Singular limit solutions for a two-dimensional semilinear elliptic Yamabe system

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Abstract: The existence of singular limit solutions are investigated by establishing a new Liouville type theorem for nonlinear elliptic Yamabe system and by using the nonlinear domain decomposition method.

Key–Words: Liouville type system, singular limit solution, nonlinear domain decomposition method.

1 Introduction

Although the real world seems in a muddle, many phenomena can be described by using nonlinear differential equations. A fundamental goal in the study of non-linear initial boundary value problems involving partial differential equations is to determine whether solutions to a given equation develop a singularity. Resolving the issue of blow-up is important, in part because it can have bearing on the physical relevance and validity of the underlying model.

For example, the nonlinear systems pose a lot of interesting but also challenging mathematical problems, which require people to develop new and deep theories and methods to treat them. For example, for the so-called BEC system, which has cubic nonlinearities and is weakly coupled, the least energy and the ground state have been attracting both physicists and mathematicians. With the deepening of the study on this line, some tough nuts remain uncracked.

Let $\Omega \subset \mathbb{R}^2$ be a regular bounded domain in $\mathbb{R}^2$. We consider the following elliptic system:

$$\begin{cases} -\mathrm{div}(a(u_1)\nabla u_1) = \rho^2 e^{\lambda u_1} e^{u_1+\gamma_1 u_2} \text{ in } \Omega \\ -\mathrm{div}(a(u_2)\nabla u_2) = \rho^2 e^{\lambda u_2} e^{u_2+\gamma_2 u_1} \text{ in } \Omega \\ u_1 = u_2 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

The function a is assumed to be positive and smooth, γ_1, γ_2 and ρ are real constants. We take $a(u_i) = e^{\lambda u_i}$, we assume that $\lambda > 0$ and $\gamma_1 \neq \gamma_2 \in (0, 1)$.

Then problem (1) take the form

$$\begin{cases} -\Delta u_1 - \lambda|\nabla u_1|^2 = \rho^2 e^{u_1+\gamma_1 u_2} \text{ in } \Omega \\ -\Delta u_2 - \lambda|\nabla u_2|^2 = \rho^2 e^{u_2+\gamma_2 u_1} \text{ in } \Omega \\ u_1 = u_2 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2}$$

Using the following transformation

$$\omega_1 = (\lambda\rho^2 e^{u_1})^{\lambda} \text{ and } \omega_2 = (\lambda\rho^2 e^{u_2})^{\lambda},$$

then the function (ω_1, ω_2) satisfies the following problem

$$\begin{cases} -\Delta\omega_1 = \mathcal{Q}_1\, \omega_1^{\frac{\lambda+1}{\lambda}} \omega_2^{\frac{\gamma_1}{\lambda}} \quad \text{in } \Omega \\ -\Delta\omega_2 = \mathcal{Q}_2\, \omega_2^{\frac{\lambda+1}{\lambda}} \omega_1^{\frac{\gamma_2}{\lambda}} \quad \text{in } \Omega \\ \omega_1 = \omega_2 = (\lambda\rho^2)^{\lambda} \quad \text{on } \partial\Omega. \end{cases}$$

With $\mathcal{Q}_i = (\lambda\rho^2)^{\gamma_i}, i = 1, 2.$

This Yamabe system has found considerable interest in recent years as it appears in a number of physical problems, for instance in nonlinear optics. There the coupled solution (ω_1, ω_2) denotes components of the beam in Kerr-like photorefractive media. We have self-focusing in both components of the beam. The nonlinear coupling constant $\mathcal{Q}_i$ is the interaction between the two components of the beam. The case in which the coupling is nonlinear has been studied extensively, which is motivated by applications to nonlinear optic and Bose-Einstein condensation. See for example [9, 10, 11, 19, 20] and references therein.

The purpose of this paper is to prove the existence of solutions (u_1, u_2) for the previous problem. More precisely, we are interested to the existence of solutions with singular limits as the parameters λ, ρ tend to 0.

We denote by ε the smallest positive parameter satisfying

$$\rho^2 = \frac{8\,\varepsilon^2}{(1+\varepsilon^2)^2}.$$

The current paper is mostly related to the papers [1, 3, 5]. We shall use the same approach, namely the nonlinear domain decomposition method, witch has already been used successfully in geometric context (constant mean curvature surfaces, constant scalar curvature metrics, extremal Kahler metrics, ...).

In this paper, we will prove the existence of some singular solution. More precisely, we prove the following result :

Theorem 1 *Let Ω be a regular bounded domain of $\mathbb{R}^2$ and $z_1, \ldots, z_n \in \Omega$ be given disjoint points. Let*

$$H(z,z') = G(z,z') + 4\log|z-z'|$$

be the regular part of G, where the Green's function G defined on $\Omega\times\Omega$ is given by

$$-\Delta G(z,z') = 8\pi\delta_{z'} \ \ in \ \Omega, \quad G(z,z') = 0 \ \ on \ \partial\Omega.$$

Suppose that $(z_1, \ldots, z_n)$ is a nondegenerate critical point of the function

$$\mathcal{F}(z_1,...,z_n) = \frac{1}{2\gamma_1}\sum_{i=1}^{p} H(z_i,z_i) + \frac{1}{2\gamma_2}\sum_{j=p+1}^{n} H(z_j,z_j) + \sum_{i=1}^{p}\sum_{j=p+1}^{n} G(z_i,z_j),$$

then there exist $\rho_0 > 0, \lambda_0 > 0$ and $(u_i^{\rho,\lambda})_{\rho\le\rho_0,\lambda\le\lambda_0}$ a family of solutions of (2), such that

$$\begin{cases} \lim\limits_{\rho,\lambda\to 0} u_1^{\rho,\lambda} = \sum\limits_{i=1}^{p} G(\cdot,z_i) \ in \ \mathcal{C}^{2,\alpha}_{loc}(\Omega\backslash\{z_1,...,z_p\}) \\ \lim\limits_{\rho,\lambda\to 0} u_2^{\rho,\lambda} = \sum\limits_{j=p+1}^{n} G(\cdot,z_j) \ in \ \mathcal{C}^{2,\alpha}_{loc}(\Omega\backslash\{z_{p+1},...,z_n\}). \end{cases}$$

To facilitate the presentation, we will look at the special case where we have only two singular points.

Theorem 2 *Let Ω be a regular bounded domain of $\mathbb{R}^2$ and $z_1,\, z_2 \in \Omega$ be given disjoint points. Let*

$$H(z,z') = G(z,z') + 4\log|z-z'|$$

be the regular part of G, where the Green's function G defined on $\Omega\times\Omega$ is given by

$$-\Delta G(z,z') = 8\pi\delta_{z'} \ \ in \ \Omega, \quad G(z,z') = 0 \ \ on \ \partial\Omega. \tag{3}$$

Suppose that $(z_1,\, z_2)$ is a nondegenerate critical point of the function

$$\mathcal{F}(z_1,z_2) = \frac{1}{2\gamma_1}H(z_1,z_1) + \frac{1}{2\gamma_2}H(z_2,z_2) + G(z_1,z_2),$$

then there exist $\rho_0 > 0, \lambda_0 > 0$ and $(u_i^{\rho,\lambda})_{\rho\le\rho_0,\lambda\le\lambda_0}$ a family of solutions of (2), such that

$$\begin{cases} \lim\limits_{\rho,\lambda\to 0} u_1^{\rho,\lambda} = G(\cdot,z_1) \quad in \ \mathcal{C}^{2,\alpha}_{loc}(\Omega\backslash\{z_1\}) \\ \lim\limits_{\rho,\lambda\to 0} u_2^{\rho,\lambda} = G(\cdot,z_2) \quad in \ \mathcal{C}^{2,\alpha}_{loc}(\Omega\backslash\{z_2\}). \end{cases}$$

2 Proof of Theorem 2

2.1 Construction of the approximate solution

We denote by ε the smallest positive parameter satisfying

$$\rho^2 = \frac{8\,\varepsilon^2}{(1+\varepsilon^2)^2}.$$

Let

$$u_\varepsilon(z) := 2\log\frac{1+\varepsilon^2}{\varepsilon^2+|z|^2} \tag{4}$$

which is a solution of

$$-\Delta u = \rho^2 e^u \ \text{ in } \mathbb{R}^2. \tag{5}$$

Hence for all $\tau > 0$ the function

$$u_{\varepsilon,\tau}(z) := 2\log\frac{\tau(1+\varepsilon^2)}{\varepsilon^2+|\tau z|^2}. \tag{6}$$

is also solution to 5.

2.1.1 Linearized operators

First we introduce some definitions and notations:

Definition 3 *Given $k\in\mathrm{N}$, $\alpha\in(0,1)$, $\mu\in\mathrm{R}$ and $|z| = r$, let $\mathcal{C}^{k,\alpha}_{\mu}(\mathbb{R}^2)$ be the space of functions in $\mathcal{C}^{k,\alpha}_{loc}(\mathbb{R}^2)$ for which the following norm*

$$\|u\|_{\mathcal{C}^{k,\alpha}_{\mu}(\mathbb{R}^2)} = \|u\|_{\mathcal{C}^{k,\alpha}(\bar{B}_1)} + \sup_{r\ge 1}\left(r^{-\mu}\,\|u(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_1\backslash B_{1/2})}\right)$$

is finite. Similarly, for given $\bar{r} \geq 1$, *let* $\mathcal{C}_\mu^{k,\alpha}(B_{\bar{r}})$ *be the space of function in* $\mathcal{C}^{k,\alpha}(B_{\bar{r}})$ *for which the following norm*

$$\|u\|_{\mathcal{C}_\mu^{k,\alpha}(B_{\bar{r}})} = \|u\|_{\mathcal{C}^{k,\alpha}(B_1)} + \sup_{1\leq r\leq \bar{r}} \left(r^{-\mu} \|u(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_1\setminus B_{1/2})} \right),$$

is finite. Finally, set $B_r^*(z) = B_r(z) - \{z\}$, *let* $\mathcal{C}_\mu^{k,\alpha}(\bar{B}_1^*)$ *be the space of functions in* $\mathcal{C}_{loc}^{k,\alpha}(\bar{B}_1^*)$ *for which the following norm*

$$\|u\|_{\mathcal{C}_\mu^{k,\alpha}(\bar{B}_1^*)} = \sup_{r\leq 1/2} \left(r^{-\mu} \|u(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_2\setminus B_1)} \right)$$

is finite.

We define the linear second order elliptic operator L by

$$\mathrm{L} := -\Delta - \frac{8}{(1+\mathrm{r}^2)^2},$$

which is the linearized operator of $-\Delta u - \rho^2 e^u = 0$ about the symmetric solutions $u_{\varepsilon=1,\tau=1}$ defined by (6). When $k \geq 2$, we let $[\mathcal{C}_\mu^{k,\alpha}(\bar{\Omega})]_0$ to be the subspace of functions $w \in \mathcal{C}_\mu^{k,\alpha}(\bar{\Omega})$ satisfying $w = 0$ on $\partial\Omega$.

For all $\lambda, \varepsilon, \tau > 0$, $\gamma_1, \gamma_2 \in (0,1)$, we define

$$\begin{aligned} r_{\varepsilon,\lambda} &:= \max(\lambda^{1/2}, \varepsilon^{1/2}, \varepsilon^{(1-\gamma_1)}, \varepsilon^{(1-\gamma_2)}) \\ R_{\varepsilon,\lambda} &:= \tfrac{\tau r_{\varepsilon,\lambda}}{\varepsilon}. \end{aligned} \tag{7}$$

Proposition 4 *[3] All bounded solutions of* $\mathrm{Lw} = 0$ *on* R^2 *are linear combination of*

$$\phi_0(z) = \frac{1-r^2}{1+r^2} \text{ and } \phi_i(z) = \frac{2z_i}{1+r^2} \text{ for } i = 1,2.$$

Moreover, for $\mu > 1$, $\mu \notin \mathrm{Z}$,

$$\mathrm{L} : \mathcal{C}_\mu^{2,\alpha}(\mathbb{R}^2) \longrightarrow \mathcal{C}_{\mu-2}^{0,\alpha}(\mathbb{R}^2)$$

is surjective.

