P_5-equicoverable graphs which contain cycles with length at least 5

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Abstract: A graph G is called H-equicoverable if every minimal H-covering of G is also a minimum H-covering of G. In this paper, we investigate the characterization of P_5-equicoverable graphs which contain cycles with length at least 5 and give some results of P_k-equicoverable graphs.

Key–Words: P_5-equicoverable,P_k-equicoverable,cycle,covering

1 Introduction

A graph G has order $|V(G)|$ and size $|E(G)|$. If vertex v is an endpoint of an edge e, then v and e are incident. The degree of vertex v in a graph G, written $d_G(v)$ or $d(v)$, is the number of edges incident to v. The path and circuit on k vertices are denoted by P_k and C_k, respectively. A star is a tree consisting of one vertex adjacent to all the others. The $(n+1)-$vertex star is the biclique $K_{1,n}$.

A graph H is a subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. Suppose that E' is a nonempty subset of E. The subgraph of G whose vertex set is the set of ends of edges in E' and whose edge set is E' is called the subgraph of G induced by E' and is denoted by $G[E']$; $G[E']$ is an edge-induced subgraph of G.

The problem that we study stems from the research of H-decomposable graphs, randomly decomposable graphs and equipackable graphs. In 2008, Zhang introduced equicoverable graph which is the dual concept of the equipackable graph and characterized all P_3-equicoverable graphs. In this paper, we investigate all P_5-equicoverable graphs which don't contain 3-cycle or 4-cycle and contain at least one cycle with length at least 5. For further definitions and results, we can refer to [1],[2],[3],[4],[5],[6].

Let H be a subgraph of a graph G. An H-covering of G is a set $L = H_1, H_2, \dots, H_k$ of subgraphs of G, where each subgraph H_i isomorphic to H, and every edge of G appeaers in at least one member of L. A graph is called H-coverable if there exists an H-covering of G. An H-covering of G with k copies $H_1, H_2, \dots, H_k$ is called minimal if, for any H_j, $\bigcup_{i=1}^{k} H_i - H_j$ is not an H-covering of G. An H-covering of $H_1, H_2, \dots, H_k$ is called minimum if there exists no H-covering with less than k copies of H. A graph is called H-equicoverable if every minimal H-covering is also a minimum H-covering. Let $C(G;H)$ denote the number of H in the minimal H-covering of G, or simply $C(G)$ for short and let $c(G;H)$ denote the number of H in the minimum H-covering of G, or simply $c(G)$ for short. For convenience, we denote by $C_n \cdot P_k$ a graph obtained from a cycle C_n and a path P_k by identifying one vertex of the cycle C_n and an endpoint of the path P_k. And we denote by $C_n \cdot K_{1,k}$ a graph obtained from a cycle C_n and a star $K_{1,k}$ by identifying one vertex of the cycle C_n and a leaf of the star $K_{1,k}$.

Then we introduce a definition and a useful proposition:

Definition 1 *[6] For a star $K_{1,k}$, we call the vertex of degree k center, and other vertices leaves. A k-extendedstar that has one vertex of degree k which is also called center, k vertices of degree 2 and k leaves is a tree obtained by inserting a vertex of degree 2 into each edge of a star $K_{1,k}$. We denote it by S_k^*. A second order k-extendedstar is a tree obtained by inserting two vertices of degree 2 into each edge of a star $K_{1,k}$, we denote it by S_k^{2*}. Similarly, an n-th order k-extendedstar is a tree obtained by inserting n vertices of degree 2 into each edge of a star $K_{1,k}$, we denote it by S_k^{n*}.*

In this paper, we denote by $C_n \cdot S_k^{n*}$ a graph obtained from a cycle C_n and an n-th order k-extendedstar by identifying one vertex of the cycle C_n

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and the center of the n-th order k-extendedstar. We denote by $P_n \cdot K_{1,k}$ a graph obtained from a path P_n and a k-star by identifying one endpoint of the path P_n and one leaf of the k-star.

