The Exponential, Gaussian and Uniform Truncated Discrete Density Functions for Discrete Time Systems Analysis

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Abstract: - Continuous density functions and their truncated versions are widely used in engineering practice. However, limited work was dedicated to the theoretical analysis and presentation in closed forms of truncated density functions of discrete random variables. The derivations of exponential, uniform and Gaussian discrete and truncated density functions and related moments, as well as their applications in the theory of discrete time stochastic processes and for the modelling of communication systems, is presented in this paper. Some imprecise solutions and common mistakes in the existing books related to discrete time stochastic signals analysis are presented and rigorous mathematical solutions are offered.

Key-Words: - truncated densities, Gaussian truncated discrete density, exponential truncated density, moments of truncated densities.

1 Introduction

1.1 Motivation

Digital technology is widely used in the development and design of electronic devices in communication systems. Therefore, the theoretical analysis of these systems assumes representation of signals in discrete time domain. Consequently, the random variables in these systems need to be represented in discrete forms having the values in a limited interval. These problems motivated us to work on truncated discrete density functions derivations. Here, we will start with presenting three cases where this kind of analysis is necessary.

FIRST CASE: In the analysis of discrete time stochastic processes, we are usually interested in calculating their mean, variance and autocorrelation function. For these calculations, we need to use the probability density function of a random variable involved. In doing this, a common mistake is that the density function is defined as a continuous function of the related random variable, which implies that the random variable is of a continuous type even though it is not. Therefore, in order to preserve mathematical exactness, we need to define and use density functions of discrete random variables. The misleading and mathematically incorrect procedures in using continuous probability density function in the theory of discrete time stochastic processes can be found in published papers and books. For example, in book [1], page 78, example 3.3.1 and book [2], pages 71 and 72, example 2.2.3, the mean and autocorrelation function of a discrete time harmonic process, which is defined as $X(m) = A\sin(\Omega m + \theta)$ with uniformly distributed phases in the interval (-π, π) having continuous density function $f_\theta(\theta)$ =1/2π, are calculated as

$$\eta(m) = E\{X(m)\} = \int_{-\pi}^{\pi} A\sin(\Omega m+\theta)\, f_\theta(\theta)d\theta$$
$$= \frac{A}{2\pi}\int_{-\pi}^{\pi}\sin(\Omega m+\theta)d\theta = 0 \qquad (1)$$

and

$$R_X(m,l) = E\{X(m)\cdot X(l)\}$$
$$= \int_{-\pi}^{\pi} A^2\sin(\Omega m+\theta)\sin(\Omega l+\theta)\, f_\theta(\theta)d\theta$$
$$= \frac{A^2}{4\pi}\int_{-\pi}^{\pi}\left[\cos\Omega(m-l)+\sin(\Omega(m+l)+2\theta)\right]d\theta \qquad (2)$$
$$= \frac{A^2}{2}\cos\Omega(m-l)$$

However, due to the discrete nature of the stochastic process $X(m)$ and its realisations, which are discrete time function $x(m)$ of argument m, the presented calculations are not mathematically correct. In this paper, these expressions will be rigorously calculated after deriving the proper density function $f_\theta(\theta)$. Namely, the random phase θ has to be of a discrete type inside the interval ($-\pi$, π) having a discrete density function. On the other hand, based on existing theory, the density function of the discrete phase need to be presented as a mass function. However, the mass function cannot be used to directly solve the integrals. Therefore, in this paper, we derived and used the density functions of the discrete random variables expressed in terms of Dirac's delta functions, which simplified the solution of related integrals, as will be shown in Section 5.

SECOND CASE: The general expression for the probability of bit error of a direct sequence spread-spectrum (DSSS) systems in a wideband channel (WBC) with white Gaussian noise, assuming existence of one primary and a set of secondary channels, was derived in [3]. It is assumed that the signals in secondary channels are transmitted with random delays τ in respect to the relative zero-delay in the primary channel. The probability of error can be expressed in this form [3]

$$P_e(\tau)=\frac{1}{2}erfc\left(\frac{(\psi-1)}{\beta}\left(\frac{1}{1-\tau/2S}\right)^2+\frac{1}{\beta}\left(\frac{\tau/2S}{1-\tau/2S}\right)^2+\frac{1}{4}\left(\frac{1}{1-\tau/2S}\right)^2\left(\frac{E_b}{N_0}\right)^{-1}\right)^{-1/2} \quad (3)$$

whereas ψ is the sequence factor, S is the number of interpolated samples in a chip, 2β is the spreading factor and E_b/N_0 is the signal-to-noise ratio in the channel. The probability function (3) is a random function conditioned on the delay τ as a random variable. Therefore, the probability of error is the mean value of this function that can be calculated using this integral

$$P_e=\int_{-\infty}^{\infty} P_e(\tau)\, f_{dt}(\tau)d\tau\,. \quad (4)$$

The probability density function of the random delay τ is generally represented by the continuous exponential density function, $f_\tau(\tau)$. However, in the above case the interval of τ values is limited and contains S possible discrete values. Therefore, the delay has to be expressed using a truncated discrete density function $f_{dt}(\tau)$. Furthermore, this function has to be expressed in a form suitable to calculate integral (4). To solve all these problems we developed the expression of a truncated discrete exponential function in closed form and demonstrated its application for this case.

THIRD CASE: In direct sequence spread-spectrum (DSSS) and code division multiple access (CDMA) systems with imperfect sequence synchronisation, the probability of error conditional on the delay τ between transmitter and the receiver spreading sequences, for a single-user system with interleavers, can be expressed as

$$P_e(\tau)=\frac{1}{2}erfc\left(\frac{4(\psi-\pi/4)}{\pi\beta}+\frac{4}{\pi\beta}X(\tau)+(1+X(\tau))\frac{2}{\pi b^2}\left(\frac{E_b}{N_0}\right)^{-1}\right)^{-1/2} \quad (5)$$

whereas $X(\tau)=|\tau|^2/(S-|\tau|)^2$, ψ is the sequence factor, S is the number of interpolated samples in a chip, $2b^2$ is the mean square value of fading coefficients, 2β is the spreading factor and E_b/N_0 is the signal-to-noise ratio in the channel.

The similar phenomenon occurs for well-investigated carrier synchronization, where a random phase difference exists between the received signal and the locally generated carrier. This random phase was usually represented by Gaussian and Tikhonov density function, as can be seen in Lo and Lam [4], Eng and Milstain [5] and Richards paper [6]. As has been pointed out by Richards, Tikhonov distribution can be expressed as Gaussian or uniform density function as its special cases. These densities are used to find the mean value of the probability of error that was conditioned on the phase random variable, as may be seen from papers of Polprasert and Ritcey [7], Song at al. [8] and Chandra at al. [9]. In all these papers, the density functions are assumed continuous and the interval of their values sometimes was beyond the interval of possible values of the phase error as explicitly noted in Richards's paper [6].