In the following, we denote by $\mathcal{G}_\mu$ to be a right inverse of L. Similarly, using the fact that any bounded harmonic function in $\mathbb{R}^2$ is constant, we claim

Proposition 5 *Let* $\delta > 0$, $\delta \notin \mathrm{Z}$ *then* Δ *is surjective from* $\mathcal{C}_\delta^{2,\alpha}(\mathrm{R}^2)$ *to* $\mathcal{C}_{\delta-2}^{0,\alpha}(\mathrm{R}^2)$.

We denote by $\mathcal{K}_\delta : \mathcal{C}_{\delta-2}^{0,\alpha}(\mathrm{R}^2) \to \mathcal{C}_\delta^{2,\alpha}(\mathrm{R}^2)$ a right inverse of Δ for $\delta > 0$, $\delta \notin \mathrm{Z}$.

Finally, we consider punctured domains. For $k \in \{1,2\}$ given $\tilde{z}_k \in \Omega$ disjoint points, we define $\tilde{\mathbf{z}} := (\tilde{z}_1, \tilde{z}_2)$ and $\bar{\Omega}^*(\tilde{\mathbf{z}}) := \bar{\Omega}\backslash\{\tilde{z}_k\}$. Let $r_0 > 0$ be small such that $\overline{B}_{r_0}(\tilde{z}_k)$ are disjoint and included in Ω. For all $r \in (0, r_0)$, we define

$$\bar{\Omega}_r(\tilde{\mathbf{z}}) := \bar{\Omega}\backslash \cup_{k=1}^2 B_r(\tilde{z}_k).$$

Definition 6 *Let* $k \in \mathrm{R}, \alpha \in (0,1)$ *and* $\nu \in \mathrm{R}$, *let*

$$\mathcal{C}_\nu^{k,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{z}})) = \mathcal{C}_{loc}^{k,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{z}})) \cap_{i=1,2} \mathcal{C}_\nu^{k,\alpha}(B_r^*(\tilde{z}_i))$$

endowed the following norm

$$\|w\|_{\mathcal{C}_\nu^{k,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{z}}))} := \|w\|_{\mathcal{C}^{k,\alpha}(\bar{\Omega}_{r_0/2}(\tilde{\mathbf{z}}))} + \sum_{i=1}^2 \sup_{0<r\leq\frac{r_0}{2}} \left(r^{-\nu} \|w(\tilde{z}_i + r.)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_2 - B_1)} \right).$$

Furthermore, for $k \geq 2$, *let* $\left[\mathcal{C}_\nu^{k,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{z}}))\right]_0$ *to be the set of* $w \in \mathcal{C}^{k,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{z}}))$ *satisfying* $w = 0$ *on* $\partial\Omega$.

We recall the following result.

Proposition 7 *[12] Let* $\nu < 0$, $\nu \notin \mathrm{Z}$, *then* Δ *is surjective from* $\left[\mathcal{C}_\nu^{2,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{z}}))\right]_0$ *to* $\mathcal{C}_{\nu-2}^{0,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{z}}))$.

We denote by $\tilde{\mathcal{G}}_\nu : \mathcal{C}_{\nu-2}^{0,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{z}})) \to \left[\mathcal{C}_\nu^{2,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{z}}))\right]_0$ a right inverse of Δ for $\nu < 0$, $\nu \notin \mathrm{Z}$.

2.1.2 Ansatz and first estimates

For all $\sigma \geq 1$, we denote by

$$\xi_\sigma : \mathcal{C}_\mu^{0,\alpha}(\bar{B}_\sigma) \longrightarrow \mathcal{C}_\mu^{0,\alpha}(\mathrm{R}^2)$$

the extension operator defined by

$$\xi_\sigma(f)(z) = \begin{cases} f(z) & \text{for } |z| \leq \sigma, \\ \chi\left(\frac{|z|}{\sigma}\right) f\left(\sigma\frac{z}{|z|}\right) & \text{for } |z| \geq \sigma. \end{cases} \tag{8}$$

Here χ is a cut-off function over $\mathbb{R}_+$, which is equal to 1 for $t \leq 1$ and equal to 0 for $t \geq 2$. It easy to check that there exists a constant $c = \bar{c}(\mu) > 0$, independent of $\sigma \geq 1$, such that

$$\|\xi_\sigma(w)\|_{\mathcal{C}_\mu^{0,\alpha}(\mathrm{R}^2)} \leq \bar{c}\,\|w\|_{\mathcal{C}_\mu^{0,\alpha}(\bar{B}_\sigma)}. \tag{9}$$

Here, we are interested in the study of the system 2 near $B(z_1, r_{\varepsilon,\lambda})$

$$\begin{cases} -\Delta u_1 - \lambda|\nabla u_1|^2 = \rho^2 e^{u_1+\gamma_1 u_2} \\ -\Delta u_2 - \lambda|\nabla u_2|^2 = \rho^2 e^{u_2+\gamma_2 u_1}. \end{cases} \tag{10}$$

Using the transformation

$$\begin{cases} v_1(z) = u_1(\frac{\varepsilon}{\tau}z) + 4\ln\varepsilon + 2\ln\frac{2}{\tau(1+\varepsilon^2)}, \\ v_2(z) = u_2(\frac{\varepsilon}{\tau}z), \end{cases} \tag{11}$$

the previous system can be written, in $B(z_1, R_{\varepsilon,\lambda})$, as

$$\begin{cases} -\Delta v_1 - \lambda|\nabla v_1|^2 = 2\, e^{v_1+\gamma_1 v_2} \text{in } B(z_1, R_{\varepsilon,\lambda}), \\ \\ -\Delta v_2 - \lambda|\nabla v_2|^2 = 2\, \dfrac{2^{2(1-\gamma_2)}\varepsilon^{4(1-\gamma_2)}}{(\tau(1+\varepsilon^2))^{2(1-\gamma_2)}}\, e^{v_2+\gamma_2 v_1}. \end{cases} \tag{12}$$

Here $\tau > 0$ is a constant which will be fixed later.

We denote by $\bar{u} = u_{\varepsilon=\tau=1}$, we look for a solution of (12) of the form

$$\begin{cases} v_1(z) &= \bar{u}(z - z_1) - \gamma_1 G(z, z_2) + h_1^1(z) \\ \\ v_2(z) &= G(z, z_2) + h_2^1(z) \end{cases}$$

this amounts to solve the equation

$$\begin{cases} \mathrm{L}h_1^1 &= \dfrac{8}{(1+r^2)^2}\left[e^{h_1^1+\gamma_1 h_2^1} - h_1^1 - 1\right] \\ &\quad +\lambda\left|\nabla(\bar{u}(z-z_1) - \gamma_1 G(z,z_2) + h_1^1)\right|^2, \\ \\ -\Delta h_2^1 &= \dfrac{8C_\varepsilon\, \varepsilon^{4(1-\gamma_2)}}{(1+r^2)^{2\gamma_2}}\, e^{h_2^1+(1-\gamma_1\gamma_2)\,G(z,z_2)+\gamma_2 h_1^1} \\ &\quad +\lambda\left|\nabla(h_2^1 + G(z,z_2))\right|^2 \end{cases} \tag{13}$$

in $B(z_1, R_{\varepsilon,\lambda})$; where $C_\varepsilon = [\tau(1+\varepsilon^2)]^{2(\gamma_2-1)}$.

Fix $\mu \in (1,2)$ and $\delta \in (0, \min\{1,\, 2(1-\gamma_1),\, 2(1-\gamma_2)\})$. To find a solution of (13), it is enough to find a fixed point (h_1^1, h_2^1) in a small ball of $\mathcal{C}^{2,\alpha}_\mu(\mathrm{R}^2) \times \mathcal{C}^{2,\alpha}_\delta(\mathrm{R}^2)$ solutions of

$$h_1^1 = \mathcal{G}_\mu \circ \xi_\mu \circ \mathcal{T}_1(h_1^1, h_2^1), \quad k_2^1 = \mathcal{K}_\delta \circ \xi_\delta \circ \mathcal{T}_2(h_1^1, h_2^1). \tag{14}$$

where

$$\mathcal{T}_1(h_1^1, h_2^1) \quad \text{and} \quad \mathcal{T}_2(h_1^1, h_2^1)$$

are the right hand side in (13). We denote by $\mathcal{N}(=\mathcal{N}_{\varepsilon,\tau})$ and $\mathcal{M}(=\mathcal{M}_{\varepsilon,\tau})$ the nonlinear operators appearing on the right hand side of the equation (14).

Lemma 8 *Given $\kappa > 0$, there exists $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\mu \in (1,2)$ and $\delta \in (0, \min\{1,\, 2(1-\gamma_1),\, 2(1-\gamma_2)\})$*

$$\|\mathcal{N}(0,0)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathrm{R}^2)} \le c_\kappa\, r^2_{\varepsilon,\lambda}, \quad \|\mathcal{M}(0,0)\|_{\mathcal{C}^{2,\alpha}_\delta(\mathrm{R}^2)} \le c_\kappa\, r^2_{\varepsilon,\lambda},$$

$$\|\mathcal{N}(h_1^1, k_2^1) - \mathcal{N}(k_1^1, k_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathrm{R}^2)} \le c_\kappa\, r^2_{\varepsilon,\lambda}\, \|(h_1^1, k_2^1) - (k_1^1, k_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathrm{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathrm{R}^2)}$$

and

$$\|\mathcal{M}(h_1^1, k_2^1) - \mathcal{M}(k_1^1, k_2^1)\|_{\mathcal{C}^{2,\alpha}_\delta(\mathrm{R}^2)} \le c_\kappa\, r^2_{\varepsilon,\lambda}\, \|(h_1^1, k_2^1) - (k_1^1, k_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathrm{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathrm{R}^2)}$$

provided $(h_1^1, k_2^1), (k_1^1, k_2^1) \in \mathcal{C}^{2,\alpha}_\mu(\mathrm{R}^2) \times \mathcal{C}^{2,\alpha}_\delta(\mathrm{R}^2)$ satisfying

$$\begin{aligned} \|(h_1^1, k_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathrm{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathrm{R}^2)} &\le 2c_\kappa r^2_{\varepsilon,\lambda}, \\ \|(k_1^1, k_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathrm{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathrm{R}^2)} &\le 2c_\kappa r^2_{\varepsilon,\lambda}. \end{aligned} \tag{15}$$

Proof: We have

$$\begin{aligned} &\sup_{r \le R_{\varepsilon,\lambda}} r^{2-\mu}\, |\, \mathcal{T}_1(0,0)\, | \\ &\le \sup_{r \le R_{\varepsilon,\lambda}} \lambda r^{2-\mu}\left|\nabla\bar{u}(z-z_1) - \gamma_1 \nabla G(z,z_2)\right|^2 \\ &\le c_\kappa\, r^2_{\varepsilon,\lambda} \end{aligned}$$

Making use of Proposition 4 together with (9), for $\mu \in (1,2)$, we get that there exists $\bar{c}_\kappa$ such that

$$\|\mathcal{N}(0,0)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)} \le c_\kappa\, r^2_{\varepsilon,\lambda}, \tag{16}$$

For the second estimate, we have

$$\begin{aligned} &\sup_{r \le R_{\varepsilon,\lambda}} r^{2-\delta}\, |\, \mathcal{T}_2(0,0)\, | \\ &\le c_\kappa \sup_{r \le R_{\varepsilon,\lambda}} \frac{8C_\varepsilon \varepsilon^{4(1-\gamma_2)}}{(1+r^2)^{2\gamma_2}} r^{2-\delta} + c_\kappa r^{2-\delta}\lambda \\ &\le c_\kappa \sup_{r \le R_{\varepsilon,\lambda}} 8C_\varepsilon \varepsilon^{4(1-\gamma_2)} S(r) + c_\kappa r^{2-\delta}\lambda \end{aligned}$$

where $S(r) = \dfrac{r^{2-\delta}}{(1+r^2)^{2\gamma_2}}$.

If $2 - \delta - 4\gamma_2 \le 0$, then S is bounded on $\mathbb{R}_+$.

If $2 - \delta - 4\gamma_2 > 0$, then $\sup_{[0,\frac{r_\varepsilon}{\varepsilon}[} S(r) = S(\frac{r_\varepsilon}{\varepsilon})$.

