Pricing vulnerable European options under a Markov-modulated jump diffusion process

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Abstract: In this paper, we investigate the pricing of vulnerable European options under a Markov-modulated jump diffusion process. The states of market economy which are described by a two-state continuous time Markov-chain are explained as a stable state and a high volatility state. The dynamic of the risky asset is described by a Markov-modulated geometry Brownian motion when the market state is stable, otherwise, it follows a Markov-modulated jump diffusion process. We consider two types of models to describe default risk: one is the structural model, the other is the reduced form model. By utilizing techniques of measure changes, some analytic formulas for pricing vulnerable European options are derived under these models.

Key–Words: Jump diffusion; Markov-modulated; Vulnerable options

1 Introduction

In a financial market, the default risk often effects every financial contract. How to quantify the default risk is very important for both market practitioners and financial researchers. there are two most popular of models to quantify it, that is structural models and reduced form models, more details see Duffie and Singleton[1] and Bielecki and Rutkowski[2]. Johnson and Stulz [3] first studied the pricing of options with the default risk, namely the so-called vulnerable options. Hull and White[4] and Jarrow and Turnbull [5] took into account the default risk when pricing options traded on the over the counter market. However, Hull and White[4] and Jarrow and Turnbull [5] assumed independence between the total assets of the option writer and the underlying asset of the option. Klein [6] extended [4] and [5] by relaxing their assumption. Since then, there exists a significant number of studies on the pricing of vulnerable options, for example, Klein and Inglis [7], Liao and Huang [8] and Capponi et al.[9].

A large number of empirical studies show that the risky asset price follows a geometric Brownian motion is not realistic. Consequently, many different financial models have been proposed to describe the dynamics of risky assets. For example, the jump diffusion processes, Markov-modulated model, stochastic volatility model, constant elasticity of variance model, stochastic interest rate model and Studies including those of Merton[10], Elliott and Osakwe [11], Wang et al.[12], Li et al.[13], Chang et al.[14] and others. Edwards [15] presented a new Markov-modulated model. He supposed that the Markov-chain had only two states which represent high volatility economic state and low volatility economic state. When the market is at high volatility state, the dynamic of risky asset follows the Lévy process, otherwise, it satisfies log normal distribution. Elliott [11] extended Edwards's [15] model to N states and introduced a Markov-modulated pure jump model. In their model, they assumed that the Markov-chain had N states and the compensator of a jump process changed its value when the Markov-chain changed.

In this paper, we are in line with Edwards's [15] idea and consider a two-state Markov-modulated jump diffusion model for the vulnerable option valuation. Our model is similar to those of Elliott[11]. However, we assume that the Markov-chain has two states and the compensator of the jump process is zero when the market is at low volatility state,i.e geometric Brownian motion. When the market is at the high volatility state, the risky asset follows a jump diffusion model.We investigate the pricing of vulnerable options under a two-state Markov-modulated jump diffusion model. Most of the literatures about the price of vulnerable options did not consider the regime switching effect. Wang and Wang [16] incorporated the regime switching into the model and obtained the price of vulnerable options. However, the work in this paper is different from Wang and Wang [16]. The major differences between their papers and this one are as follows: firstly, the model considered here is different from that in Wang and Wang [16]. Secondly, the Markov-regime switching risk is priced in this paper. Finally, we provide the analytic formulas for pricing vulnerable options under structural models and reduced form models, respectively.

This paper is structured as follows: Section 2 describes the dynamic of the risky asset price under the two-state regime switching jump diffusion model. In Section 3, we present an equivalent martingale measure. Section 4 obtains the pricing formulas of vulnerable options under the structural model. Section 5provides the pricing formulas of vulnerable options under the reduced form model. The final section gives a conclusion.

2 Financial market

Suppose that we have a given complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$, where P is a real world measure and T is a fixed investment horizon. We also have ε_t, a continuous time Markov chain, to represent the states of the market economy. Moreover, we suppose that ε has the state space $\mathcal{M} = \{e_1, e_2\}$, where states $e_1 = (1,0)' \in \mathcal{R}^2$ and $e_2 = (0,1)' \in \mathcal{R}^2$ of the chain can be explained to represent a "high volatility " state and a "stable" state, respectively. Assume that the Markov chain process ε has a generator $\mathbf{A}(t) = [a_{ij}(t)]_{i,j\in\mathcal{M}}$. From Elliott et al.[17], the semi-martingale dynamics of the Markov chain is given by

$$\varepsilon_t = \varepsilon_0 + \int_0^t \mathbf{A}(s)ds + M(t),$$

where $M(t)$ is a P martingale.

We consider a financial market with three traded assets, a riskless bond and two risky assets. The riskless bond price process $B = (B_t)$ evolves according to

$$dB_t = rB_t dt, \tag{1}$$

where r is the fixed interest rate. For sake of convenience, we assume $B_0 = 1$.

We assume that the risky asset price follows a jump diffusion process when the state of market economy is state e_1, whereas the risky asset price follows the Black-Scholes model when the state of market economy is state e_2. The dynamic of the risky asset S is specified as

$$\begin{aligned} \frac{dS_t}{S_{t-}} &= \mu dt + \sigma dW_{1t} - \lambda_S(e^{\frac{\delta^2}{2}} - 1)I_{\{\varepsilon_t=e_1\}}dt \\ &+ I_{\{\varepsilon_t=e_1\}}d(\sum_{j=1}^{N_t^1} e^{X_j} - 1), \end{aligned} \tag{2}$$

where $I_{\{\cdot\}}$ is a indictor function, μ and σ is the appreciation rate and volatility of the risky asset S and W_1 is a standard P Brownian motion, N_t^1 is a Poisson process with intensity λ_S, the jump amplitudes $\{X_j\}_{j=1,2,...}$ are independent and identical distribution with probability density function $f(x) = \frac{1}{\sqrt{2\pi}\delta}e^{-\frac{x^2}{2\delta^2}}, -\infty < x < \infty$.

Let J_t denote the occupation time of ε in state e_1 over the time period $[0, t]$, then

$$J_t = \int_0^t I_{\{\varepsilon_s=e_1\}}ds. \tag{3}$$

Consequently, the risky asset price process $S = (S_t)$ according to

$$\begin{aligned} S_t &= S_0 \exp\Big\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_{1t} + \sum_{j=1}^{N_{J_t}^1} X_j \\ &- \lambda_S(e^{\frac{\delta^2}{2}} - 1)J_t\Big\}. \end{aligned} \tag{4}$$

In addition, the dynamic of V_t is described as following

$$\begin{aligned} V_t &= V_0 \exp\Big\{(b - \frac{1}{2}\nu^2)t + \nu W_{2t}^\theta + \sum_{j=1}^{N_{J_t}^2} Y_j \\ &- \lambda_V(e^{\frac{\gamma^2}{2}} - 1)J_t\Big\}, \end{aligned} \tag{5}$$

where b and ν is the appreciation rate and volatility of risky asset V and W_2 is a standard P Brownian motion, N_t^2 is a Poisson process with intensity λ_V, the jump amplitudes $\{Y_j\}_{j=1,2,...}$ are independent and identical distribution with probability density function $g(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2\gamma^2}}, -\infty < y < \infty$. Furthermore, we let the correlation coefficient of dW_{1t} and dW_{2t} be ρ. ε, N^1, N^2, $X = (X_j)_{j=1,2,...}$ and $Y = (Y_j)_{j=1,2,...}$

are supposed that be mutually independent and independent of W_1 and W_2.