Therefore, in theoretical analysis and practice, the density function of discrete random variable τ has to be expressed using a truncated discrete density function. Furthermore, this function has to be expressed in a form suitable to solve the integral (4). For these reasons, in this paper we derived the expressions of truncated discrete exponential, Gaussian and uniform density functions and expressed them in terms of Dirac's delta functions and demonstrated their applications.

1.2 Background

The theory of continuous density functions and their truncated versions, expressed as conditional density functions, is well known. The problem of discretisation of Gaussian density function and the derivation of related discrete density function is analysed in detail in Roy's paper [10] pointing out the importance of normal distribution that makes key role in stochastic systems modelling, apostrophising that the applied discretisation method approximates the value of probability of particular event faster than in the case of simulation. This paper was a good guide for our development in spite of that the notion of the discretisation interval is not preserved and symmetry of the obtained density function around the mean value is lost. Similar analyses for geometric and hypergeometric probability functions was presented in Xekalaki work [11]. Roy and Dasgupta [12] suggested discrete approximation procedures for the evaluation of the reliability of complex systems. Roy [13] investigated the discrete Rayleigh distribution function with respect to two measures of failure rate, and used this distribution for evaluation reliability of complex systems. In Ho and Cheng's paper [14], the expression for the probability mass of a truncated exponential density function was derived. However, they did not present the related density function in closed form and did not derive the moment of the density function. Ahsanullah [15] presented his analysis of exponential distribution. Some observations on the exponential half logistic density function and related distribution were presented in Seo and Kang paper [16]. Raschke [17] pointed out the importance of applying the truncated exponential function in modelling the amplitude of an earthquake in seismology and suggested the use of the generalized truncated exponential distribution.

In this paper, we will use our unique approach to derive the expressions for discrete probability density functions, and then extend them for the derivations of related truncated density functions that are more suitable in practice when the discrete random variable exists in a limited interval of its possible values.

Specifically, the discrete truncated density functions of our interest need to fulfil these conditions: 1) Discretisation of the related continuous density functions. The discrete density function should be obtained by assigning probability values as the weights of Dirac's delta functions. 2) Preservation of the value of discretisation interval T_s that allows us to reconstruct the sampling interval and relate it to the real values in practical application. For example, in the case of defining a delay in communication systems these sampling intervals will be expressed in appropriate time units. 3) Expression of density functions in closed form: By using Dirac's delta functions the obtained density function of discrete random variable can be used to calculate the mean values of a random function according to the integral (4).

2 The Discrete Truncated Exponential Density Function and Its Moments

This section contains basic derivatives of the density function and related moment. The obtained results can be used to find mean value of the probability of error expressed as (2).

2.1 The discrete exponential density function

The procedure of a continuous exponential density function discretisation is presented in Fig. 1. The discretisation is performed at uniformly spaced discrete interval T_s. The problem is how to find the expression of this discrete density function and how to find its mean and variance. In our approach, in the process of discretisation the probability of each interval needs to be calculated and assigned to the discrete time instants nT_s. There are two possibilities in this case: Firstly, the probabilities are assigned to the left side of the interval starting with discrete value τ=0. Secondly, the probability are assigned to the right side of the interval starting with discrete value τ=1.

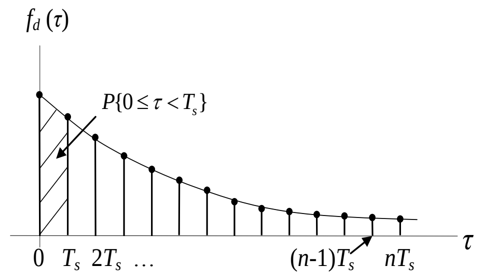

Figure 1. Discretisation of a continuous exponential density function.

These two possibilities will have some consequences to the expressions for density function, mean and variance of the truncated discrete random variable and particular care should be taken about this issue. Firstly, we will derive the discrete density and distribution functions in closed forms and related moments.

Proposition: The discrete density function, having the values at the uniformly spaced instants T_s, is expressed as

$$f_d(\tau) = (e^{\lambda T_s} - 1)\sum_{n=1}^{\infty} e^{-n\lambda T_s}\delta(\tau - nT_s)\,, \tag{6}$$

where $\delta(.)$ are Dirac's delta functions and τ is a continuous variable. The values of the density function are discrete and defined by the positions and weights of delta functions.

Proof: If this density is uniformly discretized in respect to τ, with the interval of discretisation of T_s, the probability value in any interval defined by n is

$$\begin{aligned} P\{(n-1)T_s \le \tau < nT_s\} &= \int_{(n-1)T_s}^{nT_s} f_{\mathrm{T}}(\tau)d\tau \\ &= e^{-(n-1)\lambda T_s} - e^{-n\lambda T_s} = (e^{\lambda T_s} - 1)e^{-n\lambda T_s} \end{aligned}.$$

To express the density in a closed form, these probabilities can be used as weights of Dirac's delta functions representing the discrete exponential density function at time instants $\tau = nT_s$, which results in this expression

$$\begin{aligned} f_d(\tau) &= \sum_{n=1}^{\infty} P\{(n-1)T_s \le \tau < nT_s\}\delta(\tau - nT_s) \\ &= (e^{\lambda T_s} - 1)\sum_{n=1}^{\infty} e^{-n\lambda T_s}\delta(\tau - nT_s) \end{aligned}, \tag{7}$$

which completes our proof.

Proposition: The mean and variance derived for any T_s, and for a unit interval $T_s = 1$, can be expressed in these forms

$$\begin{aligned} \eta_d &= \int_{\tau=-\infty}^{\infty} \tau \cdot f_d(\tau)d\tau = \frac{T_s}{1 - e^{-\lambda T_s}} \\ &= \frac{T_s e^{\lambda T_s}}{e^{\lambda T_s} - 1} \underset{T_s=1}{=} \frac{e^{\lambda}}{e^{\lambda} - 1} \end{aligned} \tag{8}$$

and

$$\sigma_d^2 = E\{\tau^2\} - \eta_d^2 = \frac{e^{-\lambda T_s}}{(e^{-\lambda T_s} - 1)^2} \underset{T_s=1}{=} \frac{e^{\lambda}}{(e^{\lambda} - 1)^2}\,. \tag{9}$$

Proofs: The proofs for the unit discretisation interval $T_s = 1$ will be presented. In this case the mean of discrete density function is

$$\begin{aligned} \eta_d &= \int_{\tau=-\infty}^{\infty} \tau \cdot f_d(\tau)d\tau = \int_{\tau=0}^{\infty} \tau \cdot \sum_{n=1}^{\infty}(e^{\lambda} - 1)e^{-n\lambda}\delta(\tau - n)d\tau \\ &= \sum_{n=1}^{\infty}(e^{\lambda} - 1)e^{-\lambda n}\int_{\tau=T_s}^{\infty} \tau \cdot \delta(\tau - n)d\tau \\ &= (e^{\lambda} - 1)\sum_{n=1}^{\infty} n \cdot e^{-\lambda n} = (e^{\lambda} - 1)\cdot\Sigma_1 \end{aligned} \tag{10}$$