We get

$$\begin{aligned} \|\mathcal{M}(0,0)\|_{\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)} &\le c_\kappa(\lambda + \max\{\varepsilon^{4(1-\gamma_2)}, \varepsilon^{2+\delta} r_\varepsilon^{-2-\delta}\}) \\ &\le c_\kappa r^2_{\varepsilon,\lambda} \end{aligned}$$

To derive the third estimate, for h_i, k_i, $i = 1, 2$, verifying (28), we have

$$\sup_{r\le R_{\varepsilon,\lambda}} r^{2-\mu}|\mathcal{T}_1(h_1^1,h_2^1)-\mathcal{T}_1(k_1^1,k_2^1)|$$

$$\le \sup_{r\le R_{\varepsilon,\lambda}} \frac{8r^{2-\mu}}{(1+r^2)^2}\Big|(e^{h_1^1+\gamma_1 h_2^1}-h_1^1-1) - (e^{k_1^1+\gamma_1 k_2^1}-k_1^1-1)\Big|$$

$$+\sup_{r\le R_{\varepsilon,\lambda}} \lambda r^{2-\mu}\Big(|\nabla(\bar u-\gamma_1 G(z,z_2)+h_1^1)|^2 - |\nabla(\bar u-\gamma_1 G(z,z_2)+k_1^1)|^2\Big)$$

$$\le \sup_{r\le R_{\varepsilon,\lambda}} \frac{8r^{2-\mu}}{(1+r^2)^2}\Big[(h_1^1)^2-(k_1^1)^2+\gamma_1|h_2^1-k_2^1|\Big]$$

$$+\sup_{r\le R_{\varepsilon,\lambda}} \lambda r^{2-\mu}|\nabla(h_1^1-k_1^1)| \Big(|\nabla(h_1^1+k_1^1)|+2|\nabla\bar u|+2\gamma_1|\nabla G(z,z_2)|\Big)$$

$$\le \sup_{r\le R_{\varepsilon,\lambda}} \frac{8r^{2-\mu}}{(1+r^2)^2}\Big[r^{2\mu}(\|h_1^1\|_{\mathcal{C}^{2,\alpha}_\mu}+\|k_1^1\|_{\mathcal{C}^{2,\alpha}_\mu}) \|h_1^1-k_1^1\|_{\mathcal{C}^{2,\alpha}_\mu}+\gamma_1 r^\delta\|h_2^1-k_2^1\|_{\mathcal{C}^{2,\alpha}_\delta}\Big]$$

$$+\sup_{r\le R_{\varepsilon,\lambda}} \lambda r^{2-\mu}\|h_1^1-k_1^1\|_{\mathcal{C}^{2,\alpha}_\mu} \Big(r^{2\mu}(\|h_1^1\|_{\mathcal{C}^{2,\alpha}_\mu}+\|k_1^1\|_{\mathcal{C}^{2,\alpha}_\mu})+\frac{8r}{(1+r^2)}+C\Big).$$

We conclude that

$$\begin{aligned}&\|\mathcal{N}(h_1^1,h_2^1)-\mathcal{N}(k_1^1,k_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)}\\ &\le c_\kappa\, r^2_{\varepsilon,\lambda}\,\|(h_1^1,h_2^1)-(k_1^1,k_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)}.\end{aligned} \tag{17}$$

Similarly we get the estimate for

$$\begin{aligned}&\|\mathcal{M}(h_1^1,h_2^1)-\mathcal{M}(k_1^1,k_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)}\\ &\le c_\kappa\, r^2_{\varepsilon,\lambda}\,\|(h_1^1,h_2^1)-(k_1^1,k_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)}.\end{aligned} \tag{18}$$

□

Reducing ε_κ if necessary, we can assume that $\bar c_\kappa r^2_{\varepsilon,\lambda}\le\frac12$ for all $\varepsilon\in(0,\varepsilon_\kappa)$. Therefore (16)-(18) are enough to show that

$$(h_1^1,h_2^1)\mapsto\Big(\mathcal{N}(h_1^1,h_2^1),\mathcal{M}(h_1^1,h_2^1)\Big)$$

is a contraction from the ball

$$\Big\{(h_1^1,h_2^1)\in\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2): \|(h_1^1,h_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)}\le 2c_\kappa\, r^2_{\varepsilon,\lambda}\Big\}$$

into itself and hence a unique fixed point (h_1^1,h_2^1) exists in this set, which is a solution of (14). That is

Proposition 9 *Given $\kappa>0$, there exists $\varepsilon_\kappa>0$ and $c_\kappa>0$ such that for all $\varepsilon\in(0,\varepsilon_\kappa)$, there exists a unique $(h_1^1,h_2^1):=(h^1_{1,\varepsilon,\lambda},h^1_{2,\varepsilon,\lambda})$ solution of* (26) *such that*

$$\|(h_1^1,h_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)}\le 2c_\kappa r^2_{\varepsilon,\lambda}.$$

Hence

$$\begin{cases} v_1(z):=\bar u(z-z_1)-\gamma_1 G(z,z_2)+h_1^1(z)\\ v_2(z):=G(z,z_2)+h_2^1(z).\end{cases}$$

solves (10) *in* $B_{R_{\varepsilon,\lambda}}(z_1)$.

Similarly, we get also

Proposition 10 *Given $\kappa>0$, there exists $\varepsilon_\kappa>0$ and $c_\kappa>0$ such that for any $\varepsilon\in(0,\varepsilon_\kappa)$, there exists a unique $(h_1^2,h_2^2):=(h^2_{1,\varepsilon,\lambda},h^2_{2,\varepsilon,\lambda})$ solution of* (26) *verifying*

$$\|(h_1^2,h_2^2)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)}\le 2c_\kappa r^2_{\varepsilon,\lambda}.$$

Hence

$$\begin{cases} v_1(z):=G(z,z_1)+h_1^2(z)\\ v_2(z):=\bar u(z-z_2)-\gamma_2 G(z,z_1)+h_2^2(z)\end{cases}$$

solves (10) *in* $B_{R_{\varepsilon,\lambda}}(z_2)$.

2.1.3 Harmonic extensions

Next, we will study the properties of interior and exterior harmonic extensions. Given $\varphi\in\mathcal{C}^{2,\alpha}(S^1)$, we define respectively $H^{int}=H^{int}(\varphi\,;\cdot)$ and $H_1^{ext}=H^{ext}(\tilde\varphi_1;\cdot))$ to be the solution of

$$\begin{cases} \Delta H^{int}(\varphi\,;\cdot) &=0 \quad \text{in } B_1,\\ H^{int}(\varphi\,;\cdot) &=\varphi \quad \text{on } \partial B_1.\end{cases} \tag{19}$$

$$\begin{cases} \Delta H^{ext}(\tilde\varphi;\cdot) &=0 \quad \text{in } \mathbb{R}^2\backslash B_1,\\ H^{ext}(\tilde\varphi;\cdot) &=\tilde\varphi \quad \text{on } \partial B_1,\\ \lim_{|z|\to\infty} H^{ext}(\tilde\varphi;z) &=0.\end{cases} \tag{20}$$

We will use also

Definition 11 *Given $k\in\mathbb{N}$, $\alpha\in(0,1)$ and $\nu\in\mathbb{R}$, let $\mathcal{C}^{k,\alpha}_\nu(\mathbb{R}^2\backslash B_1)$ as the space of functions $w\in\mathcal{C}^{k,\alpha}_{loc}(\mathbb{R}^2\backslash B_1)$ for which the following norm*

$$\|w\|_{\mathcal{C}^{k,\alpha}_\nu(\mathbb{R}^2\backslash B_1)}=\sup_{r\ge1}\Big(r^{-\nu}\,\|w(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}_\nu(\bar B_2\backslash B_1)}\Big),$$

is finite.

We denote by $e_1(\theta)=\cos\theta$, $e_2(\theta)=\sin\theta$.

Lemma 12 *[2] There exists* $c > 0$ *such that, for any*

$$\int_{S^1} \varphi dv_{S^1} = 0 \ \text{ and } \ \int_{S^1} \varphi\, e_\ell\, d\theta = 0; \quad \ell = 1, 2, \tag{21}$$

then $\|H^{int}(\varphi\,;\cdot)\|_{\mathcal{C}^{2,\alpha}_2(\bar{B}_1^*)} \leq c\|\varphi\|_{\mathcal{C}^{2,\alpha}(S^1)}$.
Similarly, there exists $c > 0$ *such that if*

$$\int_{S^1} \tilde{\varphi}\, d\theta = 0, \tag{22}$$

then $\|H^{ext}(\tilde{\varphi}\,;\cdot)\|_{\mathcal{C}^{2,\alpha}_{-1}(\mathbb{R}^2\setminus B_1)} \leq c\|\tilde{\varphi}\|_{\mathcal{C}^{2,\alpha}(S^1)}$.

If $F \subset L^2(S^1)$ be a subspace, we denote $F_\perp$ to be the subspace of F which are $L^2(S^1)$-orthogonal to e_1, e_2. We will need the following result:

Lemma 13 *[2] The mapping*

$$\mathcal{P} : \mathcal{C}^{2,\alpha}(S^1)_\perp \to \mathcal{C}^{1,\alpha}(S^1)_\perp$$

defined by

$$\mathcal{P}(\varphi) = \partial_r H^{int}(\varphi) - \partial_r H^{ext}(\varphi)$$

is an isomorphism.

2.2 The nonlinear interior problem

Here, we are interested in the study of the system (2) near $B(z_1, R_{\varepsilon,\lambda})$.

$$\begin{cases} -\Delta v_1 - \lambda|\nabla v_1|^2 = 2\, e^{v_1+\gamma_1 v_2} \\ \\ -\Delta v_2 - \lambda|\nabla v_2|^2 = 2\, \dfrac{2^{2(1-\gamma_2)}\varepsilon^{4(1-\gamma_2)}}{(\tau(1+\varepsilon^2))^{2(1-\gamma_2)}}\, e^{v_2+\gamma_2 v_1}. \end{cases} \tag{23}$$

Here $\tau > 0$ is a constant which will be fixed later.

Given $\varphi^1 := (\varphi_1^1, \varphi_2^1) \in (\mathcal{C}^{2,\alpha}(S^1))^2$ satisfying (21). We denote by $\bar{u} = u_{\varepsilon=\tau=1}$ and write

$$\begin{cases} v_1(z) & = \bar{u}(z - z_1) - \gamma_1 G(\frac{\varepsilon z}{\tau}, z_2) + h_1^1(z) \\ & \quad + H_1^{int,1}(\varphi_1^1, \frac{z-z_1}{R_{\varepsilon,\lambda}}) + v_1^1(z) \\ \\ v_2(z) & = h_2^1(z) + G(\frac{\varepsilon z}{\tau}, z_2) \\ & \quad + H_2^{int,1}(\varphi_2^1, \frac{z-z_1}{R_{\varepsilon,\lambda}}) + v_2^1(z). \end{cases} \tag{24}$$

Using the fact that H^{int} is harmonic and the fact that $2e^{\bar{u}} = \frac{8}{(1+r^2)^2}$, this amounts to solve the equation

$$\begin{cases} \mathbb{L}v_1^1 = \frac{8}{(1+r^2)^2}\left[e^{h_1^1+H_1^{int,1}+v_1^1+\gamma_1\,(h_2^1+H_2^{int,1}+v_2^1)} - v_1^1 - 1\right] \\ \quad + \lambda|\nabla(\bar{u} - \gamma_1 G(\frac{\varepsilon z}{\tau}, z_2) + h_1^1 + H_1^{int,1} + v_1^1)|^2 + \Delta h_1^1, \\ \\ -\Delta v_2^1 = \frac{8C_\varepsilon\, \varepsilon^{4(1-\gamma_2)}}{(1+r^2)^{2\gamma_2}}\, e^{h_2^1+H_2^{int,1}+v_2^1+(1-\gamma_1\gamma_2)G(\frac{\varepsilon z}{\tau}, z_2)+\gamma_2(h_1^1+H_1^{int,1}+v_1^1)} \\ \quad + \lambda|\nabla(h_2^1 + G(\frac{\varepsilon z}{\tau}, z_2) + H_2^{int,1} + v_2^1)|^2 + \Delta h_2^1 \end{cases} \tag{25}$$

where $C_\varepsilon = [\tau(1+\varepsilon^2)]^{2(\gamma_2-1)}$.

