Convergence Analysis of an Extended Newton-type Method for Implicit Functions and Their Solution Mappings

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Abstract: Let P , X and Y be Banach spaces. Suppose that $f : P \times X \to Y$ is continuously Fréchet differentiable function depend on the point (p, x) and $F : X \rightrightarrows 2^Y$ is a set-valued mapping with closed graph. Consider the following parametric generalized equation of the form:

$$0 \in f(p, x) + F(x). \tag{1}$$

In the present paper, we study an extended Newton-type method for solving parametric generalized equation (1). Indeed, we will analyze semi-local and local convergence of the sequence generated by extended Newton-type method under the assumptions that $f(p, x)$, the Fréchet derivative $D_x f(p, x)$ in x of $f(p, x)$ are continuously depend on (p, x) and $(f(p, \cdot) + F)^{-1}$ is Lipschitz-like at $(\bar{p}, \bar{x})$.

Key–Words: Set-valued maps, Parametric generalized equations, Semilocal convergence, Lipschitz-like mappings, Solution mapping, Extended Newton-type method.

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1 Introduction

In this study we are concerned with the problem of approximating a solution of a parametric generalized equation. Let P, X and Y are Banach spaces. Suppose that $f : P \times X \to Y$ is continuously Fréchet differentiable function and $F : X \rightrightarrows 2^Y$ is a set-valued mapping with closed graph. We consider here a parametric generalized equation of the following form to find a point $\bar{x} \in \Omega \subset X$ for $\bar{p} \in P$ satisfying

$$0 \in f(\bar{p}, \bar{x}) + F(\bar{x}). \tag{2}$$

The model of generalized equation (2) covers a huge territory. The classical case of an equation corresponds to having $F(x) \equiv 0$, whereas by taking $F(x) \equiv -K$ for a fixed set $K \subset Y$ one gets various constraint systems. When Y is the dual space X^* of X and F is the normal cone mapping N_C associated with a closed, convex set $C \subset X$, one has a variational inequality.

Let us fixing $p \in P$ and $x \in X$, and let $D_x f(p, x)$ be the Fréchet derivative of f at x. By $\mathcal{D}_p(x)$ we denote the subset of X defined by

$$\mathcal{D}_p(x) = \{d \in X : 0 \in f(p, x) + D_x f(p, x)d + F(x + d)\}.$$

According to the construction of $\mathcal{D}_p(x)$, Dontchev and Rockafellar [3] associated the following Algorithm for solving the generalized equations (2):

Algorithm 1 (Newton-type method)

Step 1. Select $x_0 \in X$ and put $k := 0$.
Step 2. If $0 \in \mathcal{D}_p(x_k)$, then stop; otherwise, go to Step 3.
Step 3. If $0 \notin \mathcal{D}_p(x_k)$, choose d_k such that $d_k \in \mathcal{D}_p(x_k)$.
Step 4. Set $x_{k+1} := x_k + d_k$.
Step 5. Replace k by $k + 1$ and go to Step 2.

Generally, for a starting point near to a solution, the sequences generated by Algorithm 1 are not uniquely defined and not every generated sequence is convergent. The result, established in [5, Theorem 2.1.], guarantees the existence of one sequence, which is convergent. Therefore, from the viewpoint of practical computations, this kind of methods would not be convenient in practical application. In accordance with the developments in computation, theoretical studies on numerical schemes are now fruitful and highly needed. That is why inspired by the works of Dontchev and Rockafellar [5] we propose " so called" extended Newton-type method as follows:

Algorithm 2 (Extended Newton-type method)

Step 1. Select $\eta \in [1, \infty)$, $x_0 \in X$ and put $k := 0$.
Step 2. If $0 \in \mathcal{D}_p(x_k)$, then stop; otherwise, go to Step 3.
Step 3. If $0 \notin \mathcal{D}_p(x_k)$, choose d_k such that $d_k \in \mathcal{D}_p(x_k)$ and

$$\|d_k\| \leq \eta \,\text{dist}\big(0, \mathcal{D}_p(x_k)\big).$$

Step 4. Set $x_{k+1} := x_k + d_k$.
Step 5. Replace k by $k + 1$ and go to Step 2.

We observe, from the Algorithm 2, that if f is explicit function, this algorithm reduces to the Algorithm introduced by [15]. A large number of fruitful works on semilocal convergence analysis have been presented for solving generalized equations in the case when f is explicit function and $F = 0$ (cf. [13, 14, 19]) or when $F = C$ (cf. [9]). In the case when f is explicit function, Rashid et al. [15] introduced Gauss-Newton method for solving the generalized equation (2) and studied its semilocal convergence analysis. However, in our best knowledge, there is no other study on semilocal convergence analysis discovered for the case (2), even for the Algorithm 1.

In this paper, our efforts will be concentrated on the role of the parameter p in generating sequences by Algorithm 2 that approach a solution of (2). Indeed, we analyze the semilocal and local convergence of the extended Newton-type method defined by Algorithm 2. The main tool is the Lipschitz-like property of set- valued mappings, which was introduced in [11] by Aubin in the context of nonsmooth analysis and studied by many mathematicians; see for example [2,7,8,10,15–18] and the references therein. Our main results are the convergence analysis, established in Section 3, which based on the attraction region around the initial point, provides some sufficient conditions ensuring the convergence to a solution of any sequence generated by Algorithm 2. As a consequence, local convergence results for the extended Newton-type method are obtained.

This paper is organized as follows: In section 2, we recall some necessary notations, notions, some preliminary results. We evoke also a fixed-point theorem which has been proved in [6] and this fixed-point theorem is the main tool to prove the existence of the sequence generated by Algorithm 2. In section 3, we consider the extended Newton-type method which has been introduced in section1. Then using the concept of Lipschitz-like property, we will show the existence and convergence of the sequence generated by Algorithm 2. In the last section, we will give a summary of the major results obtained in this paper.

2 Notations and Preliminary results

In this section we give some notations and collect some results that will be helpful to prove our main results. Throughout this paper, we suppose that P, X and Y are real or complex Banach spaces. Let $x \in X$ and $r > 0$. The closed ball centered at x with radius r is denoted by $\mathbb{B}_r(x)$ and closed unit ball denoted by $\mathbb{B}$. Let $F\colon X \rightrightarrows 2^Y$ be a set-valued mapping with $\text{dom}F \neq \emptyset$. The domain $\text{dom}F$, the inverse F^{-1} and the graph $\text{gph}F$ of F are respectively defined by

$$\text{dom}F := \{x \in X : F(x) \neq \emptyset\},$$

$$F^{-1}(y) := \{x \in X : y \in F(x)\} \quad \text{for each } y \in Y$$

and

$$\text{gph}F := \{(x, y) \in X \times Y : y \in F(x)\}.$$

Let $A \subseteq X$. The distance function of A is defined by

$$\text{dist}(x, A) := \inf\{\|x - a\| : a \in A\} \quad \text{for each } x \in X,$$

while the excess from the set A to the set $C \subseteq X$ is defined by

$$e(C, A) := \sup\{\text{dist}(x, A) : x \in C\}.$$

The norms in P, X and Y all are denoted by $\|\cdot\|$.

The rest of this section, we recall a few definitions, some results and then state the Banach contraction mapping theorem.

We begin with the definition of Lipschitz continuity for a function in a neighborhood.

