Applications of Homotopy Perturbation Method for Nonlinear Partial Differential Equations

DURMUŞ DAĞHAN
Nigde University
Department of Mathematics
51100, Nigde
TURKEY
durmusdaghan@nigde.edu.tr

H. YAVUZ MART
Nigde University
Department of Mathematics
51100, Nigde
TURKEY
halilyavuzmart@hotmail.com

GULDEM YILDIZ
Nigde University
Department of Mathematics
51100, Nigde
TURKEY
guldem.yildiz@nigde.edu.tr

Abstract: Homotopy perturbation method is simply applicable to the different non-linear partial differential equations. In this paper, Drinfeld-Sokolov and Modified Benjamin-Bona-Mahony equations are is studied perturbatively by using homotopy perturbation method.

Key-Words Homotopy Perturbation Method- Drinfeld-Sokolov equation- Modified Benjamin Bona-Mahony equation

1 Introduction

The Drinfeld-Sokolov (DS) system was first introduced by Drinfeld and Sokolov and it is a system of nonlinear partial differential equations owner of the Lax pairs of a special form [1]. The physical motivation of this system was explained in detail by Ref.[2]. Generalized form of the DS system has been studied by different authors using the various methods [3, 4, 5, 6, 7, 8, 9, 10]. In order to find solutions of DS system, we use homotopy perturbation method. To do that, we write the DS system in the following form [9],

$$u_t + (v^2)_x = 0, \tag{1}$$
$$v_t - av_{xxx} + 3bu_x v + 3kuv_x = 0, \tag{2}$$

where a, b and k are arbitrary constants.

For diverse physical systems generally require non-linear differential equations. One of the these types of equations is Benjamin- Bona- Mahony equation (BBM) which is an alternative model for the Korteweg– de Vries equation (KdV) written by Benjamin et al. in Ref. [11] . These type of equations are known as the regularized long-wave equations, and given in the following form [11]:

$$u_t + u_x + uu_x - u_{xxt} = 0. \tag{3}$$

Some of the modified versions of the Benjamin-Bona- Mahony (MBBM) equation given in Eq.3 have been investigated by many authors [12, 13, 14, 15, 16, 17]. MBBM equation can also be implemented for the solution of different physical systems, such as acoustic-gravity waves in compressible fluids, acoustic waves in enharmonic crystals, the hydromagnetic waves in cold plasma [18], [19], etc. The existence and uniqueness of the solution of initial value problems for the MBBM equation have been considered in Ref.[20]. In this paper we use following form [14]:

$$u_t + \alpha u_x + \beta u^2 u_x - \gamma u_{xxt} = 0, \tag{4}$$

where α, β and γ are arbitrary real constants. He used an effective and coincide method, (G'/G)-expansion method [21], to obtain the exact solution of the Eq.1.3. Some applications of the (G'/G)-expansion method can be seen in Ref.[22, 23].

In this paper, our aim is to present the perturbative solutions of the DS and MBBM equations by using the Homotopy Perturbation Method (HPM) [24]. To do that, the paper covers the following sections: in Section 2 we have presented HPM. We have presented application for DS system of equations and MBBM equation in Section 3. Finally, Section 4 is devoted to the conclusion of the study.

2 Homotopy Perturbation Method

For the utility of the reader, we will introduce HPM [24]. The following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{5}$$

with boundary conditions

$$B(u) - \frac{\partial u}{\partial n} = 0, \qquad r \in \Gamma,$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω. The operator A can be divided into two parts L and N, where L is linear, and N is nonlinear. Accordingly Eq. 5 can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0.$$

He constructed a homotopy $\nu(r,p) : \Omega \times [0,1] \to R$ in paper [24]. ν satisfies

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{6}$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq. 5. Expressly we have

$$\begin{aligned} H(v,0) &= L(v) - L(u_0) = 0, \\ H(v,1) &= A(v) - f(r) = 0 \end{aligned}$$

where $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic in topology. According to the Homotopy Perturbation technique, we can be written the solution Eq 6 as a power series in p small parameter:

$$v = v_0 + pv_1 + p^2 v_2 + ...$$

Setting $p = 1$ results in the approximate solution of Eq. 5

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ...$$

3 Applications of the Homotopy Perturbation Method

3.1 Application for of DS equation

System of PDEs given in Eq.1 can be converted into following system of ODEs by using transformation of $\eta = x - \beta t$, $u = U(\eta)$, $v = V(\eta)$,

$$-\beta U' + (V^2)' = 0, \tag{7}$$
$$\beta V' + aV''' - 3bU'V - 3kUV' = 0, \tag{8}$$

where $U' = \frac{dU}{d\eta}$ and $V' = \frac{dV}{d\eta}$. By integrating the first equation given in Eq.7, we get

$$U = \frac{1}{\beta}(V^2 + c), \tag{9}$$

where c is an arbitrary integration constant. We substitute Eq.9 in the second part of the Eq.7 and then integrate the obtained equation, and have found

$$V'' + \frac{(\beta^2 - 3ck)}{a\beta}V - \frac{(2b+k)}{a\beta}V^3 + \frac{e}{a\beta} = 0, \tag{10}$$

where e is an arbitrary integration constant.