Fix $\mu \in (1, 2)$ and

$$\delta \in (0, \min\{1,\ 2(1-\gamma_1),\ 2(1-\gamma_2)\}).$$

To find a solution of (25), it is enough to find a fixed point (v_1^1, v_2^1) in a small ball of $\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2) \times \mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)$ solutions of

$$\begin{cases} v_1^1 = \mathcal{G}_\mu \circ \xi_\mu \circ \Re_1(v_1^1, v_2^1), \\ v_2^1 = \mathcal{K}_\delta \circ \xi_\delta \circ \Re_2(v_1^1, v_2^1). \end{cases} \tag{26}$$

Here ξ_σ is defined in (8), $\mathcal{K}_\delta$, $\mathcal{G}_\mu$ are defined after Propositions 4 and 5; and

$$\Re_1(v_1^1, v_2^1) \ \text{ and } \ \Re_2(v_1^1, v_2^1)$$

is the right hand side given in (25).

We denote by $\aleph (= \aleph_{\varepsilon,\tau,\varphi_2^1})$ and $\Upsilon (= \Upsilon_{\varepsilon,\tau,\varphi_2^1})$ the nonlinear operators appearing on the right hand side of the equation (26).

Given $\kappa > 0$ (whose value will be fixed later on), we further assume that the functions $(\varphi_1^1, \varphi_2^1)$ satisfy

$$||(\varphi_1^1, \varphi_2^1)||_{\mathcal{C}^{2,\alpha}\times\mathcal{C}^{2,\alpha}} \leq \kappa\, r_{\varepsilon,\lambda}^2. \tag{27}$$

Then, we have the following result

Lemma 14 *Given* $\kappa > 0$*, there exists* $\varepsilon_\kappa > 0$ *and* $c_\kappa > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_\kappa)$*,* $\mu \in (1, 2)$ *and* $\delta \in (0, \min\{1,\ 2(1-\gamma_1),\ 2(1-\gamma_2)\})$

$$\|\aleph(0,0)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)} \leq c_\kappa\, r_{\varepsilon,\lambda}^2, \ \ \|\Upsilon(0,0)\|_{\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)} \leq c_\kappa\, r_{\varepsilon,\lambda}^2,$$

$$\begin{aligned} &\|\aleph(v_1^1, v_2^1) - \aleph(t_1^1, t_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)} \\ &\leq c_\kappa\, r_{\varepsilon,\lambda}^2\, \|(v_1^1, v_2^1) - (t_1^1, t_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)} \end{aligned}$$

and

$$\begin{aligned} &\|\Upsilon(v_1^1, v_2^1) - \Upsilon(t_1^1, t_2^1)\|_{\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)} \\ &\leq c_\kappa\, r_{\varepsilon,\lambda}^2\, \|(v_1^1, v_2^1) - (t_1^1, t_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)} \end{aligned}$$

provided $(v_1^1, v_2^1), (t_1^1, t_2^1) \in \mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2) \times \mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)$ *satisfying*

$$\begin{aligned} \|(v_1^1, v_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)} &\leq 2c_\kappa r_{\varepsilon,\lambda}^2, \\ \|(t_1^1, t_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)} &\leq 2c_\kappa r_{\varepsilon,\lambda}^2. \end{aligned} \tag{28}$$

Proof : The first estimate follows from Lemma 12 together with the assumption on the norms of φ_2, we have

$$\begin{aligned}\|H^{int}(\cdot/R_{\varepsilon,\lambda})\|_{\mathcal{C}^{2,\alpha}_2(\bar{B}_{R_{\varepsilon,\lambda}})} &\leq c_\kappa R^{-2}_{\varepsilon,\lambda}\|\varphi_2\|_{\mathcal{C}^{2,\alpha}(\bar{B}_{R_{\varepsilon,\lambda}})}\\ &\leq c_\kappa \varepsilon^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\sup_{r\,\leq\, R_{\varepsilon,\lambda}} r^{2-\mu} \mid \Re_1(0,0)\mid\\ &\leq \sup_{r\,\leq\, R_{\varepsilon,\lambda}} \tfrac{8r^{2-\mu}}{(1+r^2)^2}\left(e^{h_1^1+H_1^{int,1}+\gamma_1(h_2^1+H_2^{int,1})}-1\right)\\ &+\sup_{r\,\leq\, R_{\varepsilon,\lambda}} r^{2-\mu}\\ &\left(\lambda|\nabla(\bar{u}-\gamma_1 G(\tfrac{\varepsilon z}{\tau},z_2)+h_1^1+H_1^{int,1})|^2+\Delta h_1^1\right).\end{aligned}$$

Making use of Corollary 4 together with (9), for $\mu\in(1,2)$, we get that there exists $\bar{c}_\kappa$ such that

$$\|\aleph(0,0)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)} \leq c_\kappa\, r^2_{\varepsilon,\lambda}, \qquad (29)$$

For the second estimate, we have

$$\begin{aligned}&\sup_{r\leq R_{\varepsilon,\lambda}} r^{2-\delta}\mid \Re_2(0,0)\\ &\leq \sup_{r\leq R_{\varepsilon,\lambda}} r^{2-\delta}\frac{8C_\varepsilon\,\varepsilon^{4(1-\gamma_2)}}{(1+r^2)^{2\gamma_2}}\times\\ &e^{h_2^1+H_2^{int,1}+(1-\gamma_1\gamma_2)G(\frac{\varepsilon z}{\tau},z_2)+\gamma_2(h_1^1+H_1^{int,1})}\\ &+\sup_{r\leq R_{\varepsilon,\lambda}} r^{2-\delta}\times\\ &\left(\lambda|\nabla(h_2^1+G(\tfrac{\varepsilon z}{\tau},z_2)+H_2^{int,1})|^2+\Delta h_2^1\right).\end{aligned}$$

Using the same argument of $S(r)$ in proof of Lemma 8, we get

$$\|\Upsilon(0,0)\|_{\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)} \leq c_\kappa r^2_{\varepsilon,\lambda}. \qquad (30)$$

To derive the third estimate, for v_1^1, v_2^1, t_1^1, and t_2^1 verifying (28), we have

$$\begin{aligned}&\sup_{r\leq R_{\varepsilon,\lambda}} r^{2-\mu}|\Re_1(v_1^1,v_2^1)-\Re_1(t_1^1,t_2^1)|\\ &\leq \sup_{r\leq R_{\varepsilon,\lambda}} \frac{8r^{2-\mu}}{(1+r^2)^2}\times\\ &\quad\Big|(e^{v_1^1+H_1^{int,1}+h_1^1+\gamma_1\,(v_2^1+H_2^{int,1}+h_2^1)}-v_1^1-1)\\ &\quad-(e^{t_1^1+H_1^{int,1}+h_1^1+\gamma_1\,(t_2^1+H_2^{int,1}+h_2^1)}-t_1^1-1)\Big|\\ &\quad+\lambda\sup_{r\leq R_{\varepsilon,\lambda}} r^{2-\mu}\Big(|\nabla(v_1^1+\bar{u}-\gamma_1 G(\frac{\varepsilon z}{\tau},z_2)+h_1^1+H_1^{int,1})|^2\\ &\qquad-|\nabla(t_1^1+\bar{u}-\gamma_1 G(\frac{\varepsilon z}{\tau},z_2)+h_1^1+H_1^{int,1})|^2\Big)\\ &\leq \sup_{r\leq R_{\varepsilon,\lambda}} \frac{8r^{2-\mu}}{(1+r^2)^2}\Big[(v_1^1)^2-(t_1^1)^2+\gamma_1|v_2^1-t_2^1|\Big]\\ &\quad+\sup_{r\,\leq\, R_{\varepsilon,\lambda}} \lambda r^{2-\mu}|\nabla(v_1^1-t_1^1)|\times\\ &\quad\Big(|\nabla(v_1^1-t_1^1)|+2|\nabla(\bar{u}-\gamma_1 G(\frac{\varepsilon z}{\tau},z_2)+h_1^1+H_1^{int,1})|\Big)\\ &\leq c\sup_{r\leq R_{\varepsilon,\lambda}} \frac{8r^{2-\mu}}{(1+r^2)^2}\Big[r^{2\mu}(\|v_1^1\|_{\mathcal{C}^{2,\alpha}_\mu}+\|t_1^1\|_{\mathcal{C}^{2,\alpha}_\mu})\|v_1^1-t_1^1\|_{\mathcal{C}^{2,\alpha}_\mu}\\ &\qquad+\gamma_1 r^\delta\|v_2^1-t_2^1\|_{\mathcal{C}^{2,\alpha}_\delta}\Big]\\ &\quad+\sup_{r\leq R_{\varepsilon,\lambda}} \lambda r^{2-\mu}\|v_1^1-t_1^1\|_{\mathcal{C}^{2,\alpha}_\mu}\Big(r^{2\mu}(\|v_1^1\|_{\mathcal{C}^{2,\alpha}_\mu}+\|t_1^1\|_{\mathcal{C}^{2,\alpha}_\mu})+C\Big).\end{aligned}$$

We conclude that

$$\begin{aligned}&\|\aleph(v_1^1,v_2^1)-\aleph(t_1^1,t_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)}\\ &\leq c_\kappa\, r^2_{\varepsilon,\lambda}\,\|(v_1^1,v_2^1)-(t_1^1,t_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)}\end{aligned} \qquad (31)$$

Similarly we get the estimate for

$$\begin{aligned}&\|\Upsilon(v_1^1,v_2^1)-\Upsilon(t_1^1,t_2^1)\|_{\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)}\\ &\leq c_\kappa\, r^2_{\varepsilon,\lambda}\,\|(v_1^1,v_2^1)-(t_1^1,t_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)}.\end{aligned} \qquad (32)$$

□

Reducing ε_κ if necessary, we can assume that $\bar{c}_\kappa r^2_\varepsilon \leq \frac{1}{2}$ for all $\varepsilon\in(0,\varepsilon_\kappa)$. Therefore (29)-(32) and are enough to show that

$$(v_1^1,v_2^1)\mapsto\Big(\aleph(v_1^1,v_2^1),\Upsilon(v_1^1,v_2^1)\Big)$$

is a contraction from the ball

$$\Big\{(v_1^1,v_2^1)\in\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2):\\ \|(v_1^1,v_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)}\leq 2c_\kappa\, r^2_{\varepsilon,\lambda}\Big\}$$

into itself and hence a unique fixed point (v_1^1,v_2^1) exists in this set, which is a solution of (26). That is

Proposition 15 *Given $\kappa > 0$, there exists $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, for all $\tau_1 \in [\tau_1^-, \tau_1^+] \subset (0, \infty)$ and for a given φ^1 satisfying (21) and (27), there exists a unique*

$$(v_1^1, v_2^1) := (v_{1,\varepsilon,\tau_1,\varphi^1}, v_{2,\varepsilon,\tau_1,\varphi^1})$$

solution of (26) *such that*

$$\|(v_1^1, v_2^1)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)} \le 2c_\kappa r^2_{\varepsilon,\lambda}.$$

Hence

$$\begin{cases} v_1(z) & := \bar{u}(z - z_1) - \gamma_1 G(\frac{\varepsilon z}{\tau}, z_2) + h_1^1(z) \\ & \quad + H_1^{int,1}(\varphi_1^1, \frac{(z-z_1)}{R^1_{\varepsilon,\lambda}}) + v_1^1(z) \\ & \\ v_2(z) & := h_2^1(z) + G(\frac{\varepsilon z}{\tau}, z_2) \\ & \quad + H_2^{int,1}(\varphi_2^1, \frac{(z-z_1)}{R^1_{\varepsilon,\lambda}}) + v_2^1(z) \end{cases}$$

solves (23) *in $B_{R_{\varepsilon,\lambda}}(z_1)$.*

Similarly, we get also

Proposition 16 *Given $\kappa > 0$ and $\tau_2 \in [\tau_2^-, \tau_2^+] \subset (0, \infty)$, there exists $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ such that for any $\varepsilon \in (0, \varepsilon_\kappa)$, any φ^2 satisfying* (21) *and* (27)*, there exists a unique $(v_1^2, v_2^2)(:= (v_{1,\varepsilon,\tau_2,\varphi^2}, v_{2,\varepsilon,\tau_2,\varphi^2}))$ solution of* (26) *such that*

$$\|(v_1^2, v_2^2)\|_{\mathcal{C}^{2,\alpha}_\mu(\mathbb{R}^2)\times\mathcal{C}^{2,\alpha}_\delta(\mathbb{R}^2)} \le 2c_\kappa r^2_{\varepsilon,\lambda}.$$

Hence,

$$\begin{cases} v_1(z) & := h_1^2(z) + G(\frac{\varepsilon z}{\tau}, z_1) \\ & \quad + H_1^{int,2}(\varphi_1^2, \frac{(z-z_2)}{R^2_{\varepsilon,\lambda}}) + v_1^2(z) \\ & \\ v_2(z) & := \bar{u}(z - z_2) - \gamma_2 G(\frac{\varepsilon z}{\tau}, z_1) + h_2^2(z) \\ & \quad + H_2^{int,2}(\varphi_2^2, \frac{(z-z_2)}{R^2_{\varepsilon,\lambda}}) + v_2^2(z) \end{cases}$$

solves (23) *in $B_{R_{\varepsilon,\lambda}}(z_2)$.*

Remark also that the functions v_1^1, v_2^1, v_1^2 and v_2^2, obtained in the above Proposition, depend continuously on the parameter τ.