For convenience, we write (4) and (5) as the following

$$\begin{aligned} dS_t &= S_t\Big(\mu dt + \sigma dW_{1t} + \int_{-\infty}^{\infty}(e^x-1)J_S(dx,dt) \\ &- \int_{-\infty}^{\infty}(e^x-1)k_S(dx,dt)\Big), \end{aligned} \tag{6}$$

and

$$\begin{aligned} dV_t &= V_t\Big(bdt + \nu dW_{2t} + \int_{-\infty}^{\infty}(e^y-1)J_V(dy,dt) \\ &- \int_{-\infty}^{\infty}(e^y-1)k_V(dy,dt)\Big), \end{aligned} \tag{7}$$

where $< \cdot,\cdot >$ denotes the inner product operator and $J_S(dx,dt)$ and $k_V(dy,dt)$ are the random measures associated with the jump of S and V, respectively. For $m \in \{S,V\}$, $k_m(dx,dt) = \sum_{j=1}^{2} < \varepsilon_{t-}, e_j > k_{mj}(x)dxdt$ is its compensator, and $k_{mj}(x)$ is a Lévy measure which depends on the states of the Markov chain ε, $k_{S1}(x) = \lambda_S f(x)$, $k_{S2}(x) = 0$, $k_{V1}(y) = \lambda_V g(y)$ and $k_{V2}(y) = 0$.

3 Equivalent martingale measures

In this paper, we use a Markov-modulated jump diffusion model to describe the dynamics of the risky assets. Since the jumps and Markov-chain render the market is incomplete, the equivalent martingale measure is no unique. In order to pricing vulnerable options, we need to choose an equivalent martingale measure for pricing vulnerable options.

We first define

$$d\Lambda^\theta(t) = \Lambda^\theta(t)(\theta_1 dW_{1t} + \theta_2 dW_{2t}), \Lambda^\theta(0) = 1,$$

where θ_1 and θ_2 are constants.

We consider a real valued and bounded stochastic process $\mathbf{C}(t) := \{c_{ij}(t)|t \in [0,T]\}$ on $(\Omega,\mathcal{F},P)$ such that $c_{ij}(t)$ satisfy the following two conditions:

1. $c_{ij}(t) \geq 0$ for $i \neq j$;
2. $\sum_{i=1}^{N} c_{ij}(t) = 0$.

Define the following matrix:

$$\mathbf{D} = (d_{ij}(t))_{i,j=1,2} = [c_{ij}(t)/a_{ij}(t)]_{i,j=1,2}. \tag{8}$$

For $t \in [0,T]$, write $\mathbf{N}(t) = \int_0^t(\mathbf{I} - diag(\varepsilon(u-)))d\varepsilon(u)$. Here $\mathbf{N}(t)_{t\in[0,T]}$ is a vector of counting processes, its component $\mathbf{N}_i(t)$ counts the number of times that the chain ε jumps to state e_i in the time interval $[0,t]$, for each $i = 1,2$. Then we cite the following result from Dufour and Elliott [18] without proof.

Lemma 1 *For a given rate matrix $\mathbf{A}(t)$, let $\mathbf{a}(t) = (a_{11}(t), a_{22}(t))'$ and $\mathbf{A}_0(t) = \mathbf{A}(t) - \mathbf{diag}(\mathbf{a}(t))$, where $\mathbf{diag}(\mathbf{y})$ is a diagonal matrix with the diagonal elements given by the vector $\mathbf{y}$. Define*

$$\tilde{\mathbf{N}}(t) = \mathbf{N}(t) - \int_0^t \mathbf{A}_0(u)\mathbf{X}(u)du.$$

Then $\tilde{\mathbf{N}} = \tilde{\mathbf{N}}(t)_{t\in[0,T]}$ is an P martingale.

Let $\mathbf{d}(t) = (d_{11}(t), d_{22}(t))'$ and $\mathbf{D}_0(t) = \mathbf{D}(t) - \mathbf{diag}(\mathbf{d}(t))$. We consider a process $\Lambda^C(t) = 1 + \int_0^t \Lambda^C(u-)[\mathbf{D}_0(u)\mathbf{X}(u-) - \mathbf{1}](d\mathbf{N}(u) - \mathbf{A}_0(u)\mathbf{X}(u)du)$. Define $\Lambda^{\theta,C} = \Lambda^{\theta,C}(t)_{t\in[0,T]}$ as the product of the two density processes Λ^θ and Λ^C, that is $\Lambda^{\theta,C}(t) = \Lambda^\theta(t)\Lambda^C(t)$. Moreover, we present a new probability measure $Q^{\theta,C}$, which is defined by the following

$$\frac{dQ^{\theta,C}}{dP}\Big|_{\mathcal{F}_t} = \Lambda^{\theta,C}(t).$$

Lemma 2 *Let $\theta_1 = \frac{\rho(b-r)\sigma-(\mu-r)\nu}{(1-\rho^2)\sigma\nu}$, $\theta_2 = \frac{\rho(\mu-r)\nu-(b-r)\sigma}{(1-\rho^2)\sigma\nu}$. Then the measure $Q^{\theta,C}$ is a risk neutral martingale measure, and $W^\theta_{1t} = W_{1t} - \theta_1 t - \rho\theta_2 t$, $W^\theta_{2t} = W_{2t} - \theta_2 t - \rho\theta_1 t$ are two standard Brownian motions under $Q^{\theta,C}$ with $corr(dW^\theta_{1t}, dW^\theta_{2t}) = \rho$. Furthermore, $\mathbf{C}(t) := \{c_{ij}(t)|t \in [0,T]\}$ is a family of rate matrices of the Markov chain ε.*

Proof. Let $\widetilde{S}_t = e^{-rt}S_t$, $\widetilde{V}_t = e^{-rt}V_t$, $\{\mathcal{F}^\varepsilon_t\}$ denote the P augmentation of the natural filtration generated by the Markov-chain ε, and write $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}^\varepsilon_T$. From (6) and by Ito's formula, we have

$$\begin{aligned} \widetilde{S}_t &= \widetilde{S}_u \exp\Big\{\int_u^t(\mu - r - \frac{1}{2}\sigma^2)ds + \int_u^t \sigma dW_{1s} \\ &+ \int_u^t\int_{-\infty}^{\infty}(xJ_1(dx,ds) - (e^x-1)k_S(dx,ds))\Big\}. \end{aligned}$$