Definition 1 *A function $f : X \to Y$ is said to be Lipschitz continuous relative to a set $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ or on a set $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ if $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \subset \text{dom}f$ and there exists a constant $\mu \geq 0$ (a Lipschitz constant) such that, for all $x, x' \in \mathbb{B}_{r_{\bar{x}}}(\bar{x})$,*

$$\|f(x) - f(x')\| \leq \mu\|x - x'\|. \tag{3}$$

Moreover, the function f is said to be Lipschitz continuous around $\bar{x}$ when (3) holds for some neighborhood $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ of $\bar{x}$. The infimum of the set of values of μ is called the Lipschitz modulus of f at $\bar{x}$, denoted by $\text{lip}(f; \bar{x})$, for which there exists a neighborhood $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ of $\bar{x}$ such that (3) holds. Equivalently,

$$\text{lip}(f; \bar{x}) := \limsup_{\substack{x, x' \to \bar{x} \\ x \neq x'}} \frac{\|f(x) - f(x')\|}{\|x - x'\|}.$$

Definition 2 *A function $f : P \times X \to Y$ is said to be Lipschitz continuous with respect to x uniformly in p around $(\bar{p}, \bar{x}) \in \text{int dom}f$ when there are neighborhoods $\mathbb{B}_{r_{\bar{p}}}(\bar{p})$ of $\bar{p}$ and $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ of $\bar{x}$ along with a constant $\mu \geq 0$ and such that, for all $x, x' \in \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ and $p \in \mathbb{B}_{r_{\bar{p}}}(\bar{p})$,*

$$\|f(p, x) - f(p, x')\| \quad \leq \quad \mu\|x - x'\|. \tag{4}$$

The infimum of the set of values of μ is called the partial uniform Lipschitz modulus of f at $(\bar{p}, \bar{x})$, denoted by $\widehat{\text{lip}}_x(f; (\bar{p}, \bar{x}))$, for which there exist neighborhoods $\mathbb{B}_{r_{\bar{p}}}(\bar{p})$

of $\bar{p}$ and $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ of $\bar{x}$ such that (4) holds and it has the following form

$$\widehat{\mathrm{lip}}_x(f;(\bar{p},\bar{x})) := \limsup_{\substack{x,x'\to\bar{x},p\to\bar{p}\\ x\neq\bar{x}}} \frac{\|f(p,x)-f(p,x')\|}{\|x-x'\|}.$$

The following notions of pseudo-Lipschitz and Lipchitz-like set-valued mappings are due to [15]. These notions were introduced by Aubin, see for example [11,12] and have been studied extensively.

Definition 3 *Let $\Gamma : Y \rightrightarrows 2^X$ be a set-valued mapping and let $(\bar{y},\bar{x}) \in \mathrm{gph}\Gamma$. Let $r_{\bar{x}} > 0$, $r_{\bar{y}} > 0$ and $M > 0$. Then Γ is said to be*

(a) *Lipchitz-like on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with constant M if the following inequality holds, for any $y_1, y_2 \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$:*

$$e(\Gamma(y_1)\cap\mathbb{B}_{r_{\bar{x}}}(\bar{x}),\Gamma(y_2)) \le M\|y_1-y_2\|.$$

(b) *pseudo-Lipschitz around $(\bar{y},\bar{x})$ if there exist constants $r'_{\bar{y}} > 0$, $r'_{\bar{x}} > 0$ and $M' > 0$ such that Γ is Lipchitz-like on $\mathbb{B}_{r'_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r'_{\bar{x}}}(\bar{x})$ with constant M'.*

Remark 4 *Γ is Lipschitz-like on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with constant M is equivalent to the following statement: if for every $y_1, y_2 \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ and for every $x_1 \in \Gamma(y_1)\cap\mathbb{B}_{r_{\bar{x}}}(\bar{x})$, there exists $x_2 \in \Gamma(y_2)$ such that*

$$\|x_1-x_2\| \le M\|y_1-y_2\|.$$

The following lemma is useful and it has been taken from [15, Lemma 2.1].

Lemma 5 *Let $\Gamma : Y \rightrightarrows 2^X$ be a set-valued mapping and let $(\bar{y},\bar{x}) \in$ gph Γ. Assume that Γ is Lipschitz-like on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with constant M. Then*

$$\mathrm{dist}\ (x,\Gamma(y)) \le M\mathrm{dist}\ (y,\Gamma^{-1}(x))$$

holds for every $x \in \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ and $y \in \mathbb{B}_{\frac{r_{\bar{y}}}{3}}(\bar{y})$ satisfying $\mathrm{dist}(y,\Gamma^{-1}(x)) \le \frac{r_{\bar{y}}}{3}$.

The following definition is taken from [3].

Definition 6 *Let $F : X \rightrightarrows 2^Y$ be a set-valued mapping. Let $(\bar{x},\bar{y}) \in$ gphF, $r_{\bar{x}} > 0$ and $r_{\bar{y}} > 0$. Then a graphical localization of F on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ relative to $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ is a set-valued mapping $\tilde{F}$ such that*

$$\mathrm{gph}\tilde{F} = (\mathbb{B}_{r_{\bar{x}}}(\bar{x})\times\mathbb{B}_{r_{\bar{y}}}(\bar{y}))\cap\mathrm{gph}F$$

and thus

$$\tilde{F}(x) = \begin{cases} F(x)\cap\mathbb{B}_{r_{\bar{y}}}(\bar{y}) & \text{when } x\in\mathbb{B}_{r_{\bar{x}}}(\bar{x}),\\ \emptyset & \text{otherwise.}\end{cases}$$

The inverse of $\tilde{F}$ then has

$$\tilde{F}^{-1}(y) = \begin{cases} F^{-1}(y)\cap\mathbb{B}_{r_{\bar{x}}}(\bar{x}) & \text{when } y\in\mathbb{B}_{r_{\bar{y}}}(\bar{y}),\\ \emptyset & \text{otherwise,}\end{cases}$$

and is thus a graphical localization of the set-valued mapping F^{-1} on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$.

Definition 7 *Let $F : X \rightrightarrows 2^Y$ be a set-valued mapping and $\bar{y} \in F(\bar{x})$. Let $r_{\bar{x}} > 0$ and $r_{\bar{y}} > 0$ and suppose that F^{-1} is Lipschitz-like on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with Lipschitz constant $\kappa > 0$. Let s be the graphical localization of F^{-1} on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$. Then s is said to be Lipschitz-like localization on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with the same Lipschitz constant if s is Lipschitz-like and single-valued mapping.*

Definition 8 *Let $S : P \rightrightarrows 2^X$ be a set-valued mapping with closed graph. Then S is called the solution mapping associated with the generalized equation (2) if*

$$S(p) := \{x : 0 \in f(p,x) + F(x)\}\ \forall\, p\in P, x\in X.$$

The following lemma is a fixed point statement which has been proved in [6] employing the standard iterative concept for contracting mapping. This lemma is need to prove the existence of the sequence generated by Algorithm 2.

Lemma 9 *Let $\Phi : X \rightrightarrows 2^X$ be a set-valued mapping. Let $\eta_0 \in X$, $r > 0$ and $0 < \lambda < 1$ be such that*

$$\mathrm{dist}(\eta_0;\Phi(\eta_0)) < r(1-\lambda) \tag{5}$$

and, for any $x_1, x_2 \in \mathbb{B}_r(\eta_0)$,

$$e(\Phi(x_1)\cap\mathbb{B}_r(\eta_0),\Phi(x_2)) \le \lambda\|x_1-x_2\|. \tag{6}$$

Then Φ has a fixed point in $\mathbb{B}_r(\eta_0)$, that is, there exists $x \in \mathbb{B}_r(\eta_0)$ such that $x \in \Phi(x)$. If Φ is additionally single-valued, then the fixed point of Φ in $\mathbb{B}_r(\eta_0)$ is unique.

The previous lemma is a generalization of a fixed point theorem in [1], where in assertion (b) the excess e is replaced by Hausdorff distance.

Let $r_{\bar{x}} > 0$, $r_{\bar{y}} > 0$ and $r_{\bar{p}} > 0$. Further let $\kappa > 0$ and $\mu > 0$. Define

$$\bar{r} := \min\Big\{r_{\bar{y}} - \mu r_{\bar{x}}, \frac{r_{\bar{x}}(1-\kappa\mu)}{\kappa}\Big\}. \tag{7}$$

Then

$$\bar{r} > 0 \Leftrightarrow \mu < \min\Big\{\frac{r_{\bar{y}}}{r_{\bar{x}}}, \frac{1}{\kappa}\Big\}. \tag{8}$$

Applying Lemma 9, we will prove the following lemma which has been extracted from [5, Corollary 1.5].