Proposition 2 *A connected graph G is P_5-coverable if and only if it has a subgraph P_5 except the kind of graphs in Figure 1.*

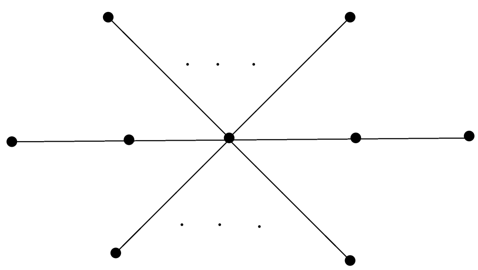


Figure 1: graphs which are not P_5-coverable

Lemma 3 *If a connected graph G can be decomposed into several connected P_k-coverable graphs and at least one component is not P_k-equicoverable, G will not be P_k-equicoverable.*

Theorem 4 *[5] Path P_n is P_k-equicoverable if and only if $k \le n \le 2k$ or $n = 3k-1$.*

Theorem 5 *[5] Cycle C_n is P_k-equicoverable if and only if*

$$\begin{cases} k \le n \le \frac{3k-1}{2} \text{ or } n = 2k-1 \; if \; k \; is \; odd, \\ k \le n \le \frac{3k-2}{2} \text{ or } n = 2k-1 \; if \; k \; is \; even. \end{cases}$$

Lemma 6 *S_k^{n*} is P_{n+2}-equicoverable and $c(S_k^{n*}; P_{n+2}) = C(S_k^{n*}; P_{n+2}) = k$.*

Proof: S_k^{n*} can be obtained by identifying the endpoints of k copies of P_{n+2}. The S_k^{n*} contains a path of length at most $2n+2$, that is, P_{2n+3}. By Theorem 4, P_{2n+3} is P_{n+2}-equicoverable and $c(P_{2n+3}; P_{n+2}) = C(P_{2n+3}; P_{n+2}) = 2$. If k is even, $c(S_k^{n*}; P_{n+2}) = C(S_k^{n*}; P_{n+2}) = \frac{k}{2} \times 2 = k$; If k is odd, $c(S_k^{n*}; P_{n+2}) = C(S_k^{n*}; P_{n+2}) = \frac{k-1}{2} \times 2 + 1 = k$. □

2 P_5-equicoverable graphs

First,we introduce P_5-equicoverable paths and cycles.

Lemma 7 *[5] The path P_n is P_5-equicoverable if and only if $n = 5, 6, 7, 8, 9, 10, 14$.*

Proof: By Theorem 4, we give the results. □

Lemma 8 *[5] The cycle C_n is P_5-equicoverable if and only if $n = 5, 6, 7, 9$.*

Proof: We can refer to Theorem 5. □

Lemma 9 *G is a connected graph that is not a cycle. If G doesn't contain any 3-cycles or 4-cycles and contains a 5-cycle, G will not be P_5-equicoverable unless G is $C_5 \cdot S_n^{3*}$ or G is obtained by adding n copies of $P_3 \cdot K_{1,t} (t \ge 3)$ to only one vertex of C_5.*

Proof: Case 1: G is obtained by adding copies of P_2 to the vertices of C_5.

(1)If each vertex of C_5 can be added to at most one P_2, G can only be one of the seven graphs shown in Figure 2. No matter which graph is in Figure 2, a minimal P_5-covering whose covering number $C(G)$ is greater than the number of the minimum P_5-covering $c(G)$. So the graphs are not P_5-equicoverable.