The First Approximation:

It is well known that the homotopy perturbation method in [24, 25, 26, 27], we use for Eq.10 this method. Eq.10 was also solved by using the HPM with different initial condition in Ref. [28]. Our aim is to find bifurcation point for same equation with different initial condition. A homotopy is constructed in Eq.10

$$V'' + \frac{(\beta^2 - 3ck)}{a\beta}V - \frac{(2b+k)}{a\beta}\, p\, V^3 + \frac{e}{a\beta} = 0, \tag{11}$$

where $p \in [0,1]$. In Eq.11 there are two observations to be considered; the first observation is that for $p = 0$, Eq.11 becomes a linear equation. The second observation is that for $p = 1$, it becomes the original nonlinear one. Due to the homotopy perturbation method, the solution of Eq.11 can be expressed in a series of p :

$$V = V_0 + p\, V_1 \tag{12}$$

Besides, the coefficient of the linear term and the constant also expend in a series of p [26, 29, 10], that is

$$\frac{(\beta^2-3ck)}{a\beta}=w^2+p\,w_1, \qquad (13)$$
$$\frac{e}{a\beta}=p\,c_1. \qquad (14)$$

Substituting Eqs. 12 and 13 into Eq. 11 and equating coefficients of like powers of p constructs a series of linear equations:

$$V_0''+w^2V_0=0, \qquad (15)$$

$$V_1''+w^2V_1+w_1V_0-\frac{(2b+k)}{a\beta}V_0^3+c_1=0. \qquad (16)$$

We apply the initial condition $V_0(0)=A$ and $V_0'(0)=0$ in the Eq.15, we get

$$V_0=A\cos w\eta. \qquad (17)$$

Substituting 17 into 16 obtain result

$$V_1''+w^2V_1+\left(w_1-\frac{3A^2(2b+k)}{4a\beta}\right)A\cos w\eta$$
$$-\frac{A^3(2b+k)}{4a\beta}\cos 3w\eta+c_1=0. \qquad (18)$$

If $w_1-\frac{3A^2(2b+k)}{4a\beta}\neq 0$ in the Eq.18 , every solution of Eq.18 will comprise a secular term. So we necessitate that

$$w_1=\frac{3A^2(2b+k)}{4a\beta}, \qquad (19)$$
$$c_1=0 \qquad (20)$$

and eliminate the $\cos w\eta$ term from Eq.18 completely. Thus Eq.18 becomes

$$V_1''+w^2V_1-\frac{A^3(2b+k)}{4a\beta}\cos 3w\eta=0. \qquad (21)$$

The solution of Eq.21 with the initial conditions $V_1(0)=0$ and $V_1'(0)=0$

$$V_1=-\frac{A^3(2b+k)}{32a\beta w^2}\left(\cos 3w\eta-\cos w\eta\right). \qquad (22)$$

Then the first-order approximate solution is enough, if putting $p=1$ in Eqs.12 and 13, we get

$$V=A\cos w\eta-\frac{A^3(2b+k)}{32a\beta w^2}\left(\cos 3w\eta-\cos w\eta\right), \qquad (23)$$

and

$$w^2=\frac{(\beta^2-3ck)}{a\beta}-\frac{3A^2(2b+k)}{4a\beta}, \qquad (24)$$
$$e=0. \qquad (25)$$

With the same logic in Ref. [10], we know that $w^2\geq 0$, the Eq.24 has no solution when

$$c>\frac{1}{3k}\beta^2-\frac{A^2}{4k}(2b+k).$$

Nevertheless, when

$$c<\frac{1}{3k}\beta^2-\frac{A^2}{4k}(2b+k),$$

Eq.24 has the solution

$$w=\sqrt{\frac{(\beta^2-3ck)}{a\beta}-\frac{3A^2(2b+k)}{4a\beta}}. \qquad (26)$$

Thus, bifurcation occurs at

$$c=\frac{1}{3k}\beta^2-\frac{A^2}{4k}(2b+k).$$

The Second Approximation:

According to the homotopy perturbation method, we assume that the solution of Eq.11 can be expressed in a series of p :

$$V=V_0+p\,V_1+p^2V_2 \qquad (27)$$

Furthermore, the coefficient of the linear term and the constant also expend in a series of p [26, 29], that is

$$\frac{(\beta^2-3ck)}{a\beta}=w^2+pw_1+p^2w_2, \qquad (28)$$
$$\frac{e}{a\beta}=pc_1+p^2c_2. \qquad (29)$$

Therefore, in the conditions $V_0(0)=A$, $V_0'(0)=0$ and $V_i(0)=0$, $V_i'(0)=0$ for $i=1,2$, using 27, 28 and repeating the process in the first approximation we get

$$V=A\cos w\eta-\frac{A^3(2b+k)}{32a\beta w^2}\left(\cos 3w\eta-\cos w\eta\right)$$

$$+\frac{A^5}{2^{10}\alpha^2\beta^2 w^4}\left(\cos 5w\eta - \cos w\eta\right)$$

and

$$w^2 = \frac{(\beta^2-3ck)}{a\beta} - \frac{3A^2(2b+k)}{4a\beta} - \frac{3A^4}{2^6}\frac{(2b+k)^2}{\alpha^2\beta^2 w^2} \quad (30)$$

$$e = 0. \quad (31)$$

Rewrite Eq.30 in the form

$$w = \mp\frac{\sqrt{\alpha\beta(8(\beta^2-3ck)-6A^2(2b+k)\mp q}}{4\alpha\beta} \quad (32)$$

where q is $\sqrt{6A^2(2b+k)(2-A^2)-16(\beta^2-3ck)}$.

Therefore, the bifurcation occurs at

$$c = \frac{1}{3k}(\beta^2 - \frac{1}{8}(-1+6A^2(2b+k)) \mp s$$

where s is

$$\sqrt{-1+6A^4(2b+k)+36A^4(2b+k)^2+54A^6(2b+k)^3}.$$

3.2 Application for MBBM equation

Eq.4 can be converted into following ordinary differential equation by using transformation of $\eta = kx + wt$, $u = U(\eta)$,

$$-\gamma k^2 w U''' + \beta k U^2 U' + (w+\alpha k)U' = 0. \quad (33)$$

Integration Eq.33 once, we get

$$U'' - \frac{(w+\alpha k)}{\gamma k^2 w}U - \frac{\beta}{3\gamma k w}U^3 - \frac{c}{\gamma k^2 w} = 0. \quad (34)$$

where c is an integration constant. Eq.4 was also solved by using the HPM with different initial condition in Ref. [30]. Our aim is to find bifurcation point for same equation with different initial condition.

The First Approximation:

We apply the homotopy perturbation method to Eq.34. We compose a homotopy in the form

$$U'' - \frac{(w+\alpha k)}{\gamma k^2 w}U - \frac{\beta}{3\gamma k w}\,p\,U^3 - \frac{c}{\gamma k^2 w} = 0, \quad (35)$$

where $p \in [0,1]$. In Eq.35 there are two observations to be considered; the first observation is that for $p = 0$, Eq.35 becomes a linear equation. The second observation is that for $p = 1$, it becomes the original nonlinear one. Due to the homotopy perturbation method, the solution of Eq.35 can be expressed in a series of p :

$$U = U_0 + p\,U_1 \quad (36)$$

Moreover, the coefficient of the linear term and the constant also expend in a series of p, that is

$$-\frac{(w+\alpha k)}{\gamma k^2 w} = \sigma^2 + p\,\sigma_1, \quad (37)$$

$$-\frac{c}{\gamma k^2 w} = p\,c_1. \quad (38)$$

Substituting Eqs 36 and 37 into Eq 35 and equating coefficients of like powers of p yields a series of linear equations:

$$U_0'' + \sigma^2 U_0 = 0, \quad (39)$$

$$U_1'' + \sigma^2 U_1 + \sigma_1 U_0 - \frac{\beta}{3\gamma k w}U_0^3 + c_1 = 0 = 0. \quad (40)$$

Solving Eq..39 in the initial condition $U_0(0) = A$ and $U_0'(0) = 0$, we get

$$U_0 = A\cos\sigma\eta. \quad (41)$$

Substituting .41 into.40 results into

$$U_1'' + \sigma^2 U_1 + \left(\sigma_1 - \frac{A^2\beta}{4\gamma k w}\right)A\cos\sigma\eta \quad (42)$$

$$-\frac{A^3\beta}{12\gamma k w}\cos 3\sigma\eta + c_1 = 0.$$

If $\sigma_1 - \frac{A^2\beta}{4\gamma k w} \neq 0$, every solution of Eq.42 will comprise a secular term. So we necessitate that

$$\sigma_1 = \frac{A^2\beta}{4\gamma k w}, \quad (43)$$

$$c_1 = 0. \quad (44)$$

and eliminate the $\cos\sigma\eta$ term from Eq.42 completely. With this requirement, Eq.42 becomes

$$U_1'' + \sigma^2 U_1 - \frac{A^3\beta}{12\gamma k w}\cos 3\sigma\eta = 0. \quad (45)$$