After calculating

$$e^{\lambda}\Sigma_1 - \Sigma_1 = \sum_{n=0}^{\infty}(e^{-\lambda})^n = \frac{1}{1 - e^{-\lambda}} = \frac{e^{\lambda}}{e^{\lambda} - 1},$$

the expression for Σ_1 can be found and used in (10) to find the mean as expressed by (8). Similarly, the mean square value can be found as

$$\begin{aligned} E\{\tau^2\} &= \int_{\tau=-\infty}^{\infty} \tau^2 \cdot f_d(\tau)d\tau \\ &= (e^{\lambda} - 1)\sum_{n=1}^{\infty} n^2 \cdot e^{-\lambda n} = (e^{\lambda} - 1)\cdot\Sigma_2 \end{aligned} \tag{11}$$

where the Σ_2 can be calculated as a function of Σ_1 as $e^{\lambda}\Sigma_2 - \Sigma_2 = \Sigma_1 + e^{\lambda}\Sigma_1$. By inserting Σ_2 into (11), we can get the variance as stated in (9).

Proposition: The mean values for the discrete random variable starting with $n = 0$ and with $n = 1$ are different as was pointed out before. Their relationship is expressed as

$$\eta_d - \eta_{d0} = \frac{e^{\lambda}}{e^{\lambda} - 1} - \frac{1}{e^{\lambda} - 1} = 1\,. \tag{12}$$

This relation needs to be taken into account, especially in the case when the sampling interval T_s is large.

2.2 The truncated discrete exponential density function

As we pointed out in Introduction, our motivation in doing this research is to develop theoretical expressions for the discrete truncated density functions that are appropriate for discrete time stochastic systems modelling. For example, the delays in discrete time communication systems are taking values in a limited interval of, say, S possible discrete values. Therefore, the function that describes the delay distribution is truncated to interval S.

Proposition: The density and distribution functions of a truncated discrete exponential random variable are given in closed form as

$$f_{dt}(\tau) = \sum_{n=1}^{S} \frac{(e^{\lambda} - 1)}{1 - e^{-S\lambda}} e^{-n\lambda}\delta(\tau - n)$$

and

$$F_{dt}(\tau)=\sum_{n=1}^{\tau}\frac{(e^{\lambda}-1)}{1-e^{-S\lambda}}e^{-n\lambda}U(\tau-n) \quad (13)$$

Proof: In order to find this truncated function, the whole domain of possible τ values from 0 to infinity, for the already analysed discrete density function, will be divided into non-overlapping intervals containing S values. All corresponding values in these intervals, starting with 1st, (S+1)th, (2S+1)th, etc. term, will be added to obtain the truncated density function values for n = 1, 2, …., S. Using expression for the density function with T_s = 1, the first values of the density function in the 1st, 2nd and m-th interval of S values will be

$$\begin{aligned} & f_{dt}(\tau=1)=(e^{\lambda}-1)e^{-\lambda}, \\ & f_{dt}(\tau=S+1)=(e^{\lambda}-1)e^{-(S+1)\lambda} \\ & \vdots \\ & f_{dt}(\tau=mS)=(e^{\lambda}-1)e^{-\lambda}e^{-mS\lambda} \end{aligned}$$

The sum of these values, when m tends to infinity, will give the first truncated value defined for unit delay τ = 1, i.e.,

$$f_{dt}(\tau=1)=\frac{(e^{\lambda}-1)}{1-e^{-S\lambda}}e^{-\lambda}$$

and the value for density function for any delay n will be

$$f_{dt}(\tau=n)=\frac{(e^{\lambda}-1)}{1-e^{-S\lambda}}e^{-n\lambda}. \quad (14)$$

If we assign these values to the weights of Dirac's delta functions in the whole interval of possible truncated values S, we can get the truncated density function in this form

$$f_{dt}(\tau)=\sum_{n=1}^{S}\frac{(e^{\lambda}-1)}{1-e^{-S\lambda}}e^{-n\lambda}\delta(\tau-n), \quad (15)$$

which completes our proof. In the process of density function discretisation, the discrete probability values were calculated at time instants nT_s, starting with n = 1. This could be also done starting with the value n = 0. However, this will not be exactly the same density function, because it will result in different mean values of the delay and imprecise simulation of the discrete delays, as we will show. For the case when the discrete values of the density function start at n = 0, the mean value is

$$\begin{aligned} \eta_{dt0} &= \sum_{n=0}^{S-1}(nT_s)\cdot f_{dt}((n+1)T_s) \\ &= 0\cdot f_{dt}(1\cdot T_s)+T_s\cdot f_{dt}(2\cdot T_s)+\ldots+ST_s\cdot f_{dt}(ST_s) \end{aligned} \quad . (16)$$

When the first density value is defined for n = 1, the mean value will be

$$\eta_{dt1}=\sum_{n=1}^{S}(nT_s)\cdot f_{dt}(nT_s)\cdot \quad (17)$$

These two mean values are not the same. The proof of their relationship is

$$\begin{aligned} \eta_{dt1} &= \sum_{n=1}^{S}(nT_s)\cdot f_{dt}(sT_s) \\ &= T_s\cdot f_{dt}(T_s)+2T_s\cdot f_{dt}(2T_s)+\ldots(ST_s)\cdot f_{dt}(ST_s) \\ &= T_s\cdot f_{dt}(T_s)+T_s\cdot f_{dt}(2T_s)+T_s\cdot f_{dt}(3T_s)+\ldots T_s\cdot f_{dt}(ST_s) \\ &\quad +T_s\cdot f_{dt}(2T_s)+2T_s\cdot f_{dt}(3T_s)+\ldots(S-1)T_s\cdot f_{dt}(ST_s) \\ &= T_s\cdot\sum_{n=1}^{S}\cdot f_{dt}(nT_s)+\sum_{n=0}^{S-1}(nT_s)\cdot f_{dt}((n+1)T_s)=T_s+\eta_{dt0} \end{aligned} \quad (18)$$

Therefore, the mean value depends on the starting value of the discrete random variable of the density function, as we said in the previous section, and can vary with the duration of discrete interval T_s. It is important to have this difference in mind especially in the case when we are doing simulation of the discrete delay values. Namely, in the case when the first density function value is defined for n = 0, and the discrete variable values (variates) need to be generated, then the variate values in the first interval from 0 to T_s will be equated with zero, and the values from T_s to $2T_s$ will be equated with T_s and so on, until (S-1)T_s is reached. Therefore, if it is not important to notify and take into account all the delays inside the first T_s interval this presentation of the density function will be used. However, if all delays in the first T_s interval need to be taken into account, then the first sample of the density truncated function should be assigned to the first discrete time instant T_s.