According to the law of condition expectation and the above equation, we obtain

$$\begin{aligned} & E_{Q^{\theta,C}}[\widetilde{S}_t|\mathcal{F}_t] \\ =\ & E_{\hat{Q}}[E_{Q^{\theta,C}}[\widetilde{S}_t|\mathcal{G}_t]|\mathcal{F}_t] \\ =\ & \widetilde{S}_u \exp\Big\{\int_0^t(\mu - r + \sigma\theta_1 + \rho\sigma\theta_2)du\Big\} \\ =\ & \widetilde{S}_u. \end{aligned}$$

The last equality follows by $\theta_1 = \frac{\rho(b-r)\sigma-(\mu-r)\nu}{(1-\rho^2)\sigma\nu}$, $\theta_2 = \frac{\rho(\mu-r)\nu-(b-r)\sigma}{(1-\rho^2)\sigma\nu}$. Using the same method we get that $E_{Q^{\theta,C}}[\widetilde{V}_t|\mathcal{F}_u] = \widetilde{V}_u$. Hence, $Q^{\theta,C}$ is a risk neutral martingale measure. By Girsanov's theorem,

$$\begin{aligned} W_{1t}^{\theta} &= W_{1t} - \theta_1 t - \rho\theta_2 t, \\ W_{2t}^{\theta} &= W_{2t} - \theta_2 t - \rho\theta_1 t, \end{aligned}$$

are two standard $Q^{\theta,C}$ Brownian motions, and $corr(dW_{1t}^{\theta}, dW_{2t}^{\theta}) = \rho$. In addition, we find that $\mathbf{C}$ is a family of rate matrices of the chain ε under $Q^{\theta,C}$ from Dufour and Elliott [18]. Then we complete the proof of Lemma 2.

From Lemma 2, we find that, under the risk neutral martingale measure $Q^{\theta,C}$, the dynamics of risky assets S_t and V_t are given by

$$\begin{aligned} dS_t &= S_t\Big(rdt + \sigma dW_{1t}^{\theta} + \int_{-\infty}^{\infty}(e^x-1)J_S(dx,dt) \\ &- \int_{-\infty}^{\infty}(e^x-1)k_S(dx,dt)\Big), \qquad (9) \\ dV_t &= V_t\Big(rdt + \nu dW_{2t}^{\theta} + \int_{-\infty}^{\infty}(e^y-1)J_V(dy,dt) \\ &- \int_{-\infty}^{\infty}(e^y-1)k_V(dy,dt)\Big). \qquad (10) \end{aligned}$$

Remark 3 *We assume that Markov chain ε, Brownian motions W_1 and W_2 are independent of random measures $J_S(dx,dt)$ and $J_V(dy,dt)$. Hence, the compensators of $J_S(dx,dt)$ and $J_V(dy,dt)$ are not changed from the measure P to the risk neutral measure $Q^{\theta,C}$.*

From (9), (10) and Remark 3, we have

$$\begin{aligned} S_t &= S_0\exp\Big\{(r-\frac{1}{2}\sigma^2)t + \sigma W_{1t}^{\theta} + \sum_{j=1}^{N_{J_t}^1} X_j \\ &- \lambda_S(e^{\frac{\delta^2}{2}}-1)J_t\Big\}. \qquad (11) \end{aligned}$$

and

$$\begin{aligned} V_t &= V_0\exp\Big\{(r-\frac{1}{2}\nu^2)t + \nu W_{2t}^{\theta} + \sum_{j=1}^{N_{J_t}^2} Y_j \\ &- \lambda_V(e^{\frac{\gamma^2}{2}}-1)J_t\Big\}. \qquad (12) \end{aligned}$$

4 Structural model

In this section, we adopt the framework in Klein [6] to describe the payoff of vulnerable European options. A vulnerable European call option has a promised payoff $(S_T - K)^+$. If the option writer's asset V_T is less than a threshold value D^*, then option writer will default and can't pay the promised payoff at the maturity time T. In the case of default, only the proportion $\frac{(1-\alpha)V_T}{D}$ of the promised payoff is paid out by the option writer. However, if V_T is greater than D^*, the option writer will pay the promised payoff at time T. Here α, D and D^* are constants.

Let $\psi^{call}(T)$ denote the payoff of a vulnerable European call option. Then it is given by

$$\begin{aligned} &\psi^{call}(T) \\ &= (S_T-K)^+ I_{\{V_T\geq D^*\}} \\ &+ \frac{(1-\alpha)V_T}{D}(S_T-K)^+ I_{\{V_T<D^*\}}. \qquad (13) \end{aligned}$$

Moreover, the payoff $\psi^{put}(T)$ of a vulnerable European put option is given by

$$\begin{aligned} &\psi^{put}(T) \\ &= (K-S_T)^+ I_{\{V_T\geq D^*\}} \\ &+ \frac{(1-\alpha)V_T}{D}(K-S_T)^+ I_{\{V_T<D^*\}}. \qquad (14) \end{aligned}$$

Before giving the option pricing formula, we present two useful Propositions. Since ε is independent of $J_S(dx,dt)$ and $J_V(dy,dt)$, measure of change from $Q^{\theta,C}$ to Q_S and Q_{SV} does not change the dynamic of ε. Furthermore, it is easy to obtain the following two propositions by Girsanov's theorem. We present them without giving the proof.

Proposition 4 *Let η_t denote the Radon-Nikodym process which is given by*

$$\begin{aligned} \eta_t &= \frac{dQ_S}{dQ^{\theta,C}}\Big|_{\mathcal{H}_t} = \frac{S_t}{E_{Q^{\theta,C}}[S_t|\mathcal{F}_T^{\varepsilon}]} \\ &= \exp\Big\{\sigma W_{1t}^{\theta} - \frac{1}{2}\sigma^2 t + \int_0^t\int_{-\infty}^{\infty} xJ_S(dx,du) \\ &- \int_0^t\int_{-\infty}^{\infty}(e^x-1)k_S(dx,du)\Big\}. \end{aligned}$$

Then under the probability measure Q_S and conditional on $\mathcal{F}_T^{\varepsilon}$,

$$\begin{aligned} W_{1t}^{S} &= W_{1t}^{\theta} - \sigma t, \\ W_{2t}^{S} &= W_{2t}^{\theta} - \rho\sigma t, \end{aligned}$$

are two standard Brownian motions and $corr(dW_{1t}^S, dW_{2t}^S) = \rho$. The compensator of random measure $J_S(dx,dt)$ is given by

$$\bar{k}_S(dx,dt) = e^x k_S(dx,dt) = \lambda_S^* \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{(x-\delta^2)^2}{2\delta^2}},$$

with $\lambda_S^* = \lambda_S e^{\frac{\delta^2}{2}}$. *Moreover, the generator of Markov chain and the compensator of random measure* $J_V(dy,dt)$ *do not be changed.*