Lemma 10 *Let $F : X \rightrightarrows Y$ be a set-valued mapping and let $(\bar{x},\bar{y}) \in \mathrm{gph}F$. Suppose that F^{-1} is Lipschitz-like on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with Lipschitz constant $\kappa > 0$ and s is the Lipschitz-like localization of F^{-1} on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$. Let $\bar{r}$ be defined in* (7) *such that* (8) *is hold. Let $f : P\times X \to Y$ be a function such that $f(p,\cdot)$ is Lipschitz continuous on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ uniformly in $p \in \mathbb{B}_{r_{\bar{p}}}(\bar{p})$ with a constant $\mu > 0$. Then, for every $p \in \mathbb{B}_{r_{\bar{p}}}(\bar{p})$, the mapping $\big(f(p,\cdot)+F\big)^{-1}$ has Lipschitz-like localization on $f(p,\bar{x})+\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with a Lipschitz constant $\dfrac{\kappa}{1-\kappa\mu}$.*

Proof: According to our assumption on s, we have that $s(y) = F^{-1}(y) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ for every $y \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$. Also, we have that

$$\|s(y) - s(y')\| \leq \kappa \|y - y'\| \; \forall \, y, y' \in \mathbb{B}_{r_{\bar{y}}}(\bar{y}). \quad (9)$$

Moreover, for each $p \in \mathbb{B}_{r_{\bar{p}}}(\bar{p})$, by the Lipschitz continuity of $f(p, \cdot)$ we have that

$$\|f(p,x) - f(p,x')\| \leq \mu \|x - x'\| \text{ for all } x, x' \in \mathbb{B}_{r_{\bar{x}}}(\bar{x}). \quad (10)$$

For any $y \in \mathbb{B}_{\bar{r}}\big(f(p,\bar{x}) + \bar{y}\big)$ and $x \in \mathbb{B}_{r_{\bar{x}}}(\bar{x})$, we have that

$$\begin{aligned} & \| - f(p,x) + y - \bar{y}\| \\ \leq \; & \|y - f(p,\bar{x}) - \bar{y}\| + \|f(p,\bar{x}) - f(p,x)\| \\ \leq \; & \bar{r} + \mu\|\bar{x} - x\| \leq \bar{r} + \mu r_{\bar{x}}. \end{aligned} \quad (11)$$

Then by relation $\bar{r} \leq r_{\bar{y}} - \mu r_{\bar{x}}$ in (7), we have that

$$\| - f(p,x) + y - \bar{y}\| \leq r_{\bar{y}}.$$

This shows that $-f(p,x) + y \in \mathbb{B}_{r_{\bar{y}}}(\bar{y}) \subseteq$ dom s. Now let $y \in \mathbb{B}_{\bar{r}}\big(f(p,\bar{x}) + \bar{y}\big)$ and define a mapping $\Phi_y : X \mapsto X$ by

$$\Phi_y(\cdot) := s\big(-f(p,\cdot) + y\big).$$

Now, we will apply Lemma 9 to the map $\Phi_y(\cdot)$ with $\eta_0 := \bar{x}$, $r := r_{\bar{x}}$ and $\lambda := \kappa\mu$ so that the assertions (5) and (6) hold. Noting that $\bar{x} \in s(\bar{y}) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x})$. Then by the relation $\kappa\bar{r} \leq r_{\bar{x}}(1 - \kappa\mu)$ in (7) and the Lipschitz continuity of s, we have that

$$\begin{aligned} & \text{dist}\,\big(\bar{x}, \Phi_y(\bar{x})\big) \\ \leq \; & e\big(s(\bar{y}) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), s\big(-f(p,\bar{x}) + y\big)\big) \\ \leq \; & \kappa\|\bar{y} + f(p,\bar{x}) - y\| \leq \kappa\|y - f(p,\bar{x}) - \bar{y}\| \\ \leq \; & \kappa\bar{r} \leq r_{\bar{x}}(1 - \kappa\mu) \\ = \; & r(1 - \lambda). \end{aligned}$$

Since $\kappa\mu < 1$ by (8), it shows that the assertion (5) of Lemma 9 is satisfied. Next we will show that the assertion (6) is also satisfied. To do this, let $x, x' \in \mathbb{B}_{r_{\bar{x}}}(\bar{x})$. Then by the definition of excess e, the Lipschitz continuity of s and $f(p, \cdot)$, we have that

$$\begin{aligned} & e\big(\Phi_y(x) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), \Phi_y(x')\big) \\ \leq \; & e\big(s(-f(p,x) + y) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), s(-f(p,x') + y)\big) \\ \leq \; & \kappa\|f(p,x) - f(p,x')\| \leq \kappa\mu\|x - x'\| \\ = \; & \lambda\|x - x'\|. \end{aligned}$$

This implies that the assertion (6) is satisfied. Since both assertions (5) and (6) are satisfied, we can deduce the existence of a fixed point $x \in \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ such that $x = \Phi_y(x)$. Now for every $x \in \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ and $p \in \mathbb{B}_{r_{\bar{p}}}(\bar{p})$, define a mapping $\psi_x : Y \mapsto X$ by

$$\psi_x(y) = \big(f(p,x) + F\big)^{-1}(y) \forall\, y \in \mathbb{B}_{\bar{r}}\big(f(p,\bar{x}) + \bar{y}\big).$$

The fixed point Lemma implies that, for every $y \in \mathbb{B}_{\bar{r}}\big(f(p,\bar{x}) + \bar{y}\big)$, there exists a unique fixed point $\psi_x(y)$ of Φ_y in $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ such that $\psi_x(y) = \Phi_y(x)$ and hence $\psi_x(y) = x$. Consequently, we have that

$$\psi_x(y) = \big(f(p,x) + F\big)^{-1}(y) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}). \quad (12)$$

Moreover, for every $y \in \mathbb{B}_{\bar{r}}\big(f(p,\bar{x}) + \bar{y}\big)$, we have that

$$\begin{aligned} & \|\psi_x(y) - \psi_x(y')\| \\ = \; & \|\Phi_y(x) - \Phi_{y'}(x)\| \\ = \; & \|\Phi_y(\psi_x(y)) - \Phi_{y'}(\psi_x(y))\| \\ = \; & \|s(y - f(p,\psi_x(y))) - s(y' - f(p,\psi_x(y')))\| \\ \leq \; & \kappa\|y - y'\| + \kappa\|f(p,\psi_x(y)) - f(p,\psi_x(y'))\| \\ \leq \; & \kappa\|y - y'\| + \kappa\mu\|\psi_x(y)) - \psi_x(y'))\|. \end{aligned}$$

Hence, it follows that

$$\|\psi_x(y) - \psi_x(y')\| \leq \frac{\kappa}{1 - \kappa\mu}\|y - y'\|. \quad (13)$$

Note that

$$\begin{aligned} \mathbb{B}_{\bar{r}}(\bar{y}) + f(p,\bar{x}) &= \{y : y - f(p,\bar{x}) \in \mathbb{B}_{\bar{r}}(\bar{y})\} \\ &= \{y : \|y - f(p,\bar{x}) - \bar{y}\| \leq \bar{r}\} \\ &= \{y : y \in \mathbb{B}_{\bar{r}}\big(f(p,\bar{x}) + \bar{y}\big)\} \\ &= \mathbb{B}_{\bar{r}}\big(f(p,\bar{x}) + \bar{y}\big). \end{aligned}$$

Thus, combining (12) and (13) we conclude that ψ_x is the Lipschitz-like localization of $\big(f(p,\cdot) + F\big)^{-1}$ with Lipschitz constant $\kappa/(1 - \kappa\mu)$ and hence, for every $p \in \mathbb{B}_{r_{\bar{p}}}(\bar{p})$, one says that $\big(f(p,\cdot) + F\big)^{-1}$ has Lipschitz-like localization on $f(p,\bar{x}) + \mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with a Lipschitz constant $\kappa/(1 - \kappa\mu)$. This completes the proof.

3 Convergence analysis of Extended Newton-type method

Throughout this section, we assume that $f : P \times X \to Y$ is continuously Fréchet differentiable in x with its derivative, denoted by $D_x f(p,x)$ and that both $f(p,x)$ and $D_x f(p,x)$ depend continuously on (p,x). Let $F : X \rightrightarrows 2^Y$ be set-valued mapping with closed graph and $S : P \rightrightarrows X$ be a solution mapping associated to the parametric generalized equation (2) such that gph$S := \big(\mathbb{B}_{r_{\bar{p}}}(\bar{p}) \times \mathbb{B}_{r_{\bar{x}}}(\bar{x})\big) \cap$ gph$S$, where $\mathbb{B}_{r_{\bar{p}}}(\bar{p})$ and $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ are neighborhoods of $\bar{p}$ and $\bar{x}$ respectively, that is, S is locally closed. Moreover, we suppose that s is the Lipschitz-like localization of S on $\mathbb{B}_{r_{\bar{p}}}(\bar{p})$ for $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ and that the point $\bar{x} \in X$ satisfy the parametric generalized equation (2) corresponding to a choice of $\bar{p}$ of $p \in P$. We prove the existence and convergence of sequences generated by the extended-Newton-type method defined by Algorithm 2.