Proposition: The mean and variance of the discrete truncated random variable, when the first discrete value with the first density value is at n = 1, are expressed in this form

$$\eta_{dt1}=\frac{1}{1-e^{-S\lambda}}\frac{Se^{-(S+1)\lambda}-(S+1)e^{-S\lambda}+1}{1-e^{-\lambda}}. \quad (19)$$

This value is for one greater than the mean value η_{dt0} of the discrete density with the first density value at

n = 0. Therefore, the mean value η_{dt0} in closed form is

$$\eta_{dt0} = \frac{e^{-\lambda}}{1-e^{-S\lambda}} \frac{(S-1)e^{-S\lambda} - Se^{-(S-1)\lambda} + 1}{1-e^{-\lambda}}. \quad (20)$$

Thus, in practical applications, for the defined mean value 1/λ of the continuous exponential distribution, the mean value of discrete exponential η_{dt} can be found and compared.

Proof: Based on the expression (13) for the discrete density function we may have

$$\eta_{dt1} = \int_{-\infty}^{\infty} \tau f_{dt}(\tau) d\tau = \frac{(e^{\lambda}-1)}{1-e^{-S\lambda}} \sum_{n=1}^{S} e^{-n\lambda} \int_{-\infty}^{\infty} \tau \cdot \delta(\tau - n) d\tau$$
$$= \frac{(e^{\lambda}-1)}{1-e^{-S\lambda}} \sum_{n=1}^{S} n \cdot e^{-n\lambda} = \frac{(e^{\lambda}-1)}{1-e^{-S\lambda}} \cdot \Sigma_S$$

The sum Σ_S can be found in a closed form from this expression

$$\Sigma_S - \Sigma_S e^{-\lambda} = \Sigma_S (1-e^{-\lambda}) = \sum_{n=1}^{S} (e^{-\lambda})^n - Se^{-(S+1)\lambda}$$
$$= e^{-\lambda} \frac{1-e^{-\lambda S}}{1-e^{-\lambda}} - Se^{-(S+1)\lambda}.$$

Having available Σ_S we may calculate the mean value as

$$\eta_{dt1} = \frac{(e^{\lambda}-1)}{1-e^{-S\lambda}} e^{-\lambda} \frac{1-e^{-\lambda S}}{(1-e^{-\lambda})^2} - \frac{(e^{\lambda}-1)}{1-e^{-S\lambda}} \frac{Se^{-(S+1)\lambda}}{1-e^{-\lambda}},$$
$$= \frac{1}{1-e^{-S\lambda}} \frac{Se^{-(S+1)\lambda} - (S+1)e^{-S\lambda} + 1}{1-e^{-\lambda}}$$

which confirms (19). Using (18) for T_s = 1, we may derive the expression for the mean of the random variable that starts with n = 0 and prove (20). In similar way, as it was presented in Giucaneanu paper [18] the variance is

$$\sigma_{dt}^2 = E\{\tau^2\} - \eta_{dt}^2$$
$$= \frac{e^{-2\lambda}}{1-e^{-S\lambda}} \left[\frac{2(1-e^{-S\lambda})}{(1-e^{-\lambda})^2} - \frac{2Se^{-(S-1)\lambda}}{1-e^{-\lambda}} - S(S-1)e^{-(S-2)\lambda} \right] \quad (21)$$

The two graphs, the mean and variance as a function of parameter $1/\lambda$, starting with $1/\lambda$ = 1 and finishing with $1/\lambda$ = 30, for S = 40, are presented in Fig. 2. Alongside with these graphs, the graphs of the mean and variance for discrete non-truncated density function, continuous (non-truncated) density function and continuous truncated density function are presented. There is obvious difference in the mean and variance values between non-truncated and truncated functions that should be taken into account in theoretical analysis and simulation of discrete time systems. To support this statement, in the next section we will present the procedure of generating variates of a discrete truncated exponential density function.

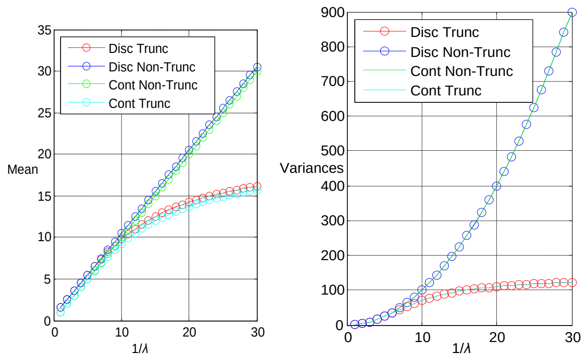

Figure 2 The mean and variance of discrete truncated exponential function.

2.3 Generating variates of the discrete truncated exponential distribution

In this section, a procedure of generating random variates of the truncated exponential discrete variable will be presented using the *inverse transformation method*. According to this method, variates of a uniform distribution F will be generated, continuous delay value τ will be calculated and then a discrete variate value τ_v will be assigned. The value of the truncated discrete exponential distribution function, for particular delay $\tau_v = \tau$, can be calculated for T_s =1 and expressed in this form

$$F_{dt}(\tau) = \frac{1}{1-e^{-S\lambda}} (1-e^{-\tau\lambda}) = F. \quad (22)$$

The delay is expressed as a function of the distribution function value as

$$\tau = -\frac{1}{\lambda} \ln(1 - F/A), \quad (23)$$

where A is a constant, i.e., $A = 1/(1-e^{-S\lambda})$. Then the uniform continuous valued variates F are generated and the delay values are calculated according to (23). Because these calculated delay values are real numbers, they need to be equated to the integer values which *are not smaller* than the real number in the argument of τ, i.e.,

$$\tau_v = \left\lceil -\frac{1}{\lambda}\ln(1 - F/A) \right\rceil \cdot \tag{24}$$

The discrete values, calculated in this way, represent the variates τ_v of an exponential truncated discrete density function that has the first discrete value at $n = 1$. However, if the first discrete value is to be at $n = 0$, the discrete truncated density and distribution functions are slightly different, the discrete delay will be generated which *is not greater* than the generated uniform variate, i.e.,

$$\tau_v = \left\lfloor -\frac{1}{\lambda}\ln(1 - F/A) \right\rfloor \cdot \tag{25}$$

3 The Discrete Truncated Gaussian Density Function and Its Moments

This section contains basic derivatives of the Gaussian discrete and truncated discrete density function and related moment.

3.1 The discrete Gaussian density function

Following the general procedure of a continuous $f_c(\tau)$ and related truncated continuous density function $f_{ct}(\tau)$ evaluation we will present here the discretisation of a Gaussian density function with a zero mean value. The related derivations for any mean value can be relatively easily obtained.