Proposition 5 *Let* ζ_t *denote the Radon-Nikodym process which is given by*

$$\begin{aligned}\zeta_t &= \frac{dQ_{SV}}{dQ^{\theta,C}}\Big|_{\mathcal{H}_t} = \frac{S_tV_t}{E_{Q^{\theta,C}}[S_tV_t|\mathcal{F}_T^\varepsilon]}\\ &= \exp\Big(\sigma W_{1t}^\theta + \nu_t W_{2t}^\theta - \frac{1}{2}\left(\sigma^2+\nu^2+2\rho\sigma\nu\right)t\\ &+ \int_0^t\int_{-\infty}^{\infty}[xJ_1(dx,du)-(e^x-1)k_S(dx,du)]\\ &+ \int_0^t\int_{-\infty}^{\infty}[yJ_2(dy,du)-(e^y-1)k_V(dy,du)]\Big).\end{aligned}$$

Then under the probability measure Q_{SV} *and conditional on* $\mathcal{F}_T^\varepsilon$ *,*

$$\begin{aligned}W_{1t}^{SV} &= W_{1t}^\theta - \int_0^t \sigma_s ds - \rho\int_0^t \nu_s ds,\\ W_{2t}^{SV} &= W_{2t}^\theta - \int_0^t \rho\sigma_s ds - \int_0^t \nu_s ds,\end{aligned}$$

are two standard Brownian motions and $corr(dW_{1t}^{SV}, dW_{2t}^{SV}) = \rho$. *Moreover, the compensators of random measures* $J_S(dx,dt)$ *and* $J_V(dy,dt)$ *are given by* $\widehat{k}_{SV}(dx,dt) = \lambda_S^*\frac{1}{\sqrt{2\pi}\delta}e^{-\frac{(x-\delta^2)^2}{2\delta^2}}$ *and* $\widehat{k}_{SV}(dx,dt) = \lambda_V^*\frac{1}{\sqrt{2\pi}\gamma}e^{-\frac{(x-\gamma^2)^2}{2\gamma^2}}$ *with* $\lambda_V^* = \lambda_V e^{\frac{\gamma^2}{2}}$ *, respectively. Moreover, the generator of Markov chain does not be changed.*

Assume that the Markov chain process ε has a generator $c = (c_{ij})$. Let $\phi^c_{J_T|\varepsilon_0=e_i}(y)$ denote the condition probability density of J_T under the measure P and given $\varepsilon_0 = e_i$, Yoon et al.[19] provided the analytic formula of $\phi^c_{J_T|\varepsilon_0=e_i}(y)$, that is, for $\forall y(0\le y\le T)$

$$\begin{cases}\phi^c_{J_T|\varepsilon_0=e_1}(y) = \exp\left[-c_{22}(T-y)-c_{11}y\right]\\ \times\Big\{\left(\frac{c_{11}c_{22}y}{T-y}\right)^{\frac{1}{2}} I_1\left(2\left(c_{11}c_{22}y(T-y)\right)^{\frac{1}{2}}\right)\\ +c_{11}I_0\left(2\left(c_{11}c_{22}y(T-y)\right)^{\frac{1}{2}}\right)\Big\},\\ \phi^c_{J_T|\varepsilon_0=e_2}(y) = \exp\left[-c_{22}(T-y)-c_{11}y\right]\\ \times\Big\{\left(\frac{c_{11}c_{22}(T-y)}{y}\right)^{\frac{1}{2}} I_1\left(2\left(c_{11}c_{22}y(T-y)\right)^{\frac{1}{2}}\right)\\ +c_{22}I_0\left(2\left(c_{11}c_{22}y(T-y)\right)^{\frac{1}{2}}\right)\Big\},\end{cases}$$

and $\phi_{J_T|\varepsilon_0=e_1}(0) = 0$, $\phi_{J_T|\varepsilon_0=e_1}(T) = e^{-c_{11}T}$, $\phi_{J_T|\varepsilon_0=e_2}(T) = e^{-c_{22}T}$, $\phi_{J_T|\varepsilon_t=e_2}(T) = 0$, where $I_b(x) := (\frac{x}{2})^b\sum_{n=0}^{\infty}\frac{(\frac{x}{2})^{2n}}{n!\Gamma(b+n+1)}$.

Under the risk neutral martingale measure $Q^{\theta,C}$, we let $c(0,S,V,i)$ denote the price of vulnerable European call options at time 0 given $S_0 = S$, $V_0 = V$, $\varepsilon_0 = e_i$. The following theorem is the main result of this section, which gives the price of a vulnerable European call option under a Markov-modulated jump diffusion model.

Theorem 6 *The price of a vulnerable European call option has the following representation:*

$$\begin{aligned}c(0,S,V,i) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_0^T \Big(S\pi(m,\lambda_S^*,y)\\ \times &\ \pi(n,\lambda_V,y)\mathcal{N}(d_1(m,y),d_2(n,y),\rho^*(m,n))\\ - &\ Ke^{-rT}\pi(m,\lambda_S,y)\pi(n,\lambda_V,y)\\ \times &\ \mathcal{N}(d_3(m,y),d_4(n,y),\rho^*(m,n))\\ + &\ \frac{(1-\alpha)V}{D}\Big[Se^{rT}\pi(m,\lambda_S^*,y)\pi(n,\lambda_V^*,y)\\ \times &\ \mathcal{N}(d_5(m,y),d_6(n,y),-\rho^*(m,n))\\ - &\ K\pi(m,\lambda_S,y)\pi(n,\lambda_V^*,y)\\ \times &\ \mathcal{N}(d_7(m,y),d_8(n,y),-\rho^*(m,n))\Big)\\ \times &\ \phi^c_{J_T|\varepsilon_0=e_i}(y)dy,\end{aligned} \tag{15}$$

where $\mathcal{N}(\cdot,\cdot,\rho)$ *represents the bivariate cumulative normal distribution with correlative coefficient* ρ, *For each* $m = 0,1,2,...,$ $n = 0,1,2...,$ *and* $i\in S,V$

$$\begin{aligned}\pi(m,\lambda_i,y) &= e^{-\lambda_i y}\frac{(\lambda_i y)^m}{m!},\\ \rho^*(m,n) &= \frac{\rho\sigma\nu T}{\sqrt{(\sigma^2T+m\delta^2)(\nu^2T+n\gamma^2)}},\end{aligned}$$

and

$$\begin{aligned}d_1(m,y) &= \frac{\log\frac{S}{K}+(r+\frac{1}{2}\sigma^2)T+m\delta^2}{\sqrt{\sigma^2T+m\delta^2}}\\ &- \frac{\lambda_S(e^{\frac{\delta^2}{2}}-1)y}{\sqrt{\sigma^2T+m\delta^2}}\\ d_2(n,y) &= \frac{\log\frac{V}{D^*}+(r-\frac{1}{2}\nu^2+\rho\sigma\nu)T}{\sqrt{\nu^2T+n\gamma^2}}\\ &- \frac{\lambda_V(e^{\frac{\gamma^2}{2}}-1)y}{\sqrt{\nu^2T+n\gamma^2}}\\ d_3(m,y) &= d_1(m,y)-\frac{\sigma^2T+m\delta^2}{\sqrt{\sigma^2T+m\delta^2}}\\ d_4(n,y) &= d_2(n,y)-\frac{\rho\sigma\nu T}{\sqrt{\nu^2T+n\gamma^2}}\\ d_5(m,y) &= d_1(m,y)+\frac{\rho\sigma\nu T}{\sqrt{\sigma^2T+m\delta^2}}\end{aligned}$$