Let $p \in P, z \in X$ and define the mapping $G_{p,z}$, for each $x \in X$, by

$$G_{p,z}(\cdot) := f(p,z) + D_x f(p,z)(\cdot - z) + F(\cdot). \quad (14)$$

Then

$$\mathcal{D}_p(x) = \Big\{ d \in X : 0 \in G_{p,z}(z+d) \Big\}. \qquad (15)$$

Moreover, the following equivalence is clear, for any $z \in X$ and $y \in Y$:

$$x \in G_{p,z}^{-1}(y) \Leftrightarrow y \in f(p,z) + D_x f(p,z)(x-z) + F(x). \qquad (16)$$

In particular,

$$\bar{x} \in G_{\bar{p},\bar{x}}^{-1}(\bar{y}),\ \forall\, (\bar{x},\bar{y}) \in \text{gph}\, \big(f(\bar{p},\cdot) + F\big).$$

Let $(\bar{x},\bar{y}) \in \text{gph}\big(f(\bar{p},\cdot)+F\big)$ and let $r_{\bar{x}} > 0$, $r_{\bar{y}} > 0$ and $r_{\bar{p}} > 0$. We assume throughout that $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \subseteq \Omega \cap \text{dom}\, F$

Lemma 11 *Let P, X and Y be Banach spaces and let $f : P \times X \to Y$ be a function such that $D_x f(p,x)$ is Lipschitz continuous on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with respect to x uniformly in p for $\mathbb{B}_{r_{\bar{p}}}(\bar{p})$ and $D_x f$ be continuous depend on (p,x). Then the following statements are equivalent:*

(i) *The mapping $\big(f(\bar{p},\cdot)+F\big)^{-1}$ is Lipschitz-like on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$.*

(ii) *The mapping $[f(\bar{p},\bar{x}) + D_x f(\bar{p},\bar{x})(\cdot - \bar{x}) + F(\cdot)]^{-1}$ is Lipschitz-like $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$.*

Proof: Define a function $g : X \to Y$ by

$$g(x) = -f(p,x) + f(\bar{p},\bar{x}) + D_x f(\bar{p},\bar{x})(x-\bar{x}).$$

To complete the proof of this lemma, according to [6, Corollary 2], we need to show that g is Lipschitz continuous on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$. Let $\kappa > 0$ and $\eta > 0$. Let us also suppose that $D_x f(p,x)$ is κ-Lipschitz continuous on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with respect to x uniformly in p for $\mathbb{B}_{r_{\bar{p}}}(\bar{p})$ and $D_x f$ is continuous depend on (p,x) with constant η. To finish this, let $\lambda > 0$ be such that $\kappa r_{\bar{x}} < \lambda - \eta$. Then, for every $x_1, x_2 \in B_{r_{\bar{x}}}(\bar{x})$, we have

$$\begin{aligned}
& \|g(x_1) - g(x_2)\| \\
= \ & \|f(p,x_2) - f(p,x_1) - D_x f(\bar{p},\bar{x})(x_2-x_1)\| \\
\le \ & \int_0^1 \Big(\|\big(D_x f(p, x_1 + t(x_2-x_1)) - D_x f(p,x_1)\big) \\
& (x_2-x_1)\| + \|\big(D_x f(p,x_1) - D_x f(\bar{p},\bar{x})\big) \\
& (x_2-x_1)\|\Big) dt \\
\le \ & \frac{\kappa}{2}\|x_2-x_1\|^2 + \eta\|x_2-x_1\| \\
\le \ & \Big(\kappa r_{\bar{x}} + \eta\Big)\|x_2-x_1\| \\
< \ & \lambda\|x_1-x_2\|.
\end{aligned}$$

This yields that g is Lipschitz continuous on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with Lipschitz constant λ and thus completes the proof of the lemma.

The following theorem can be extracted from [3, 4].

Theorem 12 *For a generalized equation (2) and its solution mapping S, let $\bar{p}$ and $\bar{x}$ be such that $\bar{x} \in S(\bar{p})$. Assume that f is Lipschitz continuous with respect to p uniformly in x at $(\bar{p},\bar{x})$ with a Lipschitz constant $\lambda > 0$ and that the inverse $G_{\bar{p},\bar{x}}^{-1}$ of the mapping $G_{\bar{p},\bar{x}}$ for which $G_{\bar{p},\bar{x}}(\bar{x}) \ni \bar{y}$ has a Lipschitz-like localization σ on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with $\kappa/(1-\kappa\mu)$. Then the mapping S has a Lipschitz-like localization s on $\mathbb{B}_{r_{\bar{p}}}(\bar{p})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with $\kappa\lambda/(1-\kappa\mu)$.*

The following lemma plays a crucial role for convergence analysis of the extended Newton-type method.

Lemma 13 *For a generalized equation (2) and its solution mapping S, let $\bar{p}$ and $\bar{x}$ be such that $\bar{x} \in S(\bar{p})$. Assume that f is Lipschitz continuous with respect to p uniform in x with Lipschitz constant $\lambda > 0$, $G_{\bar{p},\bar{x}}^{-1}(\cdot)$ has a Lipschitz-like localization σ on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with Lipschitz constant $\kappa/(1-\kappa\mu)$ and $D_x f$ is continuous at $(\bar{p},\bar{x})$ with constant $\chi > 0$. Let $\bar{r}$ be defined in* (7) *such that* (8) *is hold. Then $G_{p,u}^{-1}(\cdot)$ has a Lipschitz-like localization $\Theta_{p,u}(\cdot)$ on $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with $\kappa/(1-\kappa(\chi+\mu))$ and hence the solution mapping S has a Lipschitz-like localization s on $\mathbb{B}_{r_{\bar{p}}}(\bar{p})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with $\lambda\kappa/(1-\kappa(\chi+\mu))$.*

Proof: By the Lipschitz-like localization property of $G_{\bar{p},\bar{x}}^{-1}(\cdot)$ imply that

$$\sigma(y) = G_{\bar{p},\bar{x}}^{-1}(y) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}) \text{ for all } y \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$$

and

$$\|\sigma(y) - \sigma(y')\| \le \frac{\kappa}{1-\kappa\mu}\|y-y'\|\ \forall\, y, y' \in \mathbb{B}_{r_{\bar{y}}}(\bar{y}).$$

Furthermore, since $D_x f$ is continuous depend on (p,x), we have for any $\chi > 0$, $p \in \mathbb{B}_{r_{\bar{p}}}(\bar{p})$ and $x \in \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ that

$$\|D_x f(p,x) - D_x f(\bar{p},\bar{x})\| \le \chi. \qquad (17)$$

Define a mapping $Z_{p,u} : X \mapsto Y$ by

$$\begin{aligned}
Z_{p,u}(\cdot) :\ = \ & f(p,u) + D_x f(p,u)(\cdot - u) - f(\bar{p},\bar{x}) \\
& - D_x f(\bar{p},\bar{x})(\cdot - \bar{x}). \qquad (18)
\end{aligned}$$

Now, for every $p \in \mathbb{B}_{r_{\bar{p}}}(\bar{p})$ and $u, x \in \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ and using (17) and (18) we have

$$\begin{aligned}
& \|Z_{p,u}(x) - Z_{p,u}(\bar{x})\| \\
= \ & \|(D_x f(p,u) - D_x f(\bar{p},\bar{x}))(x-\bar{x})\| \\
\le \ & \|D_x f(p,u) - D_x f(\bar{p},\bar{x})\|\|x-\bar{x}\| \\
\le \ & \chi\|x-\bar{x}\|.
\end{aligned}$$

This yields that the function $Z_{p,u}(\cdot)$ is Lipschitz continuous on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with Lipschitz constant χ.