The procedure of discretizing a continuous Gaussian density function is illustrated in Fig. 3. We will calculate the probability value inside T_s interval and assign it to the discrete value of the random variable. The probability inside shaded area in Fig. 3 will be assigned to the discrete random variable defined for $\tau = 0$. If we use the interval T_s on the left or on the right the discrete density function will introduce the mean value that is not equal to zero. For this reason, we are starting with the interval around the origin.

***Proposition**:* The discrete density function of Gaussian random variable, having the values at the uniformly spaced instants T_s of a random variable τ, can be expressed as

$$f_d(\tau) = \frac{1}{2}\sum_{nT_s=-\infty}^{\infty}\left(erfc\frac{(2n-1)T_s/2}{\sqrt{2\sigma^2}} - erfc\frac{(2n+1)T_s/2}{\sqrt{2\sigma^2}} \right)\delta(\tau - nT_s) \tag{26}$$

where $\delta(.)$ is Dirac's delta function and the error function complementary is $erfc(\tau) = 2/\sqrt{\pi}\int_{\tau}^{\infty} e^{-x^2}dx$.

Proof: The probability value inside the interval around zero can be calculated as

$$\begin{aligned} & P\{-T_s/2 \le \tau < T_s/2\} = \int_{-T_s/2}^{T_s/2} f_c(\tau)d\tau \\ & = \int_{-T_s/2}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\tau^2/2\sigma^2} d\tau - \int_{T_s/2}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\tau^2/2\sigma^2} d\tau \\ & = \frac{1}{2} erfc\frac{-T_s/2}{\sqrt{2\sigma^2}} - \frac{1}{2} erfc\frac{T_s/2}{\sqrt{2\sigma^2}} \end{aligned} \tag{27}$$

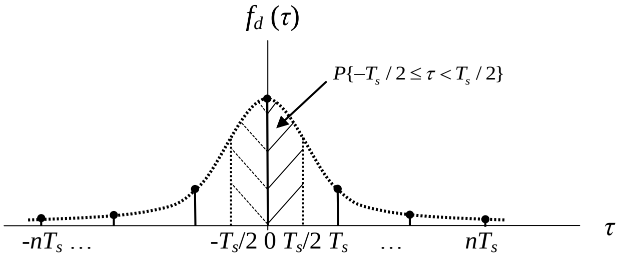

Figure 3 Discretisation of Gaussian density function.

Similarly, the probability that the random variable is inside any discrete interval n is

$$\begin{aligned} & P\{(2n-1)T_s/2 \le \tau < (2n+1)T_s/2\} \\ & = \frac{1}{2} erfc\frac{(2n-1)T_s/2}{\sqrt{2\sigma^2}} - erfc\frac{(2n+1)T_s/2}{\sqrt{2\sigma^2}} \cdot \end{aligned} \tag{28}$$

If the calculated probabilities are assigned as the weights to the Dirac's delta functions that are defined at discrete instants $\tau = nT_s$, then the obtained function represents the discrete Gaussian density function expressed as

$$\begin{aligned} & f_d(\tau) = \frac{1}{2}\sum_{nT_s=-\infty}^{\infty} P\{(2n-1)T_s/2 \le \tau < (2n+1)/2\}\delta(\tau - nT_s) \\ & = \frac{1}{2}\sum_{nT_s=-\infty}^{\infty}\left(erfc\frac{(2n-1)T_s/2}{\sqrt{2\sigma^2}} - erfc\frac{(2n+1)T_s/2}{\sqrt{2\sigma^2}} \right)\delta(\tau - nT_s) \end{aligned} . \tag{29}$$

For the unit interval, $T_s = 1$, this density is

$$\begin{aligned} & f_d(\tau) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left(erfc\frac{(2n-1)}{\sqrt{8\sigma^2}} - erfc\frac{(2n+1)}{\sqrt{8\sigma^2}} \right)\delta(\tau - n) \\ & = \frac{1}{2}\sum_{n=-\infty}^{\infty} Erfc(n)\delta(\tau - n) \end{aligned} , \tag{30}$$

where the $Erfc(n)$ is defined as

$$Erfc(n) = erfc\frac{(2n-1)}{\sqrt{8\sigma^2}} - erfc\frac{(2n+1)}{\sqrt{8\sigma^2}}, \qquad (31)$$

which completes our proof.

3.2 The discrete truncated Gaussian density function

In order to find this truncated function, the whole domain of possible discrete values τ from 0 to infinity, for the already derived functions, need to be divided into intervals containing S discrete values. All corresponding values in these intervals, starting with 1st, (S+1)th, (2S+1)th, etc. terms, need to be added to obtain the truncated density function values for n = 1, 2, …., S. However, in this case this method cannot give us the expression for density function in a closed form. For this reason, a method based on the definition of the continuous truncated density function will be used. According to this method the truncated density function is defined as the conditional density function on the interval (-S, S) and expressed as

$$f_{dt}(\tau) = f_d(\tau \| \tau | \le S) = \frac{f_d(\tau)}{P(\tau \ge -S) - P(\tau > S)}$$
$$= \frac{\frac{1}{2}\sum_{n=-S}^{n=S}\left(erfc\frac{(2n-1)}{\sqrt{8\sigma^2}} - erfc\frac{(2n+1)}{\sqrt{8\sigma^2}}\right)\delta(\tau-n)}{\frac{1}{2}\sum_{n=-S}^{n=S}\left(erfc\frac{(2n-1)}{\sqrt{8\sigma^2}} - erfc\frac{(2n+1)}{\sqrt{8\sigma^2}}\right)}, \qquad (32)$$
$$= P(S)\sum_{n=-S}^{n=S} Erfc(n)\cdot\delta(\tau-n)$$

where $Erfc(n)$ is defined in (31) and $P(S)$ is a function defined on a truncation interval

$$P(S) = \sum_{n=-S}^{n=S}\left(erfc\frac{(2n-1)}{\sqrt{8\sigma^2}} - erfc\frac{(2n+1)}{\sqrt{8\sigma^2}}\right). \qquad (33)$$

The truncated density function, for different values of the truncation interval S, is presented in Fig. 4 alongside with the continuous and discrete density functions. When the truncation interval increases the truncated variance increases and is always smaller that the variance of continuous density. The SD denotes the domain of discrete function before truncation and S defines the truncation interval, i.e., the domain of truncated variable.

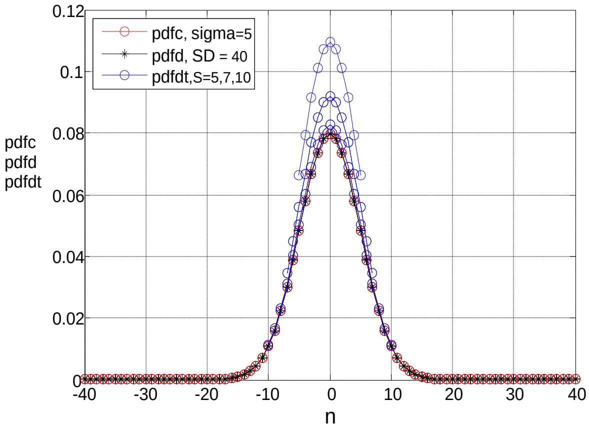

Figure 4 Continuous, discrete and discrete truncated Gaussian densities. DS is the interval of continuous and discrete random variable values. σ = 5 is the standard deviation of continuous variable and S is truncation interval.