$$d_6(n,y) = -d_2(n,y) - \frac{n\gamma^2+\nu^2 T}{\sqrt{\nu^2 T + n\gamma^2}}$$
$$d_7(m,y) = d_3(m,y) + \frac{\rho\sigma\nu T}{\sqrt{\sigma^2 T + m\delta^2}}$$
$$d_8(n,y) = d_6(n,y) + \frac{\rho\sigma\nu T}{\sqrt{\nu^2 T + n\gamma^2}}.$$

Proof. For convenience, let $\mathrm{E}^{0,S,V,i}_{Q^{\theta,C}}[\cdot]$ denote the conditional expectation given $S_0 = S$, $V_0 = V$, $\varepsilon_0 = e_i$. Based on the risk neutral pricing theorem, we have

$$\begin{aligned} & c(0,S,V,i) \quad (16)\\ = & \ e^{-rT} E^{0,S,V,i}_{Q^{\theta,C}}\left[(S_T-K)^+ I_{\{V_T \geq D^*\}}\right] + e^{-rT} \\ \times & \ \mathrm{E}^{0,S,V,i}_{Q^{\theta,C}}\left[\frac{(1-\alpha)V_T}{D}(S_T-K)^+ I_{\{V_T<D^*\}}\right]. \end{aligned}$$

Let

$$\begin{aligned} & c(0,J_T,\mathcal{F}^\varepsilon_T) \\ = & \ \mathrm{E}_{Q^{\theta,C}}\left[e^{-rT}(S_T-K)^+ I_{\{V_T\geq D^*\}}\Big|\mathcal{F}^\varepsilon_T\right] \quad (17)\\ + & \ \mathrm{E}_{Q^{\theta,C}}\left[\frac{(1-\alpha)V_T}{D}(S_T-K)^+ I_{\{V_T<D^*\}}\Big|\mathcal{F}^\varepsilon_T\right]. \end{aligned}$$

Then, apply the law of iterated expectation, we get

$$c(0,S,V,i) = \mathrm{E}^{0,S,V,i}_{Q^{\theta,C}}\left[c(0,J_T,\mathcal{F}^\varepsilon_T)\right]. \quad (18)$$

In what follows, we first derive $c(0,J_T,\mathcal{F}^\varepsilon_T)$. Using Bayes formula we get

$$\begin{aligned} & c(0,J_T,\mathcal{F}^\varepsilon_T) = S\mathrm{E}_{Q^S}\left[I_{\{S_T\geq K, V_T\geq D^*\}}\Big|\mathcal{F}^\varepsilon_T\right] \\ - & \ Ke^{-rT}\mathrm{E}_{Q^{\theta,C}}\left[I_{\{S_T\geq K, V_T\geq D^*\}}\Big|\mathcal{F}^\varepsilon_T\right] \\ + & \ \frac{(1-\alpha)V}{D}\Big(e^{rT} S\mathrm{E}_{Q^{SV}}\left[I_{\{S_T\geq K, V_T< D^*\}}\Big|\mathcal{F}^\varepsilon_T\right] \\ - & \ K\mathrm{E}_{Q^V}\left[I_{\{S_T\geq K, V_T< D^*\}}\Big|\mathcal{F}^\varepsilon_T\right]\Big) \quad (19) \end{aligned}$$

Combing the Proposition 4 with (11) and (12), we have

$$\begin{aligned} & \mathrm{E}_{Q^S}\left[I_{\{S_T\geq K, V_T\geq D^*\}}|\mathcal{F}^\varepsilon_T\right] \\ = & \ Q^S\Big((r+\frac{1}{2}\sigma^2)T + \sigma W^S_{1T} + \sum_{j=1}^{N^1_{J_T}} X_j - \lambda_S \\ \times & \ (e^{\frac{\delta^2}{2}}-1)J_T \geq \log\frac{K}{S}, (r-\frac{1}{2}\nu^2+\rho\sigma\nu)T + \nu \\ \times & \ W^S_{2T}\sum_{j=1}^{N^2_{J_T}} Y_j - \lambda_V(e^{\frac{\gamma^2}{2}}-1)J_T \geq \log\frac{D^*}{V}\Big|\mathcal{F}^\varepsilon_T\Big). \end{aligned}$$

Conditioning on $N^1_{J_T} = m$, $N^2_{J_T} = n$, and using the independence we obtain

$$\begin{aligned} & \mathrm{E}_{Q^S}\left[I_{\{S_T\geq K, V_T\geq D^*\}}\Big|\mathcal{F}^\varepsilon_T\right] \\ = & \ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\pi(m,\lambda^*_S,J_T)\pi(n,\lambda_V,J_T) \\ \times & \ \mathbb{N}(d_1(m,J_T), d_2(n,J_T), \rho^*_1(m,n)) \quad (20) \end{aligned}$$

Employing the same method, we can obtain

$$\begin{aligned} & \mathrm{E}_{Q^{\theta,C}}\left[I_{\{S_T\geq K, V_T\geq D^*\}}\Big|\mathcal{F}^\varepsilon_T\right] \quad (21)\\ = & \ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\pi(m,\lambda_S,J_T)\pi(n,\lambda_V,J_T) \\ \times & \ \mathcal{N}(d_3(m,J_T), d_4(n,J_T), \rho^*_1(m,n)), \end{aligned}$$

$$\begin{aligned} & \mathrm{E}_{Q^{SV}}\left[I_{\{S_T\geq K, V_T< D^*\}}\Big|\mathcal{F}^\varepsilon_T\right] \quad (22)\\ = & \ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\pi(m,\lambda^*_S,J_T)\pi(n,\lambda^*_V,J_T) \\ \times & \ \mathcal{N}(d_5(m,J_T), d_6(n,J_T), -\rho^*_2(m,n)), \end{aligned}$$

$$\begin{aligned} & \mathrm{E}_{Q^{V}}\left[I_{\{S_T\geq K, V_T< D^*\}}\Big|\mathcal{F}^\varepsilon_T\right] \quad (23)\\ = & \ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\pi(m,\lambda_S,J_T)\pi(n,\lambda^*_V,J_T) \\ \times & \ \mathcal{N}(d_7(m,J_T), d_8(n,J_T), -\rho^*_1(m,n)). \end{aligned}$$