Consider the parameterized mapping $G_{p,u}(\cdot)$ defined in (14). Then by using (18), we have that

$$G_{p,u}(x) = Z_{p,u}(x) + G_{\bar{p},\bar{x}}(x). \qquad (19)$$

Noting that $Z_{\bar{p},\bar{x}}(\bar{x}) = 0$. Setting $\mu := \chi$ and $\kappa := \frac{\kappa}{1-\kappa\mu}$ in Lipschitz constant for the mapping $G^{-1}_{p,u}(\cdot)$ in Lemma 10. Then we conclude that the mapping $G^{-1}_{p,u}(\cdot)$ has a Lipschitz-like localization $\Theta_{p,u}(\cdot)$ on $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with a Lipschitz constant $\frac{\kappa}{1-\kappa(\chi+\mu)}$ such that

$$\Theta_{p,u}(y) = G^{-1}_{p,u}(y) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), \tag{20}$$

and for all $y, y' \in \mathbb{B}_{\bar{r}}(\bar{y})$,

$$\|\Theta_{p,u}(y) - \Theta_{p,u}(y')\| \le \frac{\kappa}{1-\kappa(\chi+\mu)}\|y-y'\|. \tag{21}$$

Moreover, by assumption, $f(\cdot, x)$ is Lipschitz continuous with respect to p uniform in x with Lipschitz constant $\lambda > 0$. Also, we have just established that $G^{-1}_{p,u}(\cdot)$ has a Lipschitz-like localization $\Theta_{p,u}(\cdot)$ on $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with $(\chi+\mu)\kappa < 1$ and a Lipschitz constant $\frac{\kappa}{1-\kappa(\chi+\mu)}$. This reflects the fact that all the assumptions in Theorem 12 are fulfilled and hence by Theorem 12 we conclude that the solution mapping S has a Lipschitz-like localization s on $\mathbb{B}_{r_{\bar{p}}}(\bar{p})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with lipschitz constant

$$\frac{\lambda\kappa}{1-\kappa(\chi+\mu)}.$$

This completes the proof.

Before going to present our main result, we need to evoke some relations and define some notations here.

By Lemma 13 we have the solution mapping S which has a Lipschitz-like localization s on $\mathbb{B}_{r_{\bar{p}}}(\bar{p})$ for $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with Lipschitz constant $\frac{\lambda\kappa}{1-\kappa(\chi+\mu)}$. Then we have that, for all $p, p' \in \mathbb{B}_{r_{\bar{p}}}(\bar{p})$,

$$\|s(p) - s(p')\| \le \frac{\lambda\kappa}{1-\kappa(\chi+\mu)}\|p-p'\|. \tag{22}$$

and for every $p \in \mathbb{B}_{r_{\bar{p}}}(\bar{p})$,

$$s(p) = S(p) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}). \tag{23}$$

Because of (23), note by Lemma 13 that the mapping $G^{-1}_{p,s(p)}(\cdot)$ has a Lipschitz-like localization $\Theta_{p,s(p)}(\cdot)$ on $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with $\kappa(\chi+\mu) < 1$ and a Lipschitz constant $\frac{\kappa}{1-\kappa(\chi+\mu)}$ such that, for all $y, y' \in \mathbb{B}_{\bar{r}}(\bar{y})$

$$\Theta_{p,s(p)}(y) = G^{-1}_{p,s(p)}(y) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), \tag{24}$$

and

$$\|\Theta_{p,s(p)}(y) - \Theta_{p,s(p)}(y')\| \le \frac{\kappa}{1-\kappa(\chi+\mu)}\|y-y'\|. \tag{25}$$

For our convenience, we define for each $p \in \mathbb{B}_{r_{\bar{p}}}(\bar{p})$ and $u \in \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ the mapping $g_{p,u} : X \to Y$ by

$$\begin{aligned} g_{p,u}(\cdot): &= f(p,s(p)) + D_x f(p,s(p))(\cdot - s(p)) \\ &\quad - f(p,u) - D_x f(p,u)(\cdot - u). \end{aligned} \tag{26}$$

and the set-valued mapping $\Phi_x : X \rightrightarrows 2^X$ by

$$\Phi_x(\cdot) := G^{-1}_{p,s(p)}[g_{p,x}(\cdot)]. \tag{27}$$

Then, for each $x', x'' \in X$,

$$\begin{aligned} & \|g_{p,x}(x') - g_{p,x}(x'')\| \\ = \; & \|D_x f(p,s(p))((x' - s(p)) - D_x f(p,x) \\ & (x'-x) - D_x f(p,s(p))(x'' - s(p)) \\ & + D_x f(p,x)(x''-x)\| \\ \le \; & \|D_x f(p,s(p)) - D_x f(p,x)\|\|x' - x''\|. \end{aligned} \tag{28}$$

The main result of this study is as follows, which provides some sufficient conditions ensuring the convergence of the extended Newton-type method with initial point x_0.

Theorem 14 *Suppose that $\eta > 1$ and let S be the soluton mapping associated with the generalized equation (2) and let $\bar{x} \in S(\bar{p})$. Let $\bar{r}$ be defined in* (7) *such that* (8) *is hold. Assume that f is Lipschitz continuous with respect to p uniform in x with Lipschitz constant $\lambda > 0$, $D_x f$ is continuous at $(\bar{p}, \bar{x})$ with constant $\chi > 0$, $D_x f(p,x)$ is Lipschitz continuous on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with respect to x uniformly in $p \in \mathbb{B}_{r_{\bar{p}}}(\bar{p})$ with a constant $\gamma > 0$ and $G^{-1}_{\bar{p},\bar{x}}(\cdot)$ has a Lipschitz-like localization σ on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with Lipschitz constant $\kappa/(1-\kappa\mu)$. Let $\delta > 0$ be such that*

(a) $\delta \le \min\left\{\frac{r_{\bar{x}}}{2}, \frac{4r_{\bar{y}}}{9(\mu+3\gamma)}, 1, 2\kappa\bar{r}\right\}$,

(b) $3\eta\kappa(19\mu+3\gamma)\delta \le 2(1-\kappa(\chi+\mu)$,

(c) $\|\bar{y}\| < \frac{(\mu+3\gamma)\delta^2}{4}$.

Suppose that

$$\lim_{\substack{x\to\bar{x}\\ p\to\bar{p}}} \operatorname{dist}(\bar{y}, f(p,x) + F(x)) = 0. \tag{29}$$

Then there exists some $\hat{\delta} > 0$ such that any sequence $\{x_n\}$ generated by Algorithm 2 for $p \in \mathbb{B}_{r_{\bar{p}}}(\bar{p})$ with initial point in $\mathbb{B}_{\hat{\delta}}(s(p))$ converges quadratically to the value $s(p)$ of the Lipschitz-like localization s of the solution mapping S on $\mathbb{B}_{r_{\bar{p}}}(\bar{p})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ for the generalized equation (2).

Proof: Let S be the solution mapping associated with the generalized equation (2) and s be the Lipschitz-like localization of S on $\mathbb{B}_{r_{\bar{p}}}(\bar{p})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$. This implies that

$$s(p) = S(p) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}) \text{ for all } p \in \mathbb{B}_{r_{\bar{p}}}(\bar{p}). \tag{30}$$

Take $0 < \hat{\delta} \le \delta$ such that, for each $x_0 \in \mathbb{B}_{\hat{\delta}}(s(p)) \subseteq \mathbb{B}_{\delta}(\bar{x})$,

$$\operatorname{dist}(0, f(p,x_0) + F(x_0)) \le \frac{(\mu+3\gamma)\delta^2}{4}. \tag{31}$$

Noting that such $\hat{\delta}$ exists by (29) and assumption (c) and let $x_0 \in \mathbb{B}_{\hat{\delta}}(s(p))$. By assumption (b), we can write

$$\begin{aligned} \eta\kappa(\mu+3\gamma)\delta &< 3\eta\kappa(19\mu+3\gamma)\delta \\ &\leq 2(1-\kappa(\chi+\mu). \end{aligned} \tag{32}$$

Put

$$t := \frac{\eta\kappa(\mu+3\gamma)\delta}{2(1-\kappa(\chi+\gamma))}. \tag{33}$$

The above two inequalities jointly yield that

$$t \leq 1. \tag{34}$$

Because of assumption (b), we also can write

$$\begin{aligned} 3\kappa(\mu+3\gamma)\delta &< 3\eta\kappa(19\mu+3\gamma)\delta \\ &\leq 2(1-\kappa(\chi+\mu) < 2. \end{aligned} \tag{35}$$

Thus, we have from (35) that

$$(\mu+3\gamma)\delta < \frac{2}{3\kappa}. \tag{36}$$

We will proceed by mathematical induction to show that Algorithm 2 generates at least one sequence and any sequence $\{x_n\}$ generated by Algorithm 2 satisfies the following assertions:

$$\|x_n - s(p)\| \leq 2\delta \tag{37}$$

and

$$\|x_{n+1} - x_n\| \leq t\left(\frac{1}{2}\right)^{2^n}\delta. \tag{38}$$

hold for each $n = 0, 1, 2, \ldots$. For this purpose, we define, for each $x \in X$,

$$r_x := \frac{9\kappa}{10(1-\kappa(\chi+\mu))}\Big(\mu\|x-s(p)\|^2 + 2\|\bar{y}\|\Big). \tag{39}$$