The solution of Eq.45 reads in the initial conditions $U_1(0) = 0$ and $U_1'(0) = 0$

$$U_1 = -\frac{A^3\beta}{96\gamma kw\sigma^2}\left(\cos 3\sigma\eta - \cos\sigma\eta\right). \tag{46}$$

Then the first-order approximate solution is enough, then putting $p = 1$ in Eqs.36 and 37, we get

$$U = A\cos\sigma\eta - \frac{A^3\beta}{96\gamma kw\sigma^2}\left(\cos 3\sigma\eta - \cos\sigma\eta\right), \tag{47}$$

and

$$\sigma^2 = -\frac{(w+\alpha k)}{\gamma k^2 w} - \frac{A^2\beta}{4\gamma kw}, \tag{48}$$

$$c = 0. \tag{49}$$

Similarly, with the same logic in Section 3.1 and Ref. [10], since $\sigma^2 \geq 0$, the above equation has no solution when

$$\alpha > \frac{A^2\beta}{4} + \frac{w}{k}.$$

Nevertheless, when

$$\alpha < \frac{A^2\beta}{4} + \frac{w}{k},$$

Eq.48 has the solution

$$\sigma = \sqrt{-\frac{(w+\alpha k)}{\gamma k^2 w} - \frac{A^2\beta}{4\gamma kw}}. \tag{50}$$

Therefore, the bifurcation occurs at

$$\alpha = \frac{A^2\beta}{4} + \frac{w}{k}.$$

The Second Approximation:

According to the homotopy perturbation method, we assume that the solution of Eq.35 can be expressed in a series of p :

$$U = U_0 + pU_1 + p^2U_2 \tag{51}$$

Moreover, the coefficient of the linear term and the constant also expend in a series of p, that is

$$-\frac{(w+\alpha k)}{\gamma k^2 w} = \sigma^2 + p\sigma_1 + p^2\sigma_2, \tag{52}$$

$$-\frac{c}{\gamma k^2 w} = pc_1 + p^2c_2. \tag{53}$$

Therefore, in the initial conditions $U_0(0) = A$, $U_0'(0) = 0$ and $U_i(0) = 0$, $U_i'(0) = 0$ for $i = 1, 2$, using 51, 52 and repeating the process in the first approximation, then putting $p = 1$ in Eqs.51 and 52 we get

$$\begin{aligned} U &= A\cos\sigma\eta - \frac{A^3\beta}{96\gamma kw\sigma^2}\left(\cos 3\sigma\eta - \cos\sigma\eta\right) \\ &\quad + \frac{A^5\beta^2}{3^2 2^{10}\gamma^2 k^2 w^2\sigma^4}\left(\cos 5\sigma\eta - \cos\sigma\eta\right), \end{aligned}$$

and

$$\begin{aligned} \sigma^2 = {} & -\frac{(w+\alpha k)}{\gamma k^2 w} - \frac{A^2\beta}{4\gamma kw} \\ & -\frac{A^4\beta^2}{3^1 2^7\gamma^2 k^2 w^2\sigma^2}, \end{aligned} \tag{54}$$

$$c = 0. \tag{55}$$

Rewrite Eq.54 in the form

$$\sigma = \mp\frac{\sqrt{(-6w\gamma(4w + 4k\alpha + A^2k\beta) \mp w\gamma\varphi}}{4\sqrt{3}w\gamma k}$$

or

$$\sigma = \mp\frac{\sqrt{(-6w\gamma(4w + 4k\alpha + A^2k\beta) \pm w\gamma\varphi}}{4\sqrt{3}w\gamma k}$$

where φ is

$$\sqrt{6(96w^2 + 48kw(4\alpha + A^2\beta) + k^2(96\alpha^2 + 48A^2\alpha\beta + 5A^4\beta^2))}.$$

Therefore, bifurcation occurs at

$$\alpha = \frac{A^4\beta^2 k}{-32w + 8A^2k(-1+2w)\beta}.$$

4 Conclusion

The homotopy perturbation method have been used to obtain the perturbative solutions of the DS and MBBM equations. In this paper, bifurcation phenomenon in DS and MBBM equations is investigated by using HPM and then the bifurcation is observed in first and second approximations.

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