It follows from (19), (20), (21) , (22) and (23) that

$$\begin{aligned} & c(0,J_T,\mathcal{F}^\varepsilon_T) \\ = & \ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big(S\pi(m,\lambda^*_S,J_T)\pi(n,\lambda_V,J_T) \\ \times & \ \mathcal{N}(d_1(m,J_T), d_2(n,J_T), \rho^*_1(m,n)) \\ - & \ Ke^{-rT}\pi(m,\lambda_S,J_T)\pi(n,\lambda_V,J_T) \\ \times & \ \mathcal{N}(d_3(m,J_T), d_4(n,J_T), \rho^*_1(m,n)) \\ + & \ \frac{(1-\alpha)V}{D}\Big[e^{rT}S\pi(m,\lambda^*_S,J_T)\pi(n,\lambda^*_V,J_T) \\ \times & \ \mathcal{N}(d_5(m,J_T), d_6(n,J_T), -\rho^*_2(m,n)) \\ - & \ K\pi(m,\lambda_S,J_T)\pi(n,\lambda_V,J_T) \\ \times & \ \mathcal{N}(d_7(m,J_T), d_8(n,J_T), -\rho^*_1(m,n))\Big]\Big). (24) \end{aligned}$$

Note that the condition probability density of J_T under the measure $Q^{\theta,C}$ and conditional on $\varepsilon_0 = e_i$ is $\phi^c_{J_T|\varepsilon_0=e_i}(y)$, it follows from (18) and (24) yields (15). Hence, we complete the proof of Theorem 6.

Let $p(0,S,V,i)$ denote the price of vulnerable put options at time 0. Then we can obtain the following proposition.

Proposition 7 *The value of vulnerable European put option is given by*

$$\begin{aligned} & p(0,S,V,i) \\ = & c(0,S,V,i) - \int_0^T \Big(\sum_{n=0}^{\infty}\pi(n,\lambda_V,y) \\ \times & \Big[S\mathcal{N}(d_2(m,y)) - Ke^{-rT}\mathcal{N}(d_4(m,y))\Big] \\ + & \frac{(1-\alpha)V}{D}\sum_{n=0}^{\infty}\pi(n,\lambda_V^*,y)\Big[Se^{rT}\mathcal{N}(d_6(m,y)) \\ - & K\mathcal{N}(d_8(m,y))\Big]\Big)\phi^c_{J_T|\varepsilon_0=e_i}(y)dy. \end{aligned}$$

Proof. From (13) and (14), we can obtain

$$\begin{aligned} \psi^{call}(S_T) - \psi^{put}(S_T) &= (S_T-K)I_{\{V_T\geq D^*\}} \\ &+ \frac{(1-\alpha)V_T}{D}(S_T-K)I_{\{V_T<D^*\}}, \end{aligned}$$

then

$$\begin{aligned} & c(0,S,V,i) = p(0,S,V,i) + \mathrm{E}^{0,S,V,i}_{Q^{\theta,C}}\Big[\mathrm{E}_{Q^{\theta,C}} \\ & \Big[e^{-rT}\Big(\Psi^{call}(S_T) - \Psi^{put}(S_T)\Big)\Big|\mathcal{F}^{\varepsilon}_T\Big]\Big] \\ = & p(0,S,V,i) + \mathrm{E}^{0,S,V,i}_{Q^{\theta,C}}\Big\{\sum_{n=0}^{\infty}\pi(n,\lambda_V,J_T) \\ \times & \Big[S\mathcal{N}(d_2(m,J_T)) - Ke^{-rT}\mathcal{N}(d_4(m,J_T))\Big] \\ + & \frac{(1-\alpha)V}{D}\sum_{n=0}^{\infty}\pi(n,\lambda_V^*,J_T) \\ \times & \Big[Se^{rT}\mathcal{N}(d_6(m,J_T)) - K\mathcal{N}(d_8(m,J_T))\Big]\Big\}. \end{aligned}$$

Since the condition probability density of J_T under the measure $Q^{\theta,C}$ and conditional on $\varepsilon_0 = e_i$ is $\phi^c_{J_T|\varepsilon_0=e_i}(y)$, we obtain the result.

Remark 8 *If the market parameters are constants, that is, the risky asset price processes* S *and* V *follow the Merton jump diffusion model, the value* $c(0,S,V)$ *of vulnerable European call options is*

$$\begin{aligned} & c(0,S,V) \\ = & \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big(S\pi(m,\lambda_S^*,T)\pi(n,\lambda_V,T) \\ \times & \mathcal{N}(d_1(m,T),d_2(n,T),\rho_1^*(m,n)) \\ - & Ke^{-rT}\pi(m,\lambda_S,T)\pi(n,\lambda_V,T) \\ \times & \mathcal{N}(d_3(m,T),d_4(n,T),\rho_1^*(m,n)) \\ + & \frac{(1-\alpha)V}{D}\Big[Se^{-rT}\pi(m,\lambda_S^*,T)\pi(n,\lambda_V^*,T) \\ \times & \mathcal{N}(d_5(m,T),d_6(n,T),-\rho_1^*(m,n)) \\ - & K\pi(m,\lambda_S,T)\pi(n,\lambda_V^*,T) \\ \times & \mathcal{N}(d_7(m,T),d_8(n,T),-\rho_1^*(m,n))\Big). \quad (25) \end{aligned}$$

5 Reduced form model

In recent years, the reduced form approach has been widely applied to the pricing of defaultable securities. A reduced form model treats default as an unpredictable event by taking the default time as an exogenous random variable. In this section, we use the reduced form model for valuing the European options with default risk. We consider a filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_{0\leq t\leq T},Q)$, where Q is a risk neutral martingale measure. When the writer of the option defaults, an exogenously specified constant fraction times the payoff will be paid at maturity. Moreover, we let τ denote the default time of the writer of the option with default intensity process λ_t. In addition, we assume that the default intensity process λ_t is described by a Markov-modulated Vasicek model.

$$d\lambda_t = \kappa(\iota_t - \lambda_t)dt + \upsilon_t dW_{3t}, \quad (26)$$

where ι_t and υ_t are modelled by the Markov chain ε, $\iota_t = \iota_i I_{\{\varepsilon_t=e_i\}}$ and $\upsilon_t = \upsilon_i I_{\{\varepsilon_t=e_i\}}$. Furthermore, we suppose that the dynamic of the risky asset S is given by

$$\begin{aligned} S_t &= S_0\exp\Big\{(r-\frac{1}{2}\sigma^2)t + \sigma W_{1t} + \sum_{j=1}^{N^1_{J_t}} X_j \\ &- \lambda_S(e^{\frac{\delta^2}{2}}-1)J_t\Big\}, \quad (27) \end{aligned}$$

where all parameters of the above equation have been explained in Section 2 and $corr(dW_{1t},dW_{3t}) = \rho_{13}$, N^1 and $\{X_j, j=1,2,...\}$ are independent of W_3.