2.3 The nonlinear exterior problem

Given $\tilde{\mathbf{z}} := (\tilde{z}_1, \tilde{z}_2) \in \Omega^2$ close to $\mathbf{z} := (z_1, z_2)$, $\lambda_k \in \mathbb{R}$ close to 0, $\tilde{\varphi}_{\mathbf{1}} := (\tilde{\varphi}_1^1, \tilde{\varphi}_1^2) \in (\mathcal{C}^{2,\alpha}(S^1))^2$ and $\tilde{\varphi}_{\mathbf{2}} := (\tilde{\varphi}_2^1, \tilde{\varphi}_2^2) \in (\mathcal{C}^{2,\alpha}(S^1))^2$ satisfying (22). Define

$$\begin{cases} \tilde{\mathbf{w}}_{\mathbf{1}} & := (1 + \lambda_1) G(\cdot, \tilde{z}_1) \\ & \quad + \sum_{k=1}^{2} \chi_{r_0}(\cdot - \tilde{z}_k) H_1^{ext}(\tilde{\varphi}_1^k; (\cdot - \tilde{z}_k)/r_\varepsilon) \\ \tilde{\mathbf{w}}_{\mathbf{2}} & := (1 + \lambda_2) G(\cdot, \tilde{z}_2) \\ & \quad + \sum_{k=1}^{2} \chi_{r_0}(\cdot - \tilde{z}_k) H_2^{ext}(\tilde{\varphi}_2^k; (\cdot - \tilde{z}_k)/r_\varepsilon). \end{cases} \tag{33}$$

Here χ_{r_0} is a cut-off function identically equal to 1 in $B_{r_0/2}$ and identically equal to 0 outside B_{r_0}. We would like to find a solution of the system

$$\begin{cases} -\Delta u_1 - \lambda|\nabla u_1|^2 = \rho^2 e^{u_1+\gamma_1 u_2}, \\ -\Delta u_2 - \lambda|\nabla u_2|^2 = \rho^2 e^{u_2+\gamma_2 u_1} \end{cases} \tag{34}$$

in the domain $\overline{\Omega}_{r_\varepsilon}(\tilde{z})$ with $u_1 = \tilde{\mathbf{w}}_{\mathbf{1}} + \tilde{v}_1$ perturbation of $\tilde{\mathbf{w}}_{\mathbf{1}}$ and $u_2 = \tilde{\mathbf{w}}_{\mathbf{2}} + \tilde{v}_2$ perturbation of $\tilde{\mathbf{w}}_{\mathbf{2}}$.

This amounts to solve in $\overline{\Omega}_{r_\varepsilon}(\tilde{z})$

$$\begin{cases} -\Delta \tilde{v}_1 & = \rho^2 \big(e^{\tilde{\mathbf{w}}_1+\tilde{v}_1+\gamma_1(\tilde{\mathbf{w}}_2+\tilde{v}_2)}\big) \\ & \quad + \lambda|\nabla(\tilde{\mathbf{w}}_1 + \tilde{v}_1)|^2 + \Delta\tilde{\mathbf{w}}_1, \\ & \\ -\Delta \tilde{v}_2 & = \rho^2 \big(e^{\tilde{\mathbf{w}}_2+\tilde{v}_2+\gamma_2(\tilde{\mathbf{w}}_1+\tilde{v}_1)}\big) \\ & \quad + \lambda|\nabla(\tilde{\mathbf{w}}_2 + \tilde{v}_2)|^2 + \Delta\tilde{\mathbf{w}}_2. \end{cases} \tag{35}$$

For all $\sigma \in (0, r_0/2)$ and all $Y = (y_1, y_2) \in \Omega^2$ such that $\|Z - Y\| \le r_0/2$, where $Z = (z_1, z_2)$, we denote by $\tilde{\xi}_{\sigma,Y} : \mathcal{C}^{0,\alpha}_\nu(\bar{\Omega}_{\sigma,Y}) \longrightarrow \mathcal{C}^{0,\alpha}_\nu(\bar{\Omega}^*(Y))$ the extension operator Which is equal to

$$\begin{cases} f(z) & \text{in } \bar{\Omega}(Y), \\ \tilde{\chi}\Big(\dfrac{|z - y_j|}{\sigma}\Big) f\Big(y_j + \sigma\dfrac{z - y_j}{|z - y_j|}\Big) & \text{in } B_\sigma(y_j) - B_{\sigma/2}(y_j), \\ 0 & \text{in } B_{\sigma/2}(y_1) \cup B_{\sigma/2}(y_2). \end{cases}$$

Here $\tilde{\chi}$ is a cut-off function over $\mathbb{R}_+$ which is equal to 1 for $t \ge 1$ and equal to 0 for $t \le 1/2$. Obviously, there exists a constant $\bar{c} = \bar{c}(\nu) > 0$ only depending on ν, such that

$$\Big\|\tilde{\xi}_{\sigma,Y}(w)\Big\|_{\mathcal{C}^{0,\alpha}_\nu(\bar{\Omega}^*(Y))} \le \bar{c}\,\|w\|_{\mathcal{C}^{0,\alpha}_\nu(\bar{\Omega}_\sigma(Y))}. \tag{36}$$

We fix $\nu \in (-1, 0)$, to solve (35), it is enough to find $(\tilde{v}_1, \tilde{v}_2) \in (\mathcal{C}^{2,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{z}})))^2$ solution of

$$\begin{cases} \tilde{v}_1 = \widetilde{\mathcal{K}}_\nu \circ \tilde{\xi}_{r_\varepsilon} \circ \tilde{S}_1(\tilde{v}_1, \tilde{v}_2), \\ \tilde{v}_2 = \widetilde{\mathcal{K}}_\nu \circ \tilde{\xi}_{r_\varepsilon} \circ \tilde{S}_2(\tilde{v}_1, \tilde{v}_2). \end{cases} \tag{37}$$

where

$$\tilde{S}_1(\tilde{v}_1,\, \tilde{v}_2) \quad \text{and} \quad \tilde{S}_2(\tilde{v}_1,\, \tilde{v}_2)$$

is the right hand side in (35). We denote by

$$\begin{cases} \tilde{\mathcal{N}}(\tilde{v}_1,\tilde{v}_2) = \widetilde{\mathcal{K}}_\nu \circ \tilde{\xi}_{r_\varepsilon} \circ \tilde{S}_1(\tilde{v}_1,\tilde{v}_2) \\ \tilde{\mathcal{M}}(\tilde{v}_1,\tilde{v}_2) = \widetilde{\mathcal{K}}_\nu \circ \tilde{\xi}_{r_\varepsilon} \circ \tilde{S}_2(\tilde{v}_1,\tilde{v}_2). \end{cases}$$

Given $\kappa > 0$ (whose value will be fixed later on), assume that for $k \in \{1,2\}$ the functions $(\tilde{\varphi}_1^k, \tilde{\varphi}_2^k)$, the parameters λ_k and the point $\tilde{\mathbf{z}} = (\tilde{z}_1, \tilde{z}_2)$ satisfy

$$\|(\tilde{\varphi}_1^k, \tilde{\varphi}_2^k)\|_{\mathcal{C}^{2,\alpha}\times\mathcal{C}^{2,\alpha}} \leq \kappa r_\varepsilon^2 \tag{38}$$

$$|\lambda_k| \leq \kappa r_\varepsilon^2, \quad |\tilde{z}_k - z_k| \leq \kappa r_\varepsilon \tag{39}$$

Then, the following result holds

Lemma 17 *Under the above assumptions, there exists a constant $c_\kappa > 0$ such that*

$$\|\tilde{\mathcal{N}}(0,0)\|_{\mathcal{C}^{2,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{z}}))} \leq c_\kappa r_\varepsilon^2,$$

$$\|\tilde{\mathcal{M}}(0,0)\|_{\mathcal{C}^{2,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{z}}))} \leq c_\kappa r_\varepsilon^2,$$

$$\|\tilde{\mathcal{N}}(\tilde{v}_1,\,\tilde{v}_2) - \tilde{\mathcal{N}}(\tilde{v}_1',\,\tilde{v}_2')\|_{\mathcal{C}^{2,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{z}}))} \leq c\, r_\varepsilon^2\, \|(\tilde{v}_1,\,\tilde{v}_2) - (\tilde{v}_1',\,\tilde{v}_2')\|_{(\mathcal{C}^{2,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{z}})))^2}$$

and

$$\|\tilde{\mathcal{M}}(\tilde{v}_1,\,\tilde{v}_2) - \tilde{\mathcal{M}}(\tilde{v}_1',\,\tilde{v}_2')\|_{\mathcal{C}^{2,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{z}}))} \leq c\, r_\varepsilon^2\, \|(\tilde{v}_1,\,\tilde{v}_2) - (\tilde{v}_1',\,\tilde{v}_2')\|_{(\mathcal{C}^{2,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{z}})))^2}$$

provided that $\big(\tilde{v}_1,\tilde{v}_2,\tilde{v}_1',\tilde{v}_2'\big) \in \big(\mathcal{C}^{2,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{z}}))\big)^4$ satisfy

$$\|(\tilde{v}_1,\tilde{v}_2)\|_{(\mathcal{C}^{2,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{z}})))^2} \leq 2c_\kappa r_\varepsilon^2, \tag{40}$$

$$\|(\tilde{v}_1',\tilde{v}_2')\|_{(\mathcal{C}^{2,\alpha}_\nu(\bar{\Omega}^*(\tilde{\mathbf{z}})))^2} \leq 2c_\kappa r_\varepsilon^2. \tag{41}$$

Proof : As for the interior problem, the two first estimates follows from the asymptotic behavior of H^{ext} together with the assumption on the norm of the boundary data $\tilde{\varphi}_1^k, \tilde{\varphi}_2^k, k = 1,2$, given by (38). Indeed, let c_κ denote a constant depending only on κ, by Lemma 1,

$$|H^{ext}(\tilde{\varphi}^k; (z-\tilde{z}_k)/r_\varepsilon)| \leq c_\kappa r_\varepsilon^3 r^{-1}. \tag{42}$$

On the other hand,

$$\tilde{S}_1(0,0) = \rho^2\big(e^{\tilde{\mathbf{w}}_1 + \gamma_1\,\tilde{\mathbf{w}}_2}\big) + \lambda|\tilde{\mathbf{w}}_1|^2 + \,\Delta\,\tilde{\mathbf{w}}_1$$

and

$$\tilde{S}_2(0,0) = \rho^2\big(e^{\tilde{\mathbf{w}}_2 + \gamma_2\,\tilde{\mathbf{w}}_1}\big) + \lambda|\tilde{\mathbf{w}}_2|^2 + \,\Delta\,\tilde{\mathbf{w}}_2.$$

We will estimate $\tilde{S}_1(0,0)$ in different subregions of $\bar{\Omega}^*(\tilde{\mathbf{z}})$.