Then, thanks to the fact that $3\kappa(19\mu+3\gamma)\delta \leq 2(1-\kappa(\chi+\mu))$ by assumption (a) and $\|\bar{y}\| < \frac{(\mu+3\gamma)\delta^2}{4}$ by assumption (c). Then for each $x \in \mathbb{B}_{2\delta}(\bar{x})$, (39) yields that

$$\begin{aligned} r_x &\leq \frac{9\kappa}{10(1-\kappa(\chi+\mu))}\Big(\mu(\|x-\bar{x}\| \\ &\quad +\|\bar{x}-s(p)\|)^2 + 2\|\bar{y}\|\Big) \\ &\leq \frac{9\kappa}{10(1-\kappa(\chi+\mu))}\Big(9\mu\delta^2 + \frac{(\mu+3\gamma)\delta^2}{2}\Big) \\ &= \frac{9\kappa(19\mu+3\gamma)\delta^2}{20(1-\kappa(\chi+\mu))} \leq \delta. \end{aligned} \tag{40}$$

Note that (37) is trivial for $n = 0$. To show (38) holds for $n = 0$, firstly we need to show that the point x_1 exists from x_0 for p. To complete this, we have to prove that $\mathcal{D}_p(x_0) \neq \emptyset$ by applying Lemma 9 to the map Φ_{x_0} with $\eta_0 = s(p)$. Let us check that both assertions (5) and (6) of Lemma 9 hold with $r := r_{x_0}$ and $\lambda := \frac{4}{9}$. The assumed assumption of $G^{-1}_{\bar{p},\bar{x}}(\cdot)$ together with Lemma 13 implies that $G^{-1}_{p,s(p)}(\cdot)$ has a Lipschitz-like localization $\Theta_{p,s(p)}(\cdot)$ on $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with Lipschitz constant $\frac{\kappa}{1-\kappa(\chi+\mu)}$ so that the solution mapping S has a Lipschitz-like localization s on $\mathbb{B}_{r_{\bar{p}}}(\bar{p})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with lipschitz constant $\frac{\lambda\kappa}{1-\kappa(\chi+\mu)}$ which satisfies (30) and (31). Noting that $s(p) \in G^{-1}_{p,s(p)}(\bar{y}) \cap \mathbb{B}_{r_{x_0}}(s(p))$ by (17) and (30). Now, we apply the contraction mapping principle to the mapping Φ_{x_0} which is defined in (27) on $\mathbb{B}_{r_{x_0}}(s(p))$. According to the definition of the excess e, we obtain

$$\begin{aligned} &\mathrm{dist}(s(p), \Phi_{x_0}(s(p))) \\ &\leq e(G^{-1}_{p,s(p)}(\bar{y}) \cap \mathbb{B}_{r_{x_0}}(s(p)), \Phi_{x_0}(s(p))) \\ &\leq e(G^{-1}_{p,s(p)}(\bar{y}) \cap \mathbb{B}_{2\delta}(\bar{x}), \Phi_{x_0}(s(p))) \\ &\leq e(G^{-1}_{p,s(p)}(\bar{y}) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), G^{-1}_{p,s(p)}[g_{p,x_0}(s(p))]) \end{aligned} \tag{41}$$

(noting that $\mathbb{B}_{r_{x_0}}(s(p)) \subseteq \mathbb{B}_{\delta}(\bar{x}) \subseteq \mathbb{B}_{2\delta}(\bar{x}) \subseteq \mathbb{B}_{r_{\bar{x}}}(\bar{x})$). By the choice of μ, we have

$$\begin{aligned} &\|g_{p,x_0}(x) - \bar{y}\| \\ &= \|f(p,s(p)) + D_x f(p,s(p))(x-s(p)) \\ &\quad -f(p,x_0) - D_x f(p,x_0)(x-x_0) - \bar{y}\| \\ &\leq \|f(p,s(p)) - f(p,x_0) - D_x f(p,x_0) \\ &\quad (s(p)-x_0) + (D_x f(p,x_0) - D_x f(p,s(p))) \\ &\quad (x-s(p))\| + \|\bar{y}\| \\ &\leq \|f(p,s(p)) - f(p,x_0) - D_x f(p,x_0) \\ &\quad (s(p)-x_0)\| + \|D_x f(p,x_0) - D_x f(p,s(p))\| \\ &\quad \|x-s(p)\| + \|\bar{y}\| \\ &\leq \frac{1}{2}\mu\|s(p)-x_0\|^2 + \gamma\|x_0-s(p)\|\|x-s(p)\| \\ &\quad +\|\bar{y}\|. \end{aligned} \tag{42}$$

Note that $\|x_0 - s(p)\| \leq \hat{\delta} \leq \delta$, $9(\mu+3\gamma)\delta \leq 4r_{\bar{y}}$ by assumption (a) and $\|\bar{y}\| < \frac{(\mu+3\gamma)\delta^2}{4}$ by assumption (c), it follows from (42) that, for each $x \in \mathbb{B}_{2\delta}(\bar{x})$,

$$\begin{aligned} \|g_{p,x_0}(x) - \bar{y}\| &\leq \frac{\mu\delta^2}{2} + 3\gamma\delta^2 + \frac{(\mu+3\gamma)\delta^2}{4} \\ &= \frac{(3\mu+15\gamma)\delta^2}{4} \leq \frac{(3\mu+15\gamma)\delta}{4} \\ &\leq r_{\bar{y}}. \end{aligned} \tag{43}$$

In particular, letting $x = s(p)$ in (42). Then we have that

$$\begin{aligned} \|g_{p,x_0}(s(p)) - \bar{y}\| &\leq \frac{1}{2}\mu\|s(p)-x_0\|^2 \\ &\quad +\|\bar{y}\| \qquad (44) \\ &\leq \frac{1}{2}\mu\delta^2 + \frac{(\mu+3\gamma)\delta^2}{4} \\ &< \frac{(3\mu+3\gamma)\delta}{4} \leq r_{\bar{y}}; \end{aligned}$$

and hence $g_{p,x_0}(s(p)) \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$.

Hence, by (39), (41), (44) and Lipschitz-like localization of $G^{-1}_{p,s(p)}(\cdot)$, we have

$$\begin{aligned} & \operatorname{dist}(s(p), \Phi_{x_0}(s(p))) \\ \leq\ & \frac{\kappa}{1-\kappa(\chi+\mu)}\|\bar{y} - g_{p,x_0}(s(p))\| \\ =\ & (1-\frac{4}{9})r_{x_0} = (1-\lambda)r; \end{aligned} \tag{45}$$

that is, the assertion (5) of Lemma 9 is satisfied.

Now, we show that the assertion (6) of Lemma 9 holds. To end this, let $x', x'' \in \mathbb{B}_{r_{x_0}}(s(p))$. Then we have that $x', x'' \in \mathbb{B}_{r_{x_0}}(s(p)) \subseteq \mathbb{B}_{2\delta}(\bar{x}) \subseteq \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ by (40) and assumption (a), and $g_{p,x_0}(x')$, $g_{p,x_0}(x'') \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ by (43). This, together with (28) and Lipschitz-like localization $\Theta_{p,s(p)}(\cdot)$ of $G^{-1}_{p,s(p)}(\cdot)$, implies that

$$\begin{aligned} & e(\Phi_{x_0}(x') \cap \mathbb{B}_{r_{x_0}}(s(p)), \Phi_{x_0}(x'')) \\ \leq\ & e(\Phi_{x_0}(x') \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), \Phi_{x_0}(x'')) \\ =\ & e(G^{-1}_{p,s(p)}[g_{p,x_0}(x')] \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), G^{-1}_{p,s(p)}[g_{p,x_0}(x'')]) \\ \leq\ & \|\Theta_{p,s(p)}(g_{p,x_0}(x')) - \Theta_{p,s(p)}(g_{p,x_0}(x''))\| \\ \leq\ & \frac{\kappa}{1-\kappa(\chi+\mu)}\|g_{p,x_0}(x') - g_{p,x_0}(x'')\| \\ \leq\ & \frac{\kappa}{1-\kappa(\chi+\mu)}\|D_x f(p, s(p)) - \\ & D_x f(p, x_0)\|\|x' - x'\| \\ \leq\ & \frac{\gamma\kappa}{1-\kappa(\chi+\mu)}\|s(p) - x_0\|\|x' - x'\|. \end{aligned} \tag{46}$$