Proposition: The mean of the truncated discrete density function is zero.

Proof: By definition

$$\eta_{dt} = \int_{-\infty}^{\infty}\tau f_t(\tau)d\tau = \int_{-\infty}^{\infty}\tau P(S)\sum_{n=-S}^{n=S} Erfc(n)\cdot\delta(\tau-n)d\tau$$
$$= P(S)\sum_{n=-S}^{n=S} Erfc(n)\cdot\int_{-\infty}^{\infty}\tau\cdot\delta(\tau-n)d\tau = P(S)\sum_{n=-S}^{n=S} n\cdot Erfc(n) \qquad (34)$$

The term of the sum for n = 0 is zero. The corresponding terms for negative and positive n are cancelling each other. Then, we may have

$$\eta_{dt} = P(S)\sum_{n=-S}^{n=S} n\cdot Erfc(n)$$
$$= P(S)\sum_{n=-S}^{n=-1} n\cdot Erfc(n) - P(S)\sum_{n=1}^{n=S} n\cdot Erfc(n) = 0, \qquad (35)$$

which completes our proof.

Proposition: The variance of this density is

$$\sigma_{dt}^2 = P(S)\sum_{n=1}^{n=S} n^2 Erfc(n). \qquad (36)$$

Proof: By definition

$$\sigma_{dt}^2 = \int_{-\infty}^{\infty}\tau^2 f_t(\tau)d\tau = P(S)\sum_{n=-S}^{n=S} Erfc(n)\cdot\int_{-\infty}^{\infty}\tau^2\delta(\tau-n)d\tau$$
$$= P(S)\sum_{n=-S}^{n=S} n^2 Erfc(n). \qquad (37)$$

The term for n = 0 is zero. The corresponding terms for negative and positive n are added to each other. In addition, the we know that $P(S) \le 1$, thus, having in mind the variance for the discrete Gaussian random variable, we may have

$$\sigma_{dt}^2 = P(S)\sum_{n=-S}^{n=S} n^2 Erfc(n) = 2P(S)\sum_{n=1}^{n=S} n^2 Erfc(n) \le \sigma^2, \quad (38)$$

which completes our proof.

4 The Discrete Truncated Uniform Density Function and Its Moments

4.1 The discrete uniform density function

Proposition: The discrete uniform density function is defined by this expression

$$f_d(\tau) = \sum_{n=-S}^{n=S} \frac{1}{2S+1} \cdot \delta(\tau - nT_s), \quad (39)$$

and, for a unit interval $T_s = 1$, it is

$$f_d(\tau) = \sum_{n=-S}^{n=S} \frac{1}{2S+1} \cdot \delta(\tau - n). \quad (40)$$

Proof: Suppose the uniform continuous density function is expressed as $f_c(\tau) = 1/2T_c$, defined inside the interval $-T_c \le \tau < T_c$. If it is discretised in respect to τ, as shown in Fig. 5, with the interval of discretisation of T_s, the probability value in the first interval around zero, n = 0, can be expressed as

$$P\{-T_s/2 \le \tau < T_s/2\} = \frac{1}{2T_c} T_s. \quad (41)$$

Similarly, the probability value in any interval n is

$$P\{(2n-1)T_s/2 \le \tau < (2n+1)T_s/2\} = \frac{1}{2T_c} T_s. \quad (42)$$

These probabilities can be understood as the weights of Dirac's delta functions that define the discrete density function, which can be expressed as

$$f_d(\tau) = \sum_{n=-S}^{S} \frac{T_s}{2T_c} \cdot \delta(\tau - nT_s). \quad (43)$$

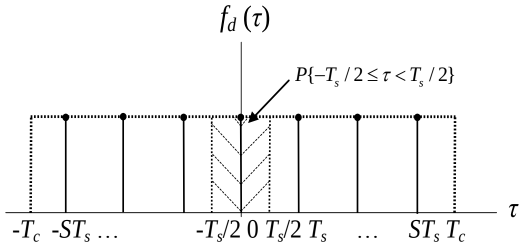

Figure 5 Discretisation of the uniform density function.

In the case the number of positive and negative intervals is S, the whole interval is $2T_c = 2ST_s + T_s$, and the relations between the values T_c, T_s and S, which will be used in this Section, can be found in these forms

$$\frac{2T_c}{T_s} = 2S + 1, \quad S = \frac{2T_c - T_s}{2T_s} = \frac{T_c}{T_s} - \frac{1}{2}. \quad (44)$$

Now, based on (42) and (44) the probability that the random variable is in the n-th interval can be expressed as

$$P\{(2n-1)T_s/2 \le \tau < (2n+1)T_s/2\} = \frac{T_s}{2T_c} = \frac{1}{(2S+1)} \quad (45)$$

Therefore, the discrete density function (43) can be expressed as stated by (39) for any T_s including T_s = 1. The calculated probability in T_s interval (for example the shaded interval in Fig. 5) is assigned as the weight of a delta function defined at the origin. The probability values can be calculate in each T_s interval and assigned to the right or left of the interval as we discussed for the exponential function, with the similar consequences related to the symmetry and moments of the truncated discrete function.

Proposition: The mean, mean square and variance are expressed as

$$\eta_d = 0, \qquad E\{\tau^2\} = T_s^2 \frac{S(S+1)}{3},$$

and

$$\sigma_d^2 = E\{\tau^2\} - \eta_d^2 = E\{\tau^2\} = T_s^2 \frac{S(S+1)}{3}, \quad (46)$$

And, for the unit interval $T_s = 1$, the variance is

$$\sigma_d^2 = \frac{S(S+1)}{3}. \tag{47}$$

Proofs: The proof for the mean is trivial. The mean square value is

$$E\{\tau^2\} = \sum_{n=-S}^{n=S} \frac{1}{2S+1} \cdot \int_{\tau=-\infty}^{\infty} \tau^2 \cdot \delta(\tau - nT_s) d\tau = \frac{T_s^2}{2S+1} \sum_{n=-S}^{n=S} n^2$$
$$= \frac{2T_s^2}{2S+1} \sum_{n=1}^{n=S} n^2 = \frac{2T_s^2}{2S+1} \frac{S(S+1)(2S+1)}{6} = T_s^2 \frac{S(S+1)}{3} \tag{48}$$

The variance is

$$\sigma_d^2 = E\{\tau^2\} - \eta_d^2 = E\{\tau^2\} = T_s^2 \frac{S(S+1)}{3}. \tag{49}$$

4.2 The discrete truncated uniform density function

In practical application the discrete delays are taking values in a limited interval defined as the truncated interval (-S + a, S - a), where $a \le S$ is a positive whole number named the truncation factor. Therefore, the function that describes the delay distribution is truncated and has the values in the truncated interval, as shown in Fig. 6.