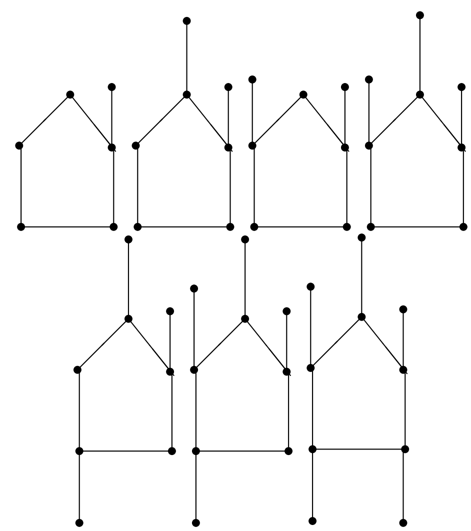


Figure 2: graphs obtained by adding at most one P_2 to each vertex of C_5

(2)If each vertex of C_5 can be added to any copies of P_2. G is obtained by adding copies of P_2 to the vertices of the 5-cycle part of G_0, where G_0 is one of the graphs in Figure 2. If the number of the copies of P_2 added is n, we can get a minimal P_5-covering whose covering number is $C(G_0) + n$(using $C(G_0)$ copies of P_5 to cover the G_0 part and n copies of P_5 to cover other parts), while the number of the minimum P_5-covering is at most $c(G_0) + n$. By (1), each of G_0 is not P_5-equicoverable, then $C(G_0) > c(G_0)$. So G is not P_5-equicoverable.

Case 2: G is obtained by adding copies of P_3 to the vertices of C_5.

Note that we identify the endpoint of each copy of P_3 with the vertices of C_5, not the center vertex. Otherwise G is the same as one of the graph in Case 1.

(1)If each vertex of C_5 can be added to at most one P_3, G can only be one of the seven graphs shown in Figure 3. No matter which graph is in Figure 3, a minimal P_5-covering whose covering number $C(G)$ is greater than the number of the minimum P_5-covering $c(G)$. So the graphs are not P_5-equicoverable.

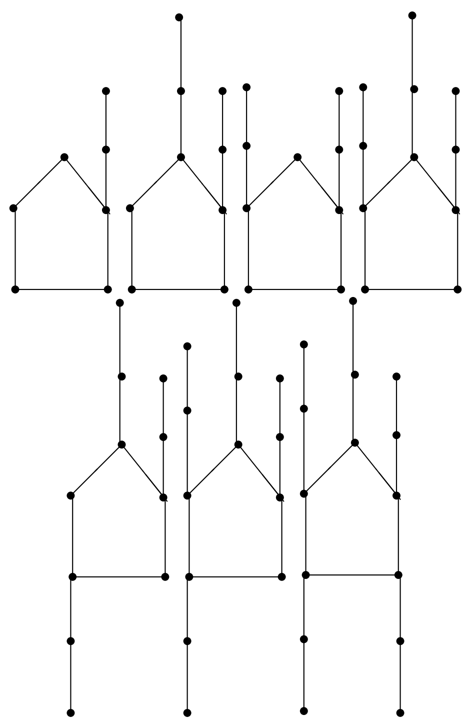


Figure 3: graphs obtained by adding at most one P_3

to each vertex of C_5

(2)If each vertex of C_5 can be added to any copies of P_3. G is obtained by adding copies of P_3 to the vertices of the 5-cycle part of G_0, where G_0 is one of the graphs in Figure 3. If the number of the copies of P_3 added is n, we can get a minimal P_5-covering whose covering number is $C(G_0) + n$(using $C(G_0)$ copies of P_5 to cover the G_0 part and n copies of P_5 to cover other parts), while the number of the minimum P_5-covering is at most $c(G_0) + n$. By (1), each of G_0 is not P_5-equicoverable, then $C(G_0) > c(G_0)$. So G is not P_5-equicoverable.

Case 3: G is obtained by adding copies of $K_{1,t}(t \geq 3)$ to the vertices of C_5.

Note that we identify one of leaves of each copy of $K_{1,t}$ with the vertices of C_5, not the center vertex. Otherwise G is the same as one of the graph in Case 1.

(1)If each vertex of C_5 can be added to at most one $K_{1,t}$, G can only be one of the seven graphs shown in Figure 4. No matter which graph is in Figure 4, a minimal P_5-covering whose covering number $C(G)$ is greater than the number of the minimum P_5-covering $c(G)$. So the graphs are not P_5-equicoverable.