• In $B_{r_0/2}(\tilde{z}_1)\backslash B_{r_\varepsilon}(\tilde{z}_1)$ we have $\chi_{r_0}(z-\tilde{z}_1) = 1$, $\chi_{r_0}(z-\tilde{z}_2) = 0$ and $\Delta\,\tilde{\mathbf{w}}_1 = 0$, so that $\tilde{S}_1(0,0) = \rho^2 e^{\tilde{w}_1+\gamma_1\tilde{w}_2}$. Then

$$\begin{aligned} &|\tilde{S}_1(0,0)| \\ &\leq c_\kappa\varepsilon^2\,|z-\tilde{z}_1|^{-4(1+\lambda_1)}\,|z-\tilde{z}_2|^{-4\gamma_1(1+\lambda_2)} + \lambda|\nabla\tilde{\mathbf{w}}_1|^2 \\ &\leq c_\kappa\varepsilon^2\,|z-\tilde{z}_1|^{-4(1+\lambda_1)} \\ &\quad + c_\kappa\lambda\Big|(1+\lambda_1)r^{-1} + (1+\lambda_1)|\nabla H(z,\tilde{z}_1)| \\ &\quad + |H^{ext}(\tilde{\varphi}_1^1;(z-\tilde{z}_1)/r_\varepsilon)|\Big|^2 \\ &\leq c_\kappa\varepsilon^2 r^{-4(1+\lambda_1)} \\ &\quad + c_\kappa\lambda\Big((1+\lambda_1)r^{-1} + (1+\lambda_1)\log r + r_{\varepsilon,\lambda}^2 r^{-2}\Big)^2. \end{aligned}$$

Hence, for $\nu \in (-1,0)$ and λ_1 small enough, we get

$$\begin{aligned} \|\tilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(B_{r_0}(\tilde{z}_1))} &\leq \sup_{r_\varepsilon\leq r\leq r_0/2} r^{2-\nu}|\tilde{S}_1(0,0)| \\ &\leq c_\kappa\varepsilon^2 r_\varepsilon^{-2} + c_\kappa\lambda. \end{aligned}$$

• In $B_{r_0}(\tilde{z}_1)\backslash B_{r_0/2}(\tilde{z}_1)$, using the estimate (42), there holds

$$\begin{aligned} &|\tilde{S}_1(0,0)| \leq c_\kappa\varepsilon^2|z-z_1|^{-4(1+\lambda_1)} \\ &+c_\kappa\lambda\Big((1+\lambda_1)r^{-1} + (1+\lambda_1)\log r + r_{\lambda,\varepsilon}^2 r^{-2}\Big)^2 \\ &+ |[\Delta,\chi_{r_0}(z-\tilde{z}_1)]|\,\big|H^{ext}(\tilde{\varphi}_1^k;(z-\tilde{z}_1)/r_\varepsilon)\big| \\ &\leq c_\kappa\varepsilon^2 + c_\kappa\lambda\Big((1+\lambda_1)r^{-1} + (1+\lambda_1)\log r + r_{\lambda,\varepsilon}^2 r^{-2}\Big)^2 \\ &+c_\kappa r^{-1} r_\varepsilon^3 \end{aligned}$$

where $[\Delta,\chi_{r_0}]\,w = \Delta w\chi_{r_0} + w\Delta\chi_{r_0} + 2\nabla w.\nabla\chi_{r_0}$. Hence, for $\nu \in (-1,0)$ and λ_1 small enough, we get

$$\begin{aligned} &\|\tilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(B_{r_0}(\tilde{z}_k)\backslash B_{r_0/2}(\tilde{z}_1))} \\ &\leq \sup_{r_0/2\leq r\leq r_0} r^{2-\nu}|\tilde{S}_1(0,0)| \\ &\leq c_\kappa\varepsilon^2 r_\varepsilon^2 + c_\kappa\lambda. \end{aligned}$$

• Similarly, we can prove that in $B_{r_0/2}(\tilde{z}_2)\backslash B_{r_\varepsilon}(\tilde{z}_2)$, for $\nu \in (-1,0)$ and λ_2 small enough, we have

$$\begin{aligned} &\|\tilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(B_{r_0}(\tilde{z}_2))} \\ &\leq \sup_{r_\varepsilon\leq r\leq r_0/2} r^{2-\nu}|\tilde{S}_1(0,0)| \\ &\leq c_\kappa\varepsilon^2 r_\varepsilon^{-2} + c_\kappa\lambda, \end{aligned}$$

and in $B_{r_0}(\tilde{z}_2)\backslash B_{r_0/2}(\tilde{z}_2)$

$$\begin{aligned} &\|\tilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(B_{r_0}(\tilde{z}_2)\backslash B_{r_0/2}(\tilde{z}_2))} \\ &\leq \sup_{r_0/2\leq r\leq r_0} r^{2-\nu}|\tilde{S}_1(0,0)| \\ &\leq c_\kappa\varepsilon^2 r_\varepsilon^2 + c_\kappa\lambda. \end{aligned}$$

• In $\Omega - \cup_{k=1}^2 \overline{B}_{r_0}(\tilde{z}_k)$, we have $\chi_{r_0}(z-\tilde{z}_k) = 0$ for $k \in \{1,2\}$ and $\Delta\tilde{\mathbf{w}}_1 = 0$. Thus

$$|\tilde{S}_1(0,0)| \leq c_\kappa\varepsilon^2 e^{(1+\lambda_1)G(z,\tilde{z}_1)} e^{\gamma_1(1+\lambda_2)G(z,\tilde{z}_2)}.$$

So for $\nu \in (-1, 0)$, we have

$$\|\tilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\overline{\Omega} - \cup_{k=1}^2 B_{r_0}(\tilde{z}_k))} \le \sup_{r_0 \le r} r^{2-\nu} |\tilde{S}_1(0,0)| \le c_\kappa \varepsilon^2.$$

So $\|\tilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-2}(\overline{\Omega}_{r_0}(\tilde{\mathbf{z}}))} \le c_\kappa r_\varepsilon^2$. Making use of Proposition 7 together with (36) we conclude that

$$\begin{aligned} \|\tilde{\mathcal{N}}(0,0)\|_{\mathcal{C}^{2,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{z}}))} \le c_\kappa r_\varepsilon^2, \\ \|\tilde{\mathcal{M}}(0,0)\|_{\mathcal{C}^{2,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{z}}))} \le c_\kappa r_\varepsilon^2. \end{aligned} \tag{43}$$

For the proof of the third estimate, let $\tilde{v}_1, \tilde{v}_2, \tilde{v}_1'$ and $\tilde{v}_2' \in \mathcal{C}^{2,\alpha}_{\nu}(\bar{\Omega}^*)$ satisfy (40), we have

$$\begin{aligned} &|\tilde{S}_1(\tilde{v}_1, \tilde{v}_2) - \tilde{S}_1(\tilde{v}_1', \tilde{v}_2')| \\ &\le c_\kappa \varepsilon^2 e^{\tilde{w}_1 + \gamma_1 \tilde{w}_2} \left| e^{\tilde{v}_1 + \gamma_1 \tilde{v}_2} - e^{\tilde{v}_1' + \gamma_1 \tilde{v}_2'} \right| \\ &\quad + c_\kappa \lambda |\nabla \tilde{v}_1 - \nabla \tilde{v}_1'| |\nabla(\tilde{v}_1 + \tilde{v}_1' + 2\tilde{w}_1)| \\ &\le c_\kappa \varepsilon^2 r^{-4(1+\lambda_k)} \Big(|\tilde{v}_1 - \tilde{v}_1'| + |\tilde{v}_2 - \tilde{v}_2'| \Big) \\ &\quad + c_\kappa \lambda |\nabla \tilde{v}_1 - \nabla \tilde{v}_1'|. \end{aligned}$$

Then for λ_k small enough, $k \in \{1, 2\}$, using the estimate (36), there exist $\overline{c}_\kappa$ (depending on κ) such that

$$\begin{aligned} &\|\tilde{\mathcal{N}}(\tilde{v}_1, \tilde{v}_2) - \tilde{\mathcal{N}}(\tilde{v}_1', \tilde{v}_2')\|_{\mathcal{C}^{2,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{z}}))} \\ &\le \overline{c}_\kappa \; r_\varepsilon^2 \Big(\|\tilde{v}_1 - \tilde{v}_1'\|_{C^{2,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{z}}))} + \|\tilde{v}_2 - \tilde{v}_2'\|_{C^{2,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{z}}))} \Big). \end{aligned} \tag{44}$$

and

$$\begin{aligned} &\|\tilde{\mathcal{M}}(\tilde{v}_1, \tilde{v}_2) - \tilde{\mathcal{M}}(\tilde{v}_1', \tilde{v}_2')\|_{\mathcal{C}^{2,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{z}}))} \\ &\le \overline{c}_\kappa r_\varepsilon^2 \Big(\|\tilde{v}_1 - \tilde{v}_1'\|_{C^{2,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{z}}))} + \|\tilde{v}_2 - \tilde{v}_2'\|_{C^{2,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{z}}))} \Big). \end{aligned} \tag{45}$$

The proof is completed. □

Reducing ε_κ if necessary, we can assume that $\overline{c}_\kappa r^2_{\varepsilon,\gamma_1,\gamma_2} \le \frac{1}{2}$ for all $\varepsilon \in (0, \varepsilon_\kappa)$. Then, (44) and (45) are enough to show that

$$(\tilde{v}_1, \tilde{v}_2) \mapsto \Big(\tilde{\mathcal{N}}(\tilde{v}_1, \tilde{v}_2), \tilde{\mathcal{M}}(\tilde{v}_1, \tilde{v}_2) \Big)$$

is a contraction from the ball

$$\Big\{ (\tilde{v}_1, \tilde{v}_2) \in \Big(C^{2,\alpha}_{\nu}(\mathbb{R}^2) \Big)^2 : \|(\tilde{v}_1, \tilde{v}_2)\|_{(C^{2,\alpha}_{\nu}(\mathbb{R}^2))^2} \le 2\overline{c}_\kappa r^2_{\varepsilon,\gamma_1,\gamma_2} \Big\}$$

into itself. Hence there exist a unique fixed point $(\tilde{v}_1, \tilde{v}_2)$ in this set, which is a solution of (37). We conclude then

Proposition 18 *Given $\kappa > 0$, there exists $\varepsilon_\kappa > 0$ (depending on κ) such that for any $\varepsilon \in (0, \varepsilon_\kappa)$; λ_k and $\tilde{z}_k$ satisfying (39); any functions $\tilde{\varphi}_1^k$ and $\tilde{\varphi}_2^k$ satisfying (22) and (38), there exists a unique $(\tilde{v}_1, \tilde{v}_2)(:= (\tilde{v}_{1,\varepsilon,\lambda,\tilde{\mathbf{z}},\tilde{\varphi}}, \tilde{v}_{2,\varepsilon,\lambda,\tilde{\mathbf{z}},\tilde{\varphi}}))$ solution of (37) so that for (v_1, v_2) defined by*

$$\left\{ \begin{aligned} &v_1(z) := (1 + \lambda_1) G(\cdot, \tilde{z}_1) \\ &\quad + \sum_{k=1}^2 \chi_{r_0}(\cdot - \tilde{z}_k) H_1^{ext}(\tilde{\varphi}_1^k; (\cdot - \tilde{z}_k)/r_\varepsilon) + \tilde{v}_1 \\ &v_2(z) := (1 + \lambda_2) G(\cdot, \tilde{z}_2) \\ &\quad + \sum_{k=1}^2 \chi_{r_0}(\cdot - \tilde{z}_k) H_2^{ext}(\tilde{\varphi}_2^k; (\cdot - \tilde{z}_k)/r_\varepsilon) + \tilde{v}_2 \end{aligned} \right.$$

solve (34) in $\overline{\Omega}_{r_\varepsilon}(\tilde{\mathbf{z}})$. In addition, we have

$$\|(\tilde{v}_1, \tilde{v}_2)\|_{C^{2,\alpha}_{\nu}(\bar{\Omega}^*(\tilde{\mathbf{z}}))} \le 2\overline{c}_\kappa r_\varepsilon^2.$$