Using (46) and the choice of x_0, we have

$$\begin{aligned} e(\Phi_{x_0}(x') \quad \cap \quad & \mathbb{B}_{r_{x_0}}(s(p)), \Phi_{x_0}(x'')) \\ \leq \quad & \frac{\gamma\kappa\delta}{1-\kappa(\chi+\mu)}\|x' - x'\|. \end{aligned} \tag{47}$$

It follows from $9\gamma\kappa\delta < 3\kappa(19\mu+3\gamma)\delta \leq 2(1-\kappa(\chi+\mu))$ as in assumption (a) together with (47) that

$$\begin{aligned} e(\Phi_{x_0}(x') \quad \cap \quad & \mathbb{B}_{r_{x_0}}(\bar{x}), \Phi_{x_0}(x'')) \leq \frac{2}{9}\|x' - x''\| \\ < \quad & \frac{4}{9}\|x' - x''\| = \lambda\|x' - x''\|. \end{aligned} \tag{48}$$

This yields that the assertion (6) of Lemma 9 is satisfied. Since both assertions of Lemma 9 are fulfilled, we can deduce the existence of a fixed point $\hat{x}_1 \in \mathbb{B}_{r_{x_0}}(s(p))$ satisfying $\hat{x}_1 \in \Phi_{x_0}(\hat{x}_1)$, which translates to $g_{p,x_0}(\hat{x}_1) \in G_{p,s(p)}(\hat{x}_1)$. This means that $0 \in f(p, x_0) + D_x f(p, x_0)(\hat{x}_1 - x_0) + F(\hat{x}_1)$. This implies that $\hat{x}_1 - x_0 \in \mathcal{D}_p(x_0)$ and thus $\mathcal{D}_p(x_0) \neq \emptyset$. Since $\eta > 1$, we can choose $d_0 \in \mathcal{D}_p(x_0)$ such that

$$\|d_0\| \leq \eta \operatorname{dist}(0, \mathcal{D}_p(x_0)).$$

By Algorithm 2, $x_1 := x_0 + d_0$ is defined. Thus, the point x_1 is generated by Algorithm 2.

Now, we show that (38) holds also for $n = 0$. By the assumed assumption of $G^{-1}_{\bar{p},\bar{x}}(\cdot)$, it follows from Lemma 13 that for each $u \in \mathbb{B}_{r_{\bar{x}}}(\bar{x})$, the mapping $G^{-1}_{p,u}(\cdot)$ has a Lipschitz localization $\Theta_{p,u}(\cdot)$ on $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with $\kappa/(1-\kappa(\chi+\mu))$. In particular, $G^{-1}_{p,x_0}(\cdot)$ has a Lipschitz localization $\Theta_{p,x_0}(\cdot)$ on $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with $\kappa/(1-\kappa(\chi+\mu))$ as $x_0 \in \mathbb{B}_{\hat{\delta}}(s(p)) \subseteq \mathbb{B}_{2\delta}(\bar{x}) \subseteq \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ by assumption (a) and by the choice of $\hat{\delta}$.

Furthermore, using (36) and the fact $\delta \leq 2\kappa\bar{r}$ by assumption (a) together with assumption (c) imply that

$$\|\bar{y}\| < \frac{(\mu+3\gamma)\delta^2}{4} < \frac{(\mu+3\gamma)\delta}{4} \cdot \delta \leq \frac{\bar{r}}{3}, \tag{49}$$

and hence (31) implies that

$$\begin{aligned} \operatorname{dist}(0, G_{p,x_0}(x_0)) \quad = \quad & \operatorname{dist}(0, f(p, x_0) + F(x_0)) \\ \leq \quad & \frac{(\mu+3\gamma)\delta^2}{4} \leq \frac{\bar{r}}{3}. \end{aligned}$$

It is noted earlier that $x_0 \in \mathbb{B}_{\delta}(\bar{x}) \subseteq \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ and $0 \in \mathbb{B}_{\bar{r}}(\bar{y})$ by (49). Thus, by utilizing Lemma 5 we get

$$\operatorname{dist}(x_0, G^{-1}_{p,x_0}(0)) \leq \frac{\kappa}{1-\kappa(\chi+\mu)}\operatorname{dist}(0, G_{p,x_0}(x_0)). \tag{50}$$

This together with (15) gives that

$$\begin{aligned} \operatorname{dist}(0, \mathcal{D}_p(x_0)) \quad = \quad & \operatorname{dist}(x_0, G^{-1}_{p,x_0}(0)) \\ \leq \quad & \frac{\kappa}{1-\kappa(\chi+\mu)}\operatorname{dist}(0, G_{p,x_0}(x_0)). \end{aligned} \tag{51}$$

According to Algorithm 2 and using (50) and (51), we have

$$\begin{aligned} \|d_0\| \quad \leq \quad & \eta \operatorname{dist}(0, \mathcal{D}_p(x_0)) \\ \leq \quad & \frac{\eta\kappa}{1-\kappa(\chi+\mu)} \operatorname{dist}(0, G_{p,x_0}(x_0)) \\ \leq \quad & \frac{\eta\kappa\delta}{1-\kappa(\chi+\mu)} \cdot (\frac{\mu+3\gamma}{4})\delta. \end{aligned} \tag{52}$$

It follows from (33), that

$$\|x_1 - x_0\| = \|d_0\| \leq t(\frac{1}{2})\delta.$$

and therefore, (38) is hold for $n = 0$.

We assume that $x_1, x_2, \ldots, x_k$ are constructed so that (37) and (38) are hold for $n = 0, 1, 2, \ldots, k-1$. We will show that there exists x_{k+1} such that (37) and (38) are also hold for $n = k$. Since (37) and (38) are true for each $n \leq k-1$, we have the following inequality

$$\begin{aligned} \|x_k - s(p)\| \quad \leq \quad & \sum_{i=0}^{k-1}\|d_i\| + \|x_0 - s(p)\| \\ \leq \quad & t\delta\sum_{i=0}^{k-1}\left(\frac{1}{2}\right)^{2^i} + \delta \\ \leq \quad & 2\delta. \end{aligned} \tag{53}$$

This shows that (37) holds for $n = k$. Finally, we will show that the assertion (38) holds for $n = k$. For doing this, we will apply again the contraction mapping principle to Φ_{x_k} as same as in (45) and (48) on the ball $\mathbb{B}_{r_{x_k}}(s(p))$. Then we can deduce the existence of a fixed point $\hat{x}_{k+1} \in \mathbb{B}_{r_{x_k}}(s(p))$ satisfying $\hat{x}_{k+1} \in \Phi_{x_k}(\hat{x}_{k+1})$, which translates to $g_{p,x_k}(\hat{x}_{k+1}) \in G_{p,s(p)}(\hat{x}_{k+1})$. This means that $0 \in f(p, x_k) + D_x f(p, x_k)(\hat{x}_{k+1} - x_k) + F(\hat{x}_{k+1})$, that is, $\mathcal{D}_p(x) \neq \emptyset$. Choose $d_k \in \mathcal{D}_p(x_k)$ such that

$$\|d_k\| \leq \eta \operatorname{dist}(0, \mathcal{D}_p(x_k)).$$

Then by Algorithm 2, set $x_{k+1} := x_k + d_k$. Moreover, applying Lemma 13 we infer that $G^{-1}_{p,x_k}(\cdot)$ has a Lipschitz localization $\Theta_{p,x_k}(\cdot)$ on $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with

$\kappa/(1-\kappa(\chi+\mu))$. Therefore, we have that

$$\begin{aligned} &\|x_{k+1}-x_k\| \\ =\ & \|d_k\| \le \eta\, \mathrm{dist}(0,\mathcal{D}_p(x_k)) \\ \le\ & \frac{\eta\kappa}{1-\kappa(\chi+\mu)}\, \mathrm{dist}(0, G_{p,x_k}(x_k)) \\ =\ & \frac{\eta\kappa}{1-\kappa(\chi+\mu)}\, \mathrm{dist}(0, f(p,x_k)+F(x_k)) \\ \le\ & \frac{\eta\kappa}{1-\kappa(\chi+\mu)}\|f(p,x_k)-f(p,x_{k-1})- \\ & D_xf(p,x_{k-1})(x_k-x_{k-1})\| \\ \le\ & \frac{\eta\mu\kappa}{2(1-\kappa(\chi+\mu))}\|x_k-x_{k-1}\|^2 \\ \le\ & \frac{\eta\mu\kappa}{2(1-\kappa(\chi+\mu))}\left(t\left(\frac{1}{2}\right)^{2^{k-1}}\delta\right)^2 \\ \le\ & t\left(\frac{1}{2}\right)^{2^k}\delta. \end{aligned}$$

This implies that (38) holds for $n=k$ and therefore the proof is completed.

