identity

$$\frac{r\pi\dim(\chi)\mathrm{vol}(Y)}{T\,\mathrm{vol}(X_d)}\int_0^s P_\sigma(w)dw=\sum_{k=0}^{\frac{n-1}{2}} c_{k-\frac{n}{2}}\Gamma\left(k-\frac{n}{2}\right)s^{n-2k}$$

applied in the proof of Lemma 5.
Since

$$\sum_{k=0}^{\frac{n-1}{2}} c_{k-\frac{n}{2}}\Gamma\left(k-\frac{n}{2}\right)s^{n-2k}$$
$$=\sum_{k=0}^{\frac{n-1}{2}} c_{k-\frac{n}{2}}\Gamma\left(k-\frac{n}{2}\right)(\sigma_1+\mathrm{i}t)^{n-2k}$$
$$=\sum_{k=0}^{\frac{n-1}{2}} c_{k-\frac{n}{2}}\Gamma\left(k-\frac{n}{2}\right)\sum_{l=0}^{n-2k}\binom{n-2k}{l}\sigma_1^{n-2k-l}\mathrm{i}^l t^l$$
$$=\sum_{k=0}^{\frac{n-1}{2}} c_{k-\frac{n}{2}}\Gamma\left(k-\frac{n}{2}\right)\sum_{l=0}^{\frac{n-1}{2}-k}\binom{n-2k}{2l}\sigma_1^{n-2k-2l}\mathrm{i}^{2l}t^{2l}$$
$$+\sum_{k=0}^{\frac{n-1}{2}} c_{k-\frac{n}{2}}\Gamma\left(k-\frac{n}{2}\right)\sum_{l=1}^{\frac{n+1}{2}-k}\binom{n-2k}{2l-1}\sigma_1^{n-2k-2l+1}\mathrm{i}^{2l-1}t^{2l-1}$$
$$=\sum_{k=0}^{\frac{n-1}{2}} c_{k-\frac{n}{2}}\Gamma\left(k-\frac{n}{2}\right)\sum_{l=0}^{\frac{n-1}{2}-k}\binom{n-2k}{2l}(-1)^l\sigma_1^{n-2k-2l}|t|^{2l}$$
$$+\mathrm{i}\sum_{k=0}^{\frac{n-1}{2}} c_{k-\frac{n}{2}}\Gamma\left(k-\frac{n}{2}\right)\sum_{l=1}^{\frac{n+1}{2}-k}\binom{n-2k}{2l-1}(-1)^{l-1}\sigma_1^{n-2k-2l+1}t^{2l-1},$$

the theorem follows. □

Theorem 12. *Let n be odd. In the case* (*b*), *Selberg zeta function $Z_{S,\chi}(s,\sigma)$ satisfies the functional equation*

$$Z_{S,\chi}(\sigma_1+\mathrm{i}t,\sigma)=g(t)e^{h(t)}Z_{S,\chi}(-\sigma_1-\mathrm{i}t,w\sigma),$$

where $w\in W$ is non-trivial and

$$g(t)=e^{\pi\mathrm{i}\eta(D_{Y,\chi}(\sigma))+\mathrm{i}\sum_{k=0}^{\frac{n-1}{2}} c_{k-\frac{n}{2}}\Gamma\left(k-\frac{n}{2}\right)\sum_{l=1}^{\frac{n+1}{2}-k}\binom{n-2k}{2l-1}(-1)^{l-1}\sigma_1^{n-2k-2l+1}t^{2l-1}},$$

$$h(t)=\sum_{k=0}^{\frac{n-1}{2}} c_{k-\frac{n}{2}}\Gamma\left(k-\frac{n}{2}\right)\sum_{l=0}^{\frac{n-1}{2}-k}\binom{n-2k}{2l}(-1)^{l}\sigma_1^{n-2k-2l}|t|^{2l}.$$

Proof. Recall the functional equation

$$Z_{S,\chi}(s,\sigma)=e^{\pi\mathrm{i}\eta(D_{Y,\chi}(\sigma))+\frac{2\pi\dim(\chi)\mathrm{vol}(Y)}{T\,\mathrm{vol}(X_d)}\int_0^s P_\sigma(w)dw}Z_{S,\chi}(-s,w\sigma)$$

applied in the proof of Lemma 7.
Now, following lines of the proof of Theorem 11, the assertion follows. □

7 Concluding remarks

Let us summarize the aspects in which Theorem 9 represents an improvement of (1).
As already mentioned in the Introduction, X is one of the following spaces:

$$H\mathbb{R}^k\,(k\geq 2),\,H\mathbb{C}^m\,(m\geq 1),H\mathbb{H}^l\,(l\geq 1),\,H\mathbb{C}a^2.$$

Hence, $n=k,2m,4l,16$ and $\rho=\frac{1}{2}(k-1),m,2l+1,11$, respectively.
Since $H\mathbb{C}^1\cong H\mathbb{R}^2$ and $H\mathbb{H}^1\cong H\mathbb{R}^4$ (see, e.g., [16]), we may assume $m\geq 2$ and $l\geq 2$.
Now, $\alpha=n+q-1=k-1,2m,4l+2,22$ respectively.
Obviously, $\alpha=2\rho$.

The size of the error term in (1) is $O\left(x^{\left(1-\frac{1}{2n}\right)2\rho}\right)$. We compare this bound to our bound $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}(\log x)^{-1}\right)$.

The factor $(\log x)^{-1}$ gives to our bound some advantage.
However, let us have a look at the corresponding powers of x.
The inequality

$$2\rho\frac{n+\rho-1}{n+2\rho-1}\leq\left(1-\frac{1}{2n}\right)2\rho$$

always holds true since the corresponding equivalent inequality $(n-1)(2\rho-1)\geq 0$ is always valid. Here, the equality sign occurs only if $X=H\mathbb{R}^2$.
Furthermore, the inequalities

$$2\rho\frac{n+\rho-1}{n+2\rho-1}\leq\frac{3}{2}\rho\leq\left(1-\frac{1}{2n}\right)2\rho$$

are always true.
Indeed, the left-hand inequality is valid, being equivalent to the inequality $n\leq 2\rho+1$. The equality occurs only if $X=H\mathbb{R}^k, k\geq 2$.
On the other side, the right-hand inequality holds also true since it reduces to $n-2\geq 0$. Clearly, the right-hand inequality becomes equality only if $X=H\mathbb{R}^2$.
Summarizing results derived above, we end up with the conclusion that the obtained bound $O\left(x^{2\rho\frac{n+\rho-1}{n+2\rho-1}}(\log x)^{-1}\right)$ is of the form $O\left(x^{\theta}(\log x)^{-1}\right)$, where $\theta<\frac{3}{2}\rho$ if $X=H\mathbb{C}^m$, $(m\geq 2)$, $H\mathbb{H}^l,(l\geq 2)$ $H\mathbb{C}a^2$ and $\theta=\frac{3}{2}\rho$ if $X=H\mathbb{R}^k$, $k\geq 2$.
Note that our result coincides with the best known results for the compact Riemann surfaces [21] and the real hyperbolic manifolds with cusps [1].
Also, note that taking $k>2n$ in the proof of Theorem 9 does not yield a better result.

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