2.4 The nonlinear Cauchy-data matching

We will gather the results of previous sections. Using the previous notations, assume that $\tilde{\mathbf{z}} := (\tilde{z}_1, \tilde{z}_2) \in \Omega^2$ are given close to $\mathbf{z} := (z_1, z_2)$. Assume also that

$$\tau_k \in [\tau_k^-, \tau_k^+] \subset (0, \infty) \;\text{ for }\; k \in \{1, 2\}$$

are given (the values of τ_k^- and τ_k^+ will be fixed later). First, we consider some set of boundary data $\varphi^{\mathbf{k}} := (\varphi_1^k, \varphi_2^k) \in (\mathcal{C}^{2,\alpha}(S^1))^2$. According to the result of Proposition 16 and provided $\varepsilon \in (0, \varepsilon_\kappa)$, we can find, $u_{int} := (u_{int,1}, u_{int,2})$ a solution of

$$\left\{ \begin{aligned} -\Delta u_1 - \lambda |\nabla u_1|^2 = \rho^2 e^{u_1 + \gamma_1 u_2} \\ -\Delta u_2 - \lambda |\nabla u_2|^2 = \rho^2 e^{u_2 + \gamma_2 u_1} \end{aligned} \right. \tag{46}$$

in $\bigcup_{k=1}^2 B_{r_\varepsilon}(\tilde{z}_k)$, which can be decomposed as $u_{int,1}(z) =$

$$\left\{ \begin{aligned} &u_{\varepsilon,\tau_1}(z - \tilde{z}_1) - \gamma_1 G(z, \tilde{z}_2) + h_1^1 + v_1^1 + H_1^{int,1} \text{ in } B_{r_\varepsilon}(\tilde{z}_1) \\ &G(z, \tilde{z}_1) + h_1^2 + v_1^2 + H_1^{int,2} \text{ in } B_{r_\varepsilon}(\tilde{z}_2) \end{aligned} \right.$$

and $u_{int,2}(z) =$

$$\left\{ \begin{aligned} &G(z, \tilde{z}_2) + h_2^1 + v_2^1 + H_2^{int,1} \text{ in } B_{r_{\varepsilon,\lambda}}(\tilde{z}_1) \\ &u_{\varepsilon,\tau_2}(z - \tilde{z}_2) - \gamma_2 G(z, \tilde{z}_1) + h_2^2 + v_2^2 + H_2^{int,2} \text{ in } B_{r_{\varepsilon,\lambda}}(\tilde{z}_2) \end{aligned} \right.$$

where the functions $h_1^1, h_1^2, v_1^1, v_1^2, h_2^1, h_2^2, v_2^1$ and v_2^2 satisfying

$$\|(h_1^1, h_1^2)\|_{(\mathcal{C}^{2,\alpha}_{\mu}(\mathbb{R}^2))^2} \le 2c_\kappa r^2_{\varepsilon,\lambda},$$

$$\|(h_2^1, h_2^2)\|_{(\mathcal{C}_\delta^{2,\alpha}(\mathbb{R}^2))^2} \leq 2c_\kappa r_{\varepsilon,\lambda}^2,$$

$$\|(v_1^1, v_1^2)\|_{(\mathcal{C}_\mu^{2,\alpha}(\mathbb{R}^2))^2} \leq 2c_\kappa r_{\varepsilon,\lambda}^2,$$

and

$$\|(v_2^1, v_2^2)\|_{(\mathcal{C}_\delta^{2,\alpha}(\mathbb{R}^2))^2} \leq 2c_\kappa r_{\varepsilon,\lambda}^2.$$

Similarly, for $k \in \{1,2\}$, given boundary data $\tilde{\varphi}_1^k, \tilde{\varphi}_2^k \in \mathcal{C}^{2,\alpha}(S^1)$ satisfying (22), $\lambda_k \in \mathbb{R}$ satisfying (39), provided $\varepsilon \in (0, \varepsilon_\kappa)$, by Proposition 18, we find a solution $u_{ext} := (u_{ext,1}, u_{ext,2})$ of (46) in $\bar{\Omega} - \bigcup_{k=1}^2 B_{r_{\varepsilon,\lambda}}(\tilde{z}_k)$, which can be decomposed as

$$\begin{cases} u_{ext,1}(z) := (1+\lambda_1)G(\cdot, \tilde{z}_1) \\ \quad + \sum_{k=1}^2 \chi_{r_0}(\cdot - \tilde{z}_k) H_1^{ext}(\tilde{\varphi}_1^k; (\cdot - \tilde{z}_k)/r_{\varepsilon,\lambda}) + \tilde{v}_1 \\ u_{ext,2}(z) := (1+\lambda_2)G(\cdot, \tilde{z}_2) \\ \quad + \sum_{k=1}^2 \chi_{r_0}(\cdot - \tilde{z}_k) H_2^{ext}(\tilde{\varphi}_2^k; (\cdot - \tilde{z}_k)/r_{\varepsilon,\lambda}) + \tilde{v}_2 \end{cases}$$

with $\tilde{v}_1, \tilde{v}_2 \in C_\nu^{2,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{z}}))$ satisfying

$$\|(\tilde{v}_1, \tilde{v}_2)\|_{(C_\nu^{2,\alpha}(\bar{\Omega}^*(\tilde{\mathbf{z}})))^2} \leq 2\bar{c}_\kappa r_{\varepsilon,\lambda}^2.$$

It remains to determine the parameters and the boundary data in such a way that the function equal to u_{int} in $\cup_{k=1}^2 B_{r_{\varepsilon,\lambda}}(\tilde{z}_k)$ and equal to u_{ext} in $\bar{\Omega}_{r_{\varepsilon,\lambda}}(\tilde{\mathbf{z}})$ is a smooth function. This amounts to find the boundary data and the parameters so that for $l = 1,2$

$$u_{int,l} = u_{ext,l} \quad \text{and} \quad \partial_r u_{int,l} = \partial_r u_{ext,l} \tag{47}$$

on $\partial B_{r_{\varepsilon,\lambda}}(\tilde{z}_k),\ k \in \{1,2\}$.

Suppose that (47) is verified, this provides that for each ε small enough, $u_\varepsilon \in \mathcal{C}^{2,\alpha}$ (which is obtained by patching together the functions u_{int} and the function u_{ext}), a weak solution of our system and elliptic regularity theory implies then this solution is in fact smooth. That will complete our proof since, as ε and λ tend to 0, the sequence of solutions we obtain satisfies the required singular limit behaviors, namely, $u_{\varepsilon,\lambda}$ converges to $G(\cdot, z_k)$.

Before we proceed, the following remarks are due. First it will be convenient to observe that the function u_{ε,τ_k}, $k \in \{1,2\}$, can be expanded, on $\partial B_{r_{\varepsilon,\lambda}}(\tilde{z}_k)$, as

$$u_{\varepsilon,\tau_k}(z) = -2\log\tau_k - 4\log|z-\tilde{z}_k| + \mathcal{O}\Big(\frac{\varepsilon^2\tau_k^{-2}}{|z-\tilde{z}_k|^2}\Big). \tag{48}$$

$*$ Thus, for z on $\partial B_{r_{\varepsilon,\lambda}}(z_1)$ we have

$$\begin{aligned} &(u_{int,1} - u_{ext,1})(z) = \\ &-2\log\tau_1 - 4\lambda_1 \log|z-\tilde{z}_1| + h_1^1(R_\varepsilon^1(z-\tilde{z}_1)/r_{\varepsilon,\lambda}) \\ &-\gamma_1 G(z,\tilde{z}_2) + v_1^1(R_\varepsilon^1(z-\tilde{z}_1)/r_{\varepsilon,\lambda}) - \tilde{v}_1 \\ &+H_1^{int,1}(\varphi_1^1, (z-\tilde{z}_1)/r_{\varepsilon,\lambda}) - H_1^{ext}(\tilde{\varphi}_1^1; (z-\tilde{z}_1)/r_{\varepsilon,\lambda}) \\ &\quad -(1+\lambda_1)H(z,\tilde{z}_1) + \mathcal{O}\Big(\frac{\varepsilon^2\tau_1^{-2}}{|z-\tilde{z}_1|^2}\Big) + \mathcal{O}(r_{\varepsilon,\lambda}^2). \end{aligned} \tag{49}$$

Next, even though all functions are defined on $\partial B_{r_{\varepsilon,\lambda}}(\tilde{z}_1)$ in (47), it will be more convenient to solve on S^1 the following set of equations

$$\begin{aligned} (u_{int,1} - u_{ext,1})(\tilde{z}_1 + r_{\varepsilon,\lambda}z) = 0 \\ \partial_r(u_{int,1} - u_{ext,1})(\tilde{z}_1 + r_{\varepsilon,\lambda}z) = 0. \end{aligned} \tag{50}$$

Indeed, all functions as considered as functions of $z \in S^1$ and we have simply used the change of variables $y = \tilde{z}_1 + r_{\varepsilon,\lambda}z$ to parameterize $\partial B_{r_{\varepsilon,\lambda}}(\tilde{z}_1)$.
Since the boundary data are chosen to satisfy (22) or (21). We decompose

$$\varphi_1^1 = \varphi_{1,0}^1 + \varphi_{1,1}^1 + \varphi_{1,\perp}^1 \quad \text{and} \quad \tilde{\varphi}_1^1 = \tilde{\varphi}_{1,0}^1 + \tilde{\varphi}_{1,1}^1 + \tilde{\varphi}_{1,\perp}^1.$$

Where $\varphi_{1,0}^1, \tilde{\varphi}_{1,0}^1 \in \mathbb{E}_0 = \mathbb{R}$ are constant on S^1, $\varphi_{1,1}^1, \tilde{\varphi}_{1,1}^1$ belong to $\mathbb{E}_1 = Span\{e_1, e_2\}$ and $\varphi_{1,\perp}^1, \tilde{\varphi}_{1,\perp}^1$ are $L^2(S^1)$ orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$.

we have for $z \in S^1$

$$\begin{aligned} &(u_{int,1} - u_{ext,1})(\tilde{z}_1 + r_{\varepsilon,\lambda}z) = -2\log\tau_1 \\ &+4\lambda_1 \log r_{\varepsilon,\lambda} - \gamma_1\mathcal{E}_1(z,\tilde{z}_1) + \mathcal{O}(r_{\varepsilon,\lambda}^2). \end{aligned} \tag{51}$$

where

$$\mathcal{E}_1(.,\tilde{\mathbf{z}}) = \frac{1}{\gamma_1}H(.,\tilde{z}_1) + G(.,\tilde{z}_2);$$

Then, the projection of the equations (50) over $\mathbb{E}_0$ will yield

$$\begin{aligned} -2\log\tau_1 + 4\lambda_1\log r_{\varepsilon,\lambda} - \gamma_1\mathcal{E}_1(\tilde{z}_1,\tilde{\mathbf{z}}) + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0, \\ 4\lambda_1 + \mathcal{O}(r_{\varepsilon,\lambda}^2) = 0. \end{aligned} \tag{52}$$

The system (52) can be simply written as

$$\frac{1}{\log r_{\varepsilon,\lambda}}\left[2\log\tau_1 + \gamma_1\mathcal{E}_1(\tilde{z}_1,\tilde{\mathbf{z}})\right] = \mathcal{O}(r_{\varepsilon,\lambda}^2) \quad \text{and} \quad \lambda_1 = \mathcal{O}(r_{\varepsilon,\lambda}^2).$$

We are now in a position to define τ_1^- and τ_1^+. In fact, according to the above analysis, as ε and λ tend to 0, we expect that $\tilde{z}_1$ will converge to z_1 and τ_1 will converge to τ_1^* satisfying

$$2\log\tau_1^* = -\gamma_1\mathcal{E}_1(z_1,\mathbf{z}).$$

Hence it is enough to choose τ_1^- and τ_1^+ in such a way that

$$2\log(\tau_1^-) < \gamma_1\mathcal{E}_1(z_1,\mathbf{z}) < 2\log(\tau_1^+).$$

Consider now the projection of (50) over $\mathbb{E}_1$. Given a smooth function f defined in Ω, we identify its gradient $\nabla f = (\partial_{x_1} f, \partial_{x_2} f)$ with the element of $\mathbb{E}_1$

$$\bar\nabla f = \sum_{l=1}^{2} \partial_{x_l} f\, e_l.$$

With these notations in mind, we obtain the system

$$\bar\nabla\mathcal{E}_1(\tilde z_1,\tilde{\mathbf{z}}) = \mathcal{O}(r^2_{\varepsilon,\lambda}),\quad \varphi^1_{1,1} = \mathcal{O}(r^2_{\varepsilon,\lambda}). \tag{53}$$

Finally, we consider the projection onto $(L^2(S^1))_\perp$. This yields the system

$$\varphi^1_{1,\perp} - \tilde\varphi^1_{1,\perp} + \mathcal{O}(r^2_{\varepsilon,\lambda}) = 0,$$

$$\partial_r\big(H^{int,1}_{1,\perp} - H^{ext}_{1,\perp}\big) + \mathcal{O}(r^2_{\varepsilon,\lambda}) = 0.$$

Thanks to Lemma 13, this last system can be rewritten as

$$\varphi^1_{1,\perp} = \mathcal{O}(r^2_{\varepsilon,\lambda}) \quad\text{and}\quad \tilde\varphi^1_{1,\perp} = \mathcal{O}(r^2_{\varepsilon,\lambda}).$$

If we define the parameters $t_1 \in \mathbb{R}$ by

$$t_1 = \frac{1}{\log r_{\varepsilon,\lambda}}\Big[2\log\tau_1 + \gamma_1\mathcal{E}_1(\tilde z_1,\tilde{\mathbf{z}})\Big],$$

then the system we have to solve reads

$$\begin{aligned} T^1_{\varepsilon,\lambda} &= \Big(t_1,\lambda_1,\varphi^1_{1,0},\tilde\varphi^1_{1,0},\varphi^1_{1,1},\tilde\varphi^1_{1,1},\\ &\quad \bar\nabla\mathcal{E}_1(\tilde z_1,\tilde{\mathbf{z}}),\varphi^1_{1,\perp},\tilde\varphi^1_{1,\perp}\Big) = \mathcal{O}(r^2_{\varepsilon,\lambda}) \end{aligned} \tag{54}$$

where as usual, the terms $\mathcal{O}(r^2_{\varepsilon,\lambda})$ depend nonlinearly on all the variables on the left side, but is bounded (in the appropriate norm) by a constant (independent of ε and κ) time $r^2_{\varepsilon,\lambda}$, provided $\varepsilon \in (0,\varepsilon_\kappa)$.