In particular, in the case when $\bar{x}$ is a solution of (2) for $\bar{p}$, that is, $\bar{y}=0$Theorem 14 is reduced to the following corollary, which gives the local convergent result for the Extended Newton-type method.

Corollary 15 *Suppose that $\eta>1$ and $\bar{x}$ satisfies (2) for $\bar{p}$. Let S be the solution mapping associated with the generalized equation (2) such that $\bar{x}\in S(\bar{p})$. Assume that f is Lipschitz continuous with respect to p uniform in x with Lipschitz constant $\lambda>0$, D_xf is continuous at $(\bar{p},\bar{x})$ with constant $\chi>0$, $D_xf(p,x)$ is Lipschitz continuous with respect to x uniformly in p with a constant $\gamma>0$ and $G^{-1}_{\bar{p},\bar{x}}(\cdot)$ has a pseudo-Lipschitz localization σ around $(0,\bar{x})$ for $\bar{p}$. Suppose that*

$$\lim_{\substack{x\to\bar{x}\\ p\to\bar{p}}} \mathrm{dist}(0, f(p,x)+F(x)) = 0. \tag{54}$$

Then there exists some $\hat{\delta}>0$ such that any sequence $\{x_n\}$ generated by Algorithm 2 for p with initial point in $\mathbb{B}_{\hat{\delta}}(s(p))$ converges quadratically to the value $s(p)$ of the pseudo-Lipschitz localization s of the solution mapping S for the generalized equation (2).

Proof: Let $G^{-1}_{\bar{p},\bar{x}}(\cdot)$ has a pseudo-Lipschitz localization σ around $(0,\bar{x})$ for $\bar{p}$. Then there exist positive constants $r_0, \hat{r}_{\bar{x}},\ \hat{r}_{\bar{p}}, \mu$ and κ satisfy the following condition:

$$\sigma(y) = G^{-1}_{\bar{p},\bar{x}}(y)\cap \mathbb{B}_{\hat{r}_{\bar{x}}}(\bar{x}) \quad \text{for every } \ y\in\mathbb{B}_{r_0}(0).$$

$$\text{and } \|\sigma(y)-\sigma(y')\| \le \frac{\kappa}{1-\kappa\mu}\|y-y'\| \forall\, y,y'\in\mathbb{B}_{r_0}(0).$$

Let $\gamma>0$ and $\chi>0$. Choose $0<\delta\le 1$. Since (54) is true and $\eta>1$, one can choose $\bar{y}$ near 0 such that $\|\bar{y}\| < \dfrac{(\mu+3\gamma)\delta^2}{4}$ and $3\eta\kappa(19\mu+3\gamma)\delta \le 2(1-\kappa(\chi+\mu)$. Then for, $0<r_{\bar{y}}\le r_0, 0<r_{\bar{x}}\le\hat{r}_{\bar{x}}, 0<r_{\bar{p}}\le\hat{r}_{\bar{p}}$, one says that $G^{-1}_{\bar{p},\bar{x}}(\cdot)$ has a Lipschitz-like localization σ on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ for $\bar{p}$ with constant $\kappa/(1-\kappa\mu)$. Let $\mu\in(0,1)$ be such $\kappa\mu<1$, and $r_{\bar{y}}-\mu r_{\bar{x}}>0$. Then

$$\bar{r} = \min\Big\{r_{\bar{y}}-\mu r_{\bar{x}}, \frac{r_{\bar{x}}(1-\kappa\mu)}{\kappa}\Big\} > 0,$$

and

$$\min\left\{\frac{r_{\bar{x}}}{2},\ \frac{4r_{\bar{y}}}{9(\mu+3\gamma)},\ 1,\ 2\kappa\bar{r}\right\} > 0. \tag{55}$$

Then we can assume that $\delta \le \min\left\{\frac{r_{\bar{x}}}{2},\ \frac{4r_{\bar{y}}}{9(\mu+3\gamma)},\ 1,\ 2\kappa\bar{r}\right\}$. Thus it is routine to check that inequalities (a)-(c) of Theorem 14 are satisfied. Therefore, Theorem 14 is applicable to complete the proof of the corollary.

4 Concluding Remarks

We have established semilocal and local convergence result for the extended Newton-type method with $\eta>1$ under the assumptions that $G^{-1}_{\bar{p},\bar{x}}(\cdot)$ has a Lipschitz-like localization σ. Indeed, under some sufficient conditions, we have presented the existence of a sequence generated by extended Newton-type method and proved this sequence converges to the value $s(p)$ of the Lipschitz-like localization s of the solution mapping S for the generalized equation (2).

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References:

[1] A. D. Ioffe and V. M. Tikhomirov, Theory of extremal problems, *Studies in Mathematics and its Applications,* North–Holland, Amsterdam, 1979

[2] A. L. Dontchev, The Graves theorem revisited, *J. Convex Anal.* 3, 1996, pp. 45–53.

[3] A. L. Dontchev and R. T. Rockafellar, *Implicit functinos and solution mappings: A view from variational analysis*, Springer Science+Business Media, LLC, New York, 2009

[4] A. L. Dontchev and R. T. Rockafellar, Robinson's implicit functions theorem and its extensions, *Math. Program Ser. B* 117, 2009, pp. 129–147.

[5] A. L. Dontchev and R. T. Rockafellar, Newton's method for generalized equations: a sequential implicit function theorem, *Math. Program Ser. B* 123, 2010, PP. 139–159.

[6] A. L. Dontchev and W. W. Hager, An inverse mapping theorem for set-valued maps, *Proc. Amer. Math. Soc.* 121, 1994, pp. 481–498.

[7] A. Pietrus, Does Newton's method for set-valued maps converges uniformly in mild differentiability context?, *Rev. Colombiana Mat.* 32, 2000, pp. 49–56.

[8] C. Jean-Alexis and A. Pietrus, On the convergence of some methods for variational inclusions, *Rev. R. Acad. Cien. serie A. Mat.* 102 (2), 2008, pp. 355–361.

[9] C. Li and K. F. Ng, Majorizing functions and convergence of the Gauss-Newton method for convex composite optimization, *SIAM J. Optim.* 18, 2007, 613–642.

[10] I. K. Argyros and S. Hilout, Local convergence of Newton-like methods for generalized equations, *Appl. Math. and Comp.* 197, 2008, pp. 507–514.

[11] J. P. Aubin, Lipschitz behavior of solutions to convex minimization problems,*Math. Oper. Res.* 9, 1984, pp. 87–111.

[12] J. P. Aubin and H. Frankowska, *Set-valued analysis*, Birkhäuser, Boston, 1990

[13] J. P. Dedieu and M. H. Kim, Newton's method for analytic systems of equations with constant rank derivatives, *J. Complexity* 18, 2002, pp. 187–209.

[14] J. P. Dedieu and M. Shub, Newton's method for overdetermined systems of equa- tions, *Math. Comp.* 69, 2000, pp. 1099–1115.

[15] M. H. Rashid, S. H. Yu, C. Li and S. Y. Wu, Convergence analysis of the Gauss-Newton method for Lipschitz-like maps,*J. Optim. Theory Appl.*, 158(1), 2013, pp. 216–233.

[16] M. H. Rashid, On the Convergence of Extended Newton-type Method for Solving Variational Inclusions, *Cogent Mathematics* 1(1), 2014, pp.–(DOI 10.1080/23311835.2014.980600).

[17] M.H. Rashid, A Convergence Analysis of Gauss-Newton-type Method for Hölder Continuous Maps,*Indian Journal of Mathematics* 57(2), 2014, pp. 181–198.

[18] S. Haliout, C. J. Alexis and A. Pietrus, A semilocal convergence of the secant-type method for solving a generalized equations, *Possitivity* 10, 2006, pp. 673–700.

[19] X. B. Xu and C. Li, Convergence criterion of Newton's method for singular systems with constant rank derivatives, *J. Math. Anal. Appl.* 345, 2008, pp. 689–701.