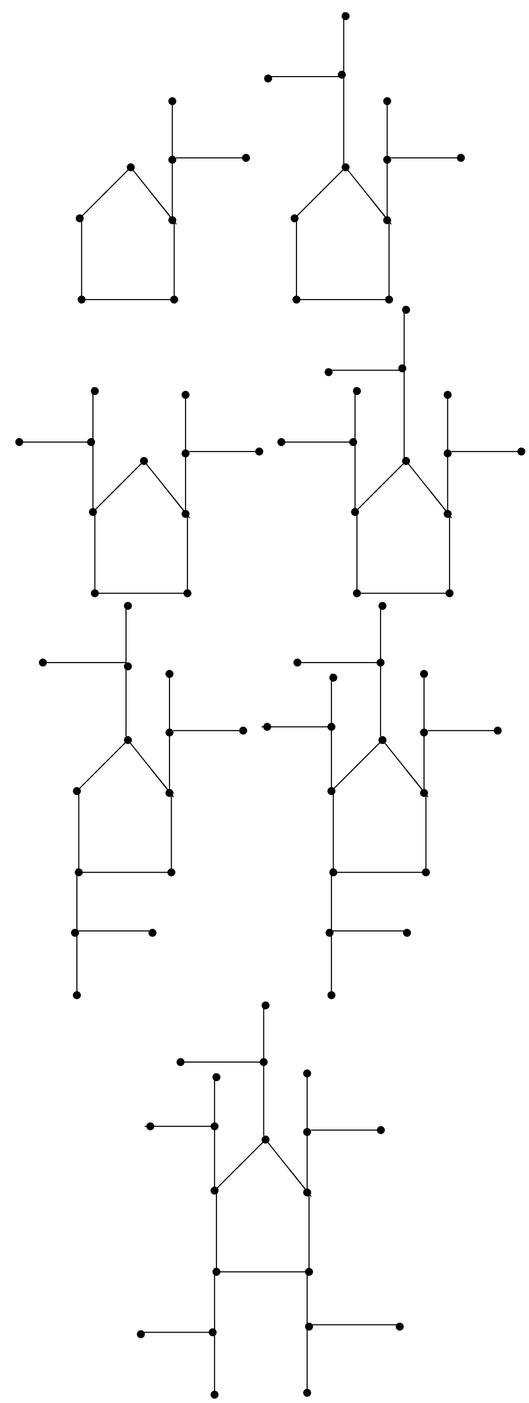


Figure 4: graphs obtained by adding at most one $K_{1,t}$

to each vertex of C_5

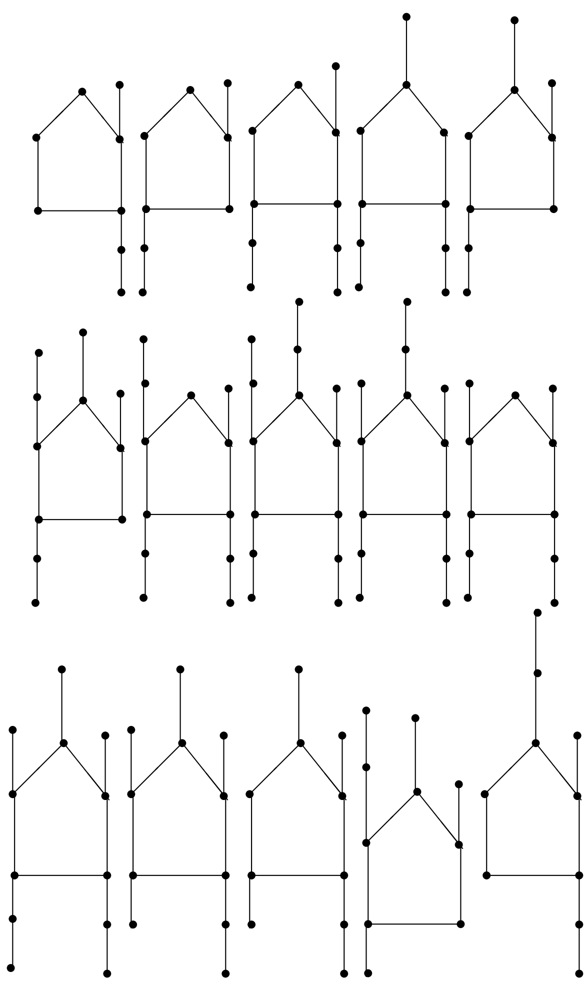
(2)If each vertex of C_5 can be added to any copies of $K_{1,t}$. G is obtained by adding copies of $K_{1,t}$ to the vertices of the 5-cycle part of G_0, where G_0 is one of the graphs in Figure 4. If the number of the copies of $K_{1,t}$ added is n, we can get a minimal P_5-covering whose covering number is $C(G_0) + n(t-1)$(using $C(G_0)$ copies of P_5 to cover the G_0 part and $n(t-1)$ copies of P_5 to cover other parts), while the number of the minimum P_5-covering is at most $c(G_0)+n(t-1)$. By (1), each of G_0 is not P_5-equicoverable, then $C(G_0) > c(G_0)$. So G is not P_5-equicoverable.