$*$ Similarly, when ε tends to 0, we expect that $\tilde z_2$ converges to z_2 and τ_2 converges to τ_2^* satisfying

$$2\log\tau_2^* = -\gamma_2\mathcal{E}_2(z_2,\mathbf{z}).$$

So we choose τ_2^- and τ_2^+ to satisfy

$$2\log(\tau_2^-) < -\gamma_2\mathcal{E}_2(z_2,\mathbf{z}) < 2\log(\tau_2^+),$$

where

$$\mathcal{E}_2(.,\tilde{\mathbf{z}}) = \frac{1}{\gamma_2}H(.,\tilde z_2) + G(.,\tilde z_1).$$

Using the decomposition $\mathbb{E}_0 \oplus \mathbb{E}_1 \oplus (L^2(S^1))_\perp$

$$\varphi_2^2 = \varphi^2_{2,0}+\varphi^2_{2,1}+\varphi^2_{2,\perp} \quad\text{and}\quad \tilde\varphi_2^2 = \tilde\varphi^2_{2,0}+\tilde\varphi^2_{2,1}+\tilde\varphi^2_{2,\perp},$$

we can prove that

$$\begin{aligned}(u_{int,2}-u_{ext,2})(\tilde z_2 + r_{\varepsilon,\lambda}z) &= 0\\ \partial_r(u_{int,2}-u_{ext,2})(\tilde z_2 + r_{\varepsilon,\lambda}z) &= 0\end{aligned}$$

on S^1 yield to

$$\begin{aligned} T^2_{\varepsilon,\lambda} &= \Big(t_2,\lambda_2,\varphi^2_{2,0},\tilde\varphi^2_{2,0},\varphi^2_{2,1},\tilde\varphi^2_{2,1},\\ &\quad \bar\nabla\mathcal{E}_2(\tilde z_2,\tilde{\mathbf{z}}),\varphi^2_{2,\perp},\tilde\varphi^2_{2,\perp}\Big) = \mathcal{O}(r^2_{\varepsilon,\lambda}) \end{aligned} \tag{55}$$

where

$$t_2 = \frac{1}{\log r_{\varepsilon,\lambda}}\Big[2\log\tau_2 + \gamma_2\mathcal{E}_2(\tilde z_2,\tilde{\mathbf{z}})\Big].$$

$*$ On the other hand, on $\partial B_{r_\varepsilon}(\tilde z_2)$ we have

$$\begin{aligned}&(u_{int,1}-u_{ext,1})(z) = -\lambda_1 G(z,\tilde z_1)\\ &+ h_1^2(R^2_\varepsilon(z-\tilde z_2)/r_{\varepsilon,\lambda}) + H_1^{int,2}(\varphi_1^2,(z-\tilde z_2)/r_{\varepsilon,\lambda})\\ &- H_1^{ext}(\tilde\varphi_1^2,(z-\tilde z_2)/r_{\varepsilon,\lambda}) + \mathcal{O}(r^2_{\varepsilon,\lambda}).\end{aligned}$$

As above, we will solve on S^1 the following system:

$$\begin{aligned}(u_{int,1}-u_{ext,1})(\tilde z_2 + r_{\varepsilon,\lambda}z) &= 0\\ \partial_r(u_{int,1}-u_{ext,1})(\tilde z_2 + r_{\varepsilon,\lambda}z) &= 0.\end{aligned} \tag{56}$$

We decompose

$$\varphi_1^2 = \varphi^2_{1,0}+\varphi^2_{1,1}+\varphi^2_{1,\perp} \quad\text{and}\quad \tilde\varphi_1^2 = \tilde\varphi^2_{1,0}+\tilde\varphi^2_{1,1}+\tilde\varphi^2_{1,\perp} \tag{57}$$

where $\varphi^2_{1,0},\tilde\varphi^2_{1,0} \in \mathbb{E}_0$, $\varphi^2_{1,1},\tilde\varphi^2_{1,1} \in \mathbb{E}_1$ and $\varphi^2_{1,\perp},\tilde\varphi^2_{1,\perp}$ belong to $(L^2(S^1))_\perp$.

Projecting the set of equations (56) over $\mathbb{E}_0$, we get $\lambda_1 = \mathcal{O}(r^2_{\varepsilon,\lambda})$. From the L^2-projection of (56) over $\mathbb{E}_1$, we obtain the equation $\varphi^2_{1,1} = \mathcal{O}(r^2_{\varepsilon,\lambda})$. Finally, the L^2-projection onto $(L^2(S^1))_\perp$ yields

$$\varphi^2_{1,\perp} - \tilde\varphi^2_{1,\perp} + \mathcal{O}(r^2_{\varepsilon,\lambda}) = 0,$$

$$\partial_r\big(H^{int,2}_{1,\perp} - H^{ext}_{1,\perp}\big) + \mathcal{O}(r^2_{\varepsilon,\lambda}) = 0.$$

Using again Lemma 13, the above system can be rewritten as

$$\varphi^2_{1,\perp} = \mathcal{O}(r^2_{\varepsilon,\lambda}) \quad\text{and}\quad \tilde\varphi^2_{1,\perp} = \mathcal{O}(r^2_{\varepsilon,\lambda}).$$

Then the system we have to solve reads

$$\Big(\lambda_1,\varphi^2_{1,0},\tilde\varphi^2_{1,0},\varphi^2_{1,1},\tilde\varphi^2_{1,1},\varphi^2_{1,,\perp},\tilde\varphi^2_{1,\perp}\Big) = \mathcal{O}(r^2_{\varepsilon,\lambda}). \tag{58}$$

By exactly the same arguments for (54), we can claim a solution of equation (58) in the ball of radius $\kappa r^2_{\varepsilon,\lambda}$ of the corresponding product space.

$*$ Similarly, using the fact that

$\varphi^1_2 = \varphi^1_{2,0} + \varphi^1_{2,1} + \varphi^1_{2,\perp}$ and $\tilde{\varphi}^2_2 = \tilde{\varphi}^1_{2,0} + \tilde{\varphi}^1_{2,1} + \tilde{\varphi}^1_{2,\perp}$

with $\varphi^1_{2,0}, \tilde{\varphi}^1_{2,0} \in \mathbb{E}_0$, $\varphi^1_{2,1}, \tilde{\varphi}^1_{2,1} \in \mathbb{E}_1$ and $\varphi^1_{2,\perp}, \tilde{\varphi}^1_{2,\perp} \in (L^2(S^1))_\perp$, we can prove that

$$\begin{array}{c}(u_{int,2} - u_{ext,2})(\tilde{z}_1 + r_{\varepsilon,\lambda} z) = 0\\ \partial_r(u_{int,2} - u_{ext,2})(\tilde{z}_1 + r_{\varepsilon,\lambda} z) = 0\end{array}$$

on S^1, yield to

$$\Big(\lambda_2, \varphi^1_{2,0}, \tilde{\varphi}^1_{2,0}, \varphi^1_{2,1}, \tilde{\varphi}^1_{2,1}, \varphi^1_{2,,\perp}, \tilde{\varphi}^1_{2,\perp}\Big) = \mathcal{O}(r^2_{\varepsilon,\lambda}). \tag{59}$$

Finally, recall that $\mathbf{x} = r_{\varepsilon,\lambda}(\tilde{\mathbf{z}} - \mathbf{z})$, in addition the previous systems can be written as:

$$\Big(\mathbf{x},\ t_k,\ \lambda_k,\ \varphi^k,\ \tilde{\varphi}^k,\ \bar{\nabla}\mathcal{E}_k\Big) = \mathcal{O}(r^2_{\varepsilon,\lambda}).$$

Combining(54) and (55), we have

$$T_{\varepsilon,\lambda} = (T^1_{\varepsilon,\lambda}, T^2_{\varepsilon,\lambda}) = \mathcal{O}(r^2_{\varepsilon,\lambda}). \tag{60}$$

Then the nonlinear mapping which appears on the right hand side of (60) is continuous, compact. In addition, reducing ε_κ if necessary, this nonlinear mapping sends the ball of radius $\kappa\, r^2_{\varepsilon,\lambda}$ (for the natural product norm) into itself, provided κ is fixed large enough. Applying Schauder's fixed point Theorem in the ball of radius $\kappa r^2_{\varepsilon;\lambda}$ in the product space where the entries live, we obtain the existence of a solution of equation (60). . □

Remark. We recall that

$$\mathcal{E}_1(.,\tilde{\mathbf{z}}) = \frac{1}{\gamma_1} H(.,\tilde{z}_1) + G(.,\tilde{z}_2)$$

and

$$\mathcal{E}_2(.,\tilde{\mathbf{z}}) = \frac{1}{\gamma_2} H(.,\tilde{z}_2) + G(.,\tilde{z}_1).$$

In order to inverse problem (54) and (55), we remark that the fact that z_i is a nondegenerate critical point of $\mathcal{E}_i, i = 1,2$ is equivalent to say that (z_1, z_2) is a nondegenerate critical point of the function $\mathcal{F}$ defined by

$$\mathcal{F}(z_1,z_2) = \frac{1}{2\gamma_1} H(z_1,z_1) + \frac{1}{2\gamma_2} H(z_2,z_2) + G(z_1,z_2).$$

Indeed, we have

$$\nabla\mathcal{F}(z_1,z_2) = \Big(\frac{\partial\mathcal{F}}{\partial z_1}(z_1,z_2), \frac{\partial\mathcal{F}}{\partial z_2}(z_1,z_2)\Big). \tag{61}$$

On the other hand,

$$\mathcal{E}_1(z,\mathbf{z}) = \frac{1}{\gamma_1} H(z,z_1) + G(z,z_2)$$

and

$$\mathcal{E}_2(z,\mathbf{z}) := \frac{1}{\gamma_2} H(z,z_2) + G(z,z_1)$$

then

$$\begin{aligned}\frac{\partial\mathcal{E}_1}{\partial z}(z_1,\mathbf{z}) &= \frac{1}{\gamma_1}\frac{\partial H}{\partial z}(z_1,z_1) + \frac{\partial G}{\partial z}(z_1,z_2)\\ &= \frac{\partial\mathcal{F}}{\partial z_1}(z_1,z_2)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\mathcal{E}_2}{\partial z}(z_2,\mathbf{z}) &= \frac{1}{\gamma_2}\frac{\partial H}{\partial z}(z_2,z_2) + \frac{\partial G}{\partial z}(z_2,z_1)\\ &= \frac{\partial\mathcal{F}}{\partial z_2}(z_1,z_2).\end{aligned}$$

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