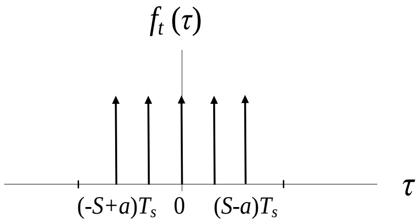

Figure 6 Discrete truncated uniform density function presented using Dirac's delta functions.

Proposition: The density function is given in closed form by these expressions

$$f_{dt}(\tau) = \frac{1}{2(S-a)+1} \sum_{n=-S+a}^{n=S-a} \delta(\tau - nT_s),$$

and

$$f_{dt}(\tau) \underset{T_s=1}{=} \frac{1}{2(S-a)+1} \sum_{n=-S+a}^{n=S-a} \delta(\tau - n). \tag{50}$$

Proof: Based on the definition of a truncated density function as a conditional density function, the truncated discrete uniform density function can be expressed as

$$f_{dt}(\tau) = f_d(\tau \mid -S+a \le \tau \le S-a)$$
$$= \frac{f_d(\tau)}{P(-S+a \le \tau \le S-a)} = \frac{\sum_{n=-S+a}^{S-a} \frac{1}{2S+1} \cdot \delta(\tau - nT_s)}{P(S)} \cdot \tag{51}$$

The value $P(S)$ can be calculated as

$$P(S) = \sum_{n=-S+a}^{S-a} \frac{1}{2S+1}$$
$$= \frac{1}{2S+1}(S-a+S-a+1) = \frac{2S-2a+1}{2S+1}. \tag{52}$$

By inserting this expression into (51), the density function can be expressed as in (50). In the same way, for a unit sampling interval $T_s = 1$, the random variable τ is a whole number from the interval (-S+a, S-a), for $a \ge 0$, and the density function (51) is expressed as in (50)

The variances of continuous, discrete and truncated discrete uniform density functions are presented in Table 1. Due to discretisation, the values of all truncated variances are smaller than the variance of continuous density. This fact needs to be taken into account in theoretical analysis and simulation of discrete time systems.

Table 1 Variance expressions

Uniform distributions	**Variances**
Continuous	$\sigma_c^2 = T_c^2 / 3$
Continuous truncated	$\sigma_{ct}^2 = \sigma_c^2 \left(1 - \frac{2a}{2S+1}\right)^2$
Discrete	$\sigma_d^2 = \sigma_c^2 \left(1 - \frac{1}{(2S+1)^2}\right)$
Discrete truncated	$\sigma_{dt}^2 = \sigma_c^2 \left(1 - \frac{1-4a(2S-a+1)}{(2S+1)^2}\right)$

4.3 The discrete asymmetric uniform density function

The above-mentioned discrete density functions preserved the symmetry in respect to the mean value. However, in discrete time signal analysis, due to the nature of analogy-to-digital conversion, we are dealing with asymmetric densities. These density functions can be obtained using the same procedure as for the symmetric densities described above. The only difference is in the assignment of discrete probability values that will start at the point -T_c = -ST_s that corresponds to the discrete value S as shown in Fig. 7. In simple words we are assigning the probabilities to the right side of the intervals defined by T_c and making S discrete values on the negative τ axis and (S-1) discrete values at the positive τ axis including also a component for $\tau = 0$.

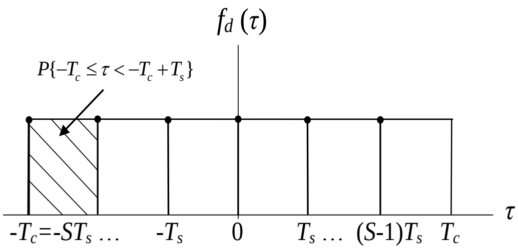

Figure 7 Discretisation of the uniform density function to obtain an asymmetric discrete density

Following the procedure in subsection 4.1 it can be proven that the discrete probability density function can be expressed in terms of Dirac's delta functions as

$$f_d(\tau) = \sum_{n=-S}^{n=S-1} \frac{1}{2S} \cdot \delta(\tau - nT_s), \tag{53}$$

Where the whole interval of continuous random variable values can be expressed as $2T_c = 2ST_s$. Therefore, the corresponding discrete variable here can have S negative values and (S-1) positive values, i.e. it will have the density function that is asymmetric in respect to the zero value. The consequence of this is that the mean value of the discrete density will be different from zero due to this asymmetry. Following the procedures presented in subsections 4.1 and 4.2 it is simple to derive the moments of this discrete density function and the related truncated density function.

5 Applications of Derived Expressions

FIRST CASE: The application of the presented theory will be demonstrated on the solution for the first problem mentioned in the Introduction of this paper. Because the defined stochastic process $X(m)$ is a discrete time process the number of phases is to be finite and needs to be represented by an asymmetric density function expressed by (53). Suppose that the number of samples inside one period of the sinusoidal realisation of stochastic process $x(m)$ have N values. Therefore the number of random phases θ_n will be N all of them having the possible random values $\theta_n = 2\pi n/N$, for $n = -N/2, \ldots, N/2-1$.

Therefore, the density function of the phase is defined by expression (53) with the following redefined parameters: $T_c = \pi$, $S = N/2$, $T_c = T_s N/2$. Then, the discrete density function (53) has this form

$$f_d(\theta) = \sum_{n=-N/2}^{n=N/2-1} \frac{1}{N} \cdot \delta(\theta - \theta_n) = \sum_{n=-N/2}^{n=N/2-1} \frac{1}{N} \cdot \delta(\theta - 2\pi n / N)$$

and the mean value of the stochastic process can be calculated as

$$\begin{aligned}
\eta(m) &= E\{X(m,\theta)\} = \int_{-\infty}^{\infty} x(m,\theta) f_d(\theta) d\theta \\
&= \int_{-\infty}^{\infty} x(m,\theta) \sum_{n=-N/2}^{n=N/2-1} \frac{1}{N} \cdot \delta(\theta - 2\pi n / N) d\theta \\
&= \frac{A}{N} \sum_{n=-N/2}^{n=N/2-1} \int_{-\infty}^{\infty} \sin(\Omega m + \theta) \delta(\theta - 2\pi n / N) d\theta \\
&= \frac{A}{N} \sum_{n=-N/2}^{n=N/2-1} \sin(\Omega m + 2\pi n / N)
\end{aligned} \tag{54}$$