Actually, this case is similar to Case 2.

Case 4: G is obtained by adding copies of P_2 and P_3 to the vertices of C_5.

If only copies of P_2 or only copies of P_3 are added, G has been discussed in Case 1 or Case 2. Otherwise, we have:

(1)If each vertex of C_5 can be added to only one P_2 or one P_3, G can only be one of the 24 graphs shown in Figure 5. No matter which graph is in Figure 5, a minimal P_5-covering whose covering number $C(G)$ is greater than the number of the minimum P_5-covering $c(G)$. So the graphs are not P_5-equicoverable.



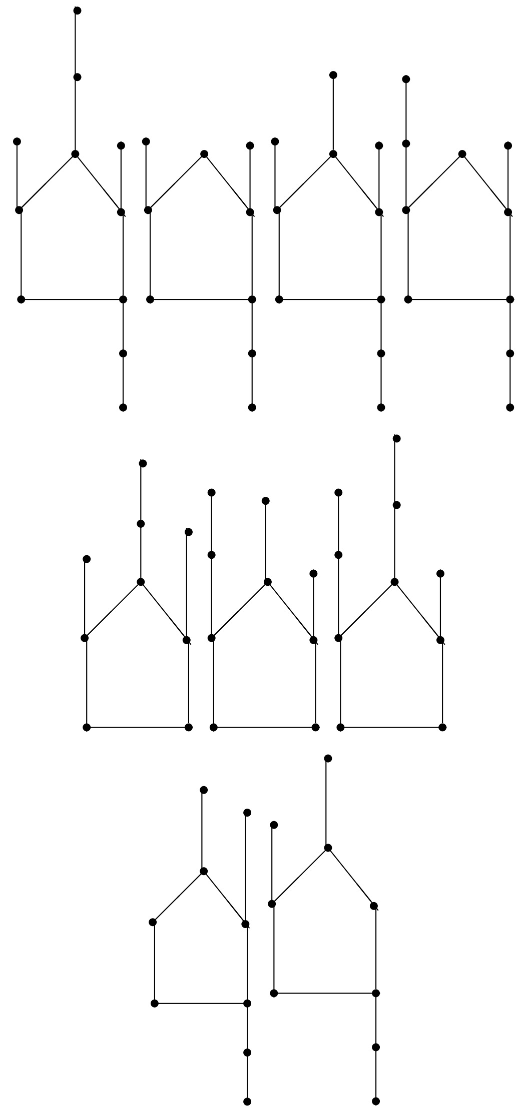


Figure 5: graphs obtained by adding only one P_2 or one P_3 to each vertex of C_5