Let $\mathcal{F}^{\lambda}_t = \sigma(\lambda_s, 0\leq s\leq t)$ be the right continuous P complete filtration generated by the default intensity process λ_t, and $\mathcal{G}_t = \sigma\left(I_{\{\tau\leq s\}}, 0\leq s\leq t\right)$. Furthermore, we define a new filtration $\mathcal{I}_t = \mathcal{F}_t \vee \mathcal{F}^{\lambda}_T \vee \mathcal{G}_T \vee \mathcal{F}^{\varepsilon}_T$, which the minimal σ-field containing $\mathcal{F}_t$, $\mathcal{F}^{\lambda}_T$, $\mathcal{G}_T$ and $\mathcal{F}^{\varepsilon}_T$. Then the conditional and unconditional distribution of τ are given by

$$\begin{aligned} P(\tau>t|\mathcal{I}_0) &= \exp\left(-\int_0^t \lambda_u du\right), \quad (28) \\ P(\tau>t) &= \mathrm{E}_Q\left[\exp\left(-\int_0^t \lambda_u du\right)\right]. \end{aligned}$$

As in Jarrow and Yu [20], we assume that the recovery rate is a constant ω. The payoff is given by ω times the payoff of the default free option at maturity if the writer of the European option defaults.

A vulnerable European call option in reduced form has payoff $\Psi^{call}(S_T)$ at time T which is given by

$$\begin{aligned} \Psi^{call}(S_T) &= \omega(S_T-K)^+ I_{\{\tau\leq T\}} \\ &+ (S_T-K)^+ I_{\{\tau>T\}}, \quad (29) \end{aligned}$$

where the parameter $1 > \omega > 0$ is a constant, K represents the strike price. The random variable $(S_T - K)^+$ is the value at time T of the option to buy one share of stock at the price K. If $S_T < K$, this option should be exercised by its holder. If $S_T \geq K$, this option should not be exercised, it is worthless to its holder. When $\tau < T$, it means that the writer of option will default, the holder of option will receive $\omega(S_T - K)^+$ at time T, whereas if $\tau > T$, default case will not occur, the holder of option will receive $(S_T - K)^+$ at time T.

Moreover, the payoff $\Psi^{put}(S_T)$ of a vulnerable put option in reduced form is given by

$$\begin{aligned} \Psi^{put}(S_T) &= \omega(K - S_T)^+ I_{\{\tau \leq T\}} \\ &+ (K - S_T)^+ I_{\{\tau > T\}}. \end{aligned} \tag{30}$$

Let $\widehat{c}(0, S, i)$ denote the valuation of the vulnerable European call option at time 0 and given $S_0 = S$ and $\varepsilon_0 = e_i$. Then by the risk neutral pricing theorem we obtain

$$\begin{aligned} \widehat{c}(0, S, i) &= \mathrm{E}_Q^{0,S,i}\Big[e^{-rT}\Big(\omega(S_T - K)^+ I_{\{\tau \leq T\}} \\ &+ (S_T - K)^+ I_{\{\tau > T\}}\Big)\Big]. \end{aligned}$$

In terms of the equation (28) and the property of condition expectation, we have

$$\begin{aligned} &\widehat{c}(0, S, i) \\ =\ &\omega \mathrm{E}_Q^{0,S,i}\left[e^{-rT}(S_T - K)^+\right] \\ +\ &(1-\omega)\mathrm{E}_Q^{0,S,i}\left[e^{-\int_0^T (r+\lambda_s)ds}(S_T - K)^+\right] \end{aligned} \tag{31}$$

Details can be found in the Proposition 1 in Jarrow and Yu [20].

Let $P(t, T) = \mathrm{E}_Q\left[e^{-\int_t^T \lambda_s ds}|\mathcal{F}_t\right]$. From the Shen and Siu [21], we have the regime switching exponential affine form solution

$$P(t, T) = \exp\{-B(t, T)r + C(t, T, \varepsilon)\}, \tag{32}$$

where the terminal conditions are given by $C(T, T, \varepsilon_T) = B(T, T) = 0$, $B(t, T) = \frac{1-e^{-\kappa(T-t)}}{\kappa}$ and

$$\begin{aligned} C(t, T, \varepsilon) &= log\Big\{\mathrm{E}_Q\Big[\exp\Big\{\int_t^T \Big(\kappa\iota_s B(s, T) \\ &- \frac{1}{2}\upsilon_s^2 B^2(s, T)\Big)\Big\}\Big|\varepsilon_t = \varepsilon\Big]\Big\}. \end{aligned}$$

Define a new measure Q^λ equivalent to Q by the following Radon-Nikodym derivative

$$\frac{dQ^\lambda}{dQ}\Big|_{\mathcal{F}_T} = \zeta_T^\lambda = \frac{e^{-\int_0^T \lambda_s ds}}{\mathrm{E}_Q\left[e^{-\int_0^T \lambda_s ds}\right]},$$

where

$$\begin{aligned} \frac{d\zeta_t^\lambda}{\zeta_t^\lambda} &= -\upsilon_t B(t, T)dW_t \\ &+ \tilde{C}(t, T, \varepsilon_t)^{-1} < \tilde{C}(t, T), dM_t >, \end{aligned}$$

with $\tilde{C}(t, T, \varepsilon_t) = e^{C(t,T,\varepsilon_t)}$ and $\tilde{C}(t, T) = (\tilde{C}(t, T, e_1), \tilde{C}(t, T, e_2))'$.

According to Girsanov's theorem, we have

$$\begin{aligned} W_{1t}^\lambda &= W_{1t} + \rho_{13}\int_0^t \upsilon_s B(s, T)ds, \\ W_{3t}^\lambda &= W_{3t} + \int_0^t \upsilon_s B(s, T)ds \end{aligned}$$

are two standard Q^λ Brownian motions with $corr(dW_{1t}^\lambda, dW_{3t}^\lambda) = \rho_{13}$. Since N^1 and $\{Y_j, j = 1, 2, ...\}$ are independent of W_1, W_3, the measure of change from Q to Q^λ does not alter the intensity of N^1 and the distribution of Y_j. In addition, by Lemma 3.2 in Shen and Siu [21], we can obtain, under the measure Q^λ, the rate matrix $A^\lambda = (a_{ij}^\lambda)_{i,j\in\mathcal{A}}$ of the Markov chain ε is given by

$$a_{ij}^\lambda = \begin{cases} a_{ij}\frac{\tilde{C}(t,T,e_j)}{\tilde{C}(t,T,e_i)}, & i \neq j, \\ -\sum\limits_{k\neq i} a_{ik}\frac{\tilde{C}(t,T,e_k)}{\tilde{C}(t,T,e_i)}, & i = j, \end{cases} \tag{33}$$

and the semi-martingale decomposition of the Markov chain ε is given by

$$\varepsilon_t = \varepsilon_0 + \int_0^t A_s^\lambda \varepsilon_s ds + M_t^\lambda,$$

where M_t^λ is a Q^λ martingale.