Due to the properties of sinusoidal function the sum in (54) is zero and the mean values of the stochastic process is zero for each m. The autocorrelation function needs to be calculated as

$$\begin{aligned}
R_X(m,l) &= E\{X(m,\theta)X(l,\theta)\} = \int_{-\infty}^{\infty} X(m,\theta)X(l,\theta) f_d(\theta) d\theta \\
&= \int_{-\infty}^{\infty} X(m,\theta)X(l,\theta) \sum_{n=-N/2}^{n=N/2-1} \frac{1}{N} \cdot \delta(\theta - 2\pi n / N) d\theta \\
&= \frac{A^2}{2N} \sum_{n=-N/2}^{n=N/2-1} \int_{-\infty}^{\infty} [\cos(\Omega(m-l) - \cos\Omega(m+l+2\theta)\delta(\theta - 2\pi n / N) d\theta \\
&= \frac{A^2}{2N} \cos(\Omega(m-l) \sum_{n=-N/2}^{n=N/2-1} \int_{-\infty}^{\infty} \delta(\theta - 2\pi n / N) d\theta \\
&\quad - \frac{A^2}{2N} \sum_{n=-N/2}^{n=N/2-1} \cos(\Omega(m+l+2\pi n / N)
\end{aligned}$$

The sum in the first addend is equal to N and the sum in the second term is zero which result in this expression for the autocorrelation function

$$R_X(m,l) = \frac{A^2}{2} \cos(\Omega(m-l)\,. \tag{55}$$

Of course, the solutions presented in (1) and (2) are formally the same as in (54) and (55). However, the right calculations are demonstrated in (54) and (55) because they are based on rigorous mathematical presentation of the density function and strict mathematical procedure in the related solutions of integrals. Therefore, the method of presenting discrete random variables I am proposing in this paper can be used widely. Moreover, using the densities in the presented forms the integrals involved can be solved relatively easy. In addition to this, we need to take care of various derivatives of discrete densities for the same continuous random variable. For example, in the analysis of discrete time stochastic process, like above, we need to use an asymmetric density. However, in the case when the variable represents random phase in communication systems, which can take negative and corresponding positive random values with the same probability, we need to use a symmetric density function as can be mentioned in the third case.

SECOND CASE: We present here how to find the mean of the random function (3) by using and solving integral (4). Suppose that the random delay inside the chip interval τ is distributed according to the truncated discrete exponential density function. Inserting expression (3) for the conditional probability of error and the expression for the truncated discrete exponential density function (13) into the expression for the mean value of the probability of error (4), we may get the expression for the probability of error in a closed form as

$$P_e = \int_{-\infty}^{\infty} P_e(\tau) f_{dt}(\tau) d\tau = \sum_{n=1}^{S} P_e(n) \frac{(e^{\lambda}-1)}{1-e^{-S\lambda}} e^{-n\lambda} = \sum_{n=1}^{S} \frac{1}{2} \frac{(e^{\lambda}-1)}{1-e^{-S\lambda}} e^{-n\lambda} \cdot erfc\left(\frac{(\psi-1)}{\beta}\left(\frac{1}{1-n/2S}\right)^2 + \frac{1}{\beta}\left(\frac{n/2S}{1-n/2S}\right)^2 + \frac{1}{4}\left(\frac{1}{1-n/2S}\right)^2\left(\frac{E_b}{N_0}\right)^{-1}\right)^{-1/2} \quad (56)$$

Therefore, using the density functions expressions as suggested in this paper it is relatively easy to solve these integrals and avoid the use of numerical integration.

THIRD CASE: We can also find the probability of error in a DSSS system as the average value of the conditional probability of error presented in (4). For this system, it is impossible to achieve perfect synchronisation of the spreading sequences. Suppose the random delay τ in the system with imperfect synchronisation is characterised by the uniform truncated density function expressed as (50). By inserting (5) and (50) into (4) the average value for probability of error can be found

$$P_e = \int_{-\infty}^{\infty} P_e(\tau) f_{dt}(\tau) d\tau = \frac{1}{2(S-a)+1} \sum_{n=-S+a}^{n=S-a} \int_{-\infty}^{\infty} P_e(\tau)\delta(\tau-n) d\tau = \frac{1}{2(S-a)+1}\sum_{n=1}^{S} P_e(n) = \frac{1}{4(S-a)+2} \cdot \sum_{n=1}^{S} erfc\left(\frac{4(\psi-\pi/4)}{\pi\beta} + \frac{4}{\pi\beta}X(n) + (1+X(n))\frac{2}{\pi b^2}\left(\frac{E_b}{N_0}\right)^{-1}\right)^{-1/2} \quad (57)$$

Because the possible random delays can have any positive and negative value around zero we have been using discrete symmetric density function.

The expression (5) presents the probability of error in DSSS system, which is equivalent to a single-user code division multiple access (CDMA) system. If the CDMA system operates with N users, the expression for the probability of error is

$$P_e(\tau) = \frac{1}{2} erfc\left(\theta + X(\tau)\frac{4N}{\beta\pi} + \left[1 + X(\tau)\right]\frac{2}{b^2\pi}\left(\frac{E_b}{N_0}\right)^{-1}\right)^{-1/2} \quad (58)$$

whereas $\theta = 4(\psi - \pi/4 + N - 1)/\beta\pi$. By inserting (58) and (50) into (4) the average value of the probability of error can be calculated following the procedure presented in (57), resulting in this expression

$$P_e = \int_{-\infty}^{\infty} P_e(\tau) f_{dt}(\tau) d\tau = \frac{1}{4(S-a)+2} \cdot \sum_{n=1}^{S} erfc\left(\theta + \frac{4N}{\pi\beta}X(n) + (1+X(n))\frac{2}{\pi b^2}\left(\frac{E_b}{N_0}\right)^{-1}\right)^{-1/2}. \quad (59)$$

The final notes we may make: Firstly, the expressions for Gaussian and uniform density functions are derived for the assumed zero mean value of random variable. However, it is easy to derive the corresponding expressions for any mean value of the discrete random variables, as was done for an example of asymmetric uniform distribution in subsection 4.3. Secondly, in this paper we presented density functions of discrete random variables in terms of Dirac's delta functions. It is important to note that it is possible to develop and use their expressions using Kronecker's delta functions. Thirdly, in practice, the discrete random variables take the values in limited intervals. Therefore, the use of truncated density functions is necessary. This necessity supports our motivation for writing this paper. Namely, the mean values and variances are generally changing depending on the size of truncating interval, which can have significant influence on our theoretical analysis, simulation and practical design of digital devices.

6 Conclusions

In this paper the expressions for exponential, Gaussian and uniform discrete and truncated discrete density functions, and their first and second moments, are derived. The expressions for the moments are compared with related moments of the continuous and truncated continuous density functions. It was confirmed that the density functions could be expressed in terms of Dirac's delta functions in order to be applied for the calculation of the mean value of a function of random variable. The application of derived densities of discrete random variables is demonstrated on three examples. One example presents a rigorous calculation of the mean and autocorrelation function of a discrete time harmonic process, and two examples demonstrate precise calculations of the probability of error in DSSS communication systems where all signals are represented in discrete time domain.

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