Theorem 9 *The value of vulnerable European call option in reduced form model is given by*

$$\begin{aligned} &\widehat{c}(0, S, i) \\ =\ &\int_0^T \Big(\sum_{m=0}^\infty \pi(m, \lambda_S^*, y)S\Big[\omega\mathcal{N}(d_1(m, y)) \\ +\ &(1-\omega)P(0, T)\mathcal{N}(\widehat{d}_1(m, y))\Big] - \sum_{m=0}^\infty \pi(m, \lambda_S, y) \\ \times\ &Ke^{-rT}\Big[\omega\mathcal{N}(d_3(m, y)) + (1-\omega)P(0, T) \\ \times\ &\mathcal{N}(\widehat{d}_3(m, y))\Big]\Big)\phi_{J_T|\varepsilon_0=e_i}^{a^\lambda}(y)dy, \end{aligned} \tag{34}$$

where $\mathcal{N}(\cdot)$ denotes the cumulative normal distribution function, and

$$\begin{aligned} \widehat{d}_1(m, y) &= d_1(m, y) - \frac{\rho_{13}\int_0^T \upsilon_u B(u, T)du}{\sqrt{\int_0^T \sigma_u^2 du + m\delta^2}}, \\ \widehat{d}_3(m, y) &= d_3(m, y) - \frac{\rho_{13}\int_0^T \upsilon_u B(u, T)du}{\sqrt{\int_0^T \sigma_u^2 du + m\delta^2}}, \end{aligned}$$

where $d_1(m,y)$ and $d_2(m,y)$ are defined in Theorem 7.

Proof. Firstly, we can obtain

$$\begin{aligned} & \mathrm{E}_Q^{0,S,i}\left[e^{-rT}(S_T-K)^+\Big|\mathcal{F}_T^\varepsilon\right] \\ = \ & S\sum_{m=0}^{\infty}\pi(m,\lambda_S^*,J_T)\mathcal{N}(d_1(m,J_T)) \quad (35) \\ - \ & Ke^{-rT}\sum_{m=0}^{\infty}\pi(m,\lambda_S,J_T)\mathcal{N}(d_3(m,J_T)). \end{aligned}$$

Moreover, using the Bayes formula we have

$$\begin{aligned} & \mathrm{E}_Q^{0,S,i}\left[e^{-\int_0^T(r+\lambda_u)du}(S_T-K)^+\Big|\mathcal{F}_T^\varepsilon\right] \\ = \ & P(0,T)\mathrm{E}_{Q^\lambda}\left[e^{-rT}(S_T-K)^+\Big|\mathcal{F}_T^\varepsilon\right]. \quad (36) \end{aligned}$$

Changing measure from Q to Q^λ yields

$$\begin{aligned} S_T \ = \ & S\exp\Big\{\int_0^T(r-\frac{1}{2}\sigma^2-\rho_{13}\upsilon_u B(u,T))du \\ + \ & \sigma W_{1T}^\lambda+\sum_{j=1}^{N^1_{J_T}}Y_j-\lambda_S(e^{\frac{\delta^2}{2}}-1)J_T\Big\}. \end{aligned}$$

Hence

$$\begin{aligned} & \mathrm{E}_{Q^\lambda}\left[e^{-rT}(S_T-K)^+\Big|\mathcal{F}_T^\varepsilon\right] \\ = \ & \Big[S\sum_{m=0}^{\infty}\pi(m,\lambda_S^*,J_T)\mathcal{N}(\widehat{d}_1(m,J_T)) \quad (37) \\ - \ & Ke^{-rT}\sum_{m=0}^{\infty}\pi(m,\lambda_S,J_T)\mathcal{N}(\widehat{d}_3(m,J_T))\Big]. \end{aligned}$$

Combing (31), (35), (36), and (37), we obtain the result.

Let $\widehat{p}(0,S,i)$ denote the valuation of the vulnerable European put option at time 0 and given $S_0=S$ and $\varepsilon_0=e_i$. It is provided in the following Proposition.

Proposition 10 *The value of vulnerable European put option is given by*

$$\begin{aligned} \widehat{p}(0,S,i) \ = \ & \widehat{c}(0,S,i)-\omega S_0+Ke^{-rT} \\ - \ & (1-\omega)S_0e^{-\rho_{13}\int_0^T\upsilon_u B(u,T)du}. \end{aligned}$$

Proof. In light of (29) and (30), we obtain

$$\begin{aligned} \psi^{put}(T)-\psi^{call}(T) \ = \ & \omega(K-S_T) \\ + \ & (1-\omega)(K-S_T)I_{\{\tau>T\}}. \end{aligned}$$

Using the risk neutral pricing theorem,

$$\begin{aligned} & \widehat{p}(0,S,i)-\widehat{c}(0,S,i) \\ = \ & \mathrm{E}_Q^{0,S,i}\left[e^{-rT}(\Psi^{put}(T)-\Psi^{call}(T))\right] \\ = \ & \omega\mathrm{E}_Q^{0,S,i}\left[e^{-rT}(K-S_T)\right] \\ + \ & (1-\omega)\mathrm{E}_Q^{0,S,i}\left[e^{-rT}(K-S_T)I_{\{\tau>T\}}\right] \end{aligned}$$

Since Q is a risk neutral martingale measure, then

$$\mathrm{E}_Q^{0,S,i}\left[e^{-rT}(K-S_T)\Big|\mathcal{F}_T^\varepsilon\right]=Ke^{-rT}-S_0.$$

Moreover,

$$\begin{aligned} & \mathrm{E}_Q^{0,S,i}\left[e^{-rT}(K-S_T)I_{\{\tau>T\}}\Big|\mathcal{F}_T^\varepsilon\right] \\ = \ & \mathrm{E}_Q^{0,S,i}\left[e^{-\int_0^T(r+\lambda_u)du}(K-S_T)\Big|\mathcal{F}_T^\varepsilon\right] \\ = \ & \mathrm{E}_{Q^\lambda}^{0,S,i}\left[e^{-rT}(K-S_T)\Big|\mathcal{F}_T^\varepsilon\right] \\ = \ & Ke^{-rT}-S_0e^{-\rho_{13}\int_0^T\upsilon_u B(u,T)du}. \end{aligned}$$

Hence

$$\begin{aligned} \widehat{p}(0,S,i) \ = \ & \widehat{c}(0,S,i)-\omega S_0+Ke^{-rT} \\ - \ & (1-\omega)S_0e^{-\rho_{13}\int_0^T\upsilon_u B(u,T)du}. \end{aligned}$$

6 Conclusion

We first have introduced a two-state Markov-modulated jump diffusion model to describe the dynamics of risk assets. A key feature of this financial model is that a regime switch will induce a high volatility state of economic, it maybe bring a jump in the price of risky asset. Then, since the market in this paper is incomplete, the regime switching risk is considered in this paper, and we present an risk neutral martingale measure to pricing options. In the end, we investigate the pricing of vulnerable European options and provide some analytical pricing formulas of these derivatives.

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