(2)If each vertex of C_5 can be added to any copies of P_2 or P_3. G is obtained by adding copies of P_2 and P_3 to the vertices of the 5-cycle part of G_0, where G_0 is one of the graphs in Figure 5. If the number of the copies of P_2 and P_3 added is n, we can get a minimal P_5-covering whose covering number is $C(G_0)+n$(using $C(G_0)$ copies of P_5 to cover the G_0 part and n copies of P_5 to cover other parts), while the number of the minimum P_5-covering is at most $c(G_0)+n$. By (1), each of G_0 is not P_5-equicoverable, then $C(G_0) > c(G_0)$. So G is not P_5-equicoverable.

(3)If each vertex of C_5 can be added to at most one $P_2 \cdot P_3$, G can only be one of the seven graphs shown in Figure 6. No matter which graph is in Figure 6, a minimal P_5-covering whose covering number $C(G)$ is greater than the number of the minimum P_5-covering $c(G)$. So the graphs are not P_5-equicoverable; If each vertex of C_5 can be added to any copies of $P_2 \cdot P_3$, G can be decomposed several components which can be P_5-coverable. While there is at least one component which is similar to Case 1 or Case 4(2) not P_5-equicoverable. G is not P_5-equicoverable.

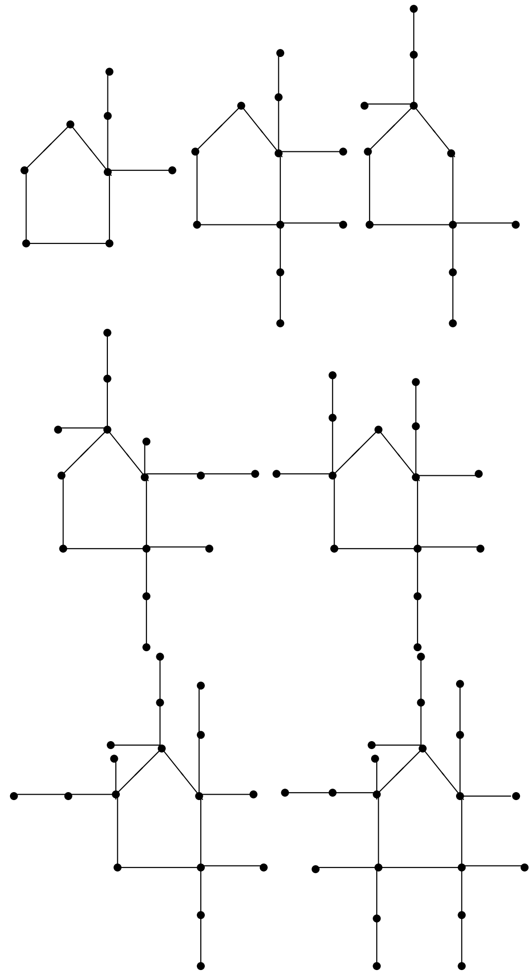


Figure 6: graphs obtained by adding at most one

$P_2 \cdot P_3$ to each vertex of C_5

Case 5: G is obtained by adding copies of P_2 and $K_{1,t}(t \geq 3)$ to the vertices of C_5.

The case is similar to Case 4. G is not P_5-equicoverable.

Case 6: G is obtained by adding copies of P_3 and $K_{1,t}(t \geq 3)$ to the vertices of C_5.

The case is similar to Case 2. G is not P_5-equicoverable.

Case 7: G is obtained by adding copies of P_2 and P_3 and $K_{1,t}(t \geq 3)$ to the vertices of C_5.

The case is similar to Case 4. G is not P_5-equicoverable.

Case 8: G is obtained by adding copies of P_4 to the vertices of C_5.

(1)If each vertex of C_5 can be added to at most one P_4, G can only be one of the seven graphs shown in Figure 7. No matter which graph is in Figure 7, a minimal P_5-covering whose covering number $C(G)$ is greater than the number of the minimum P_5-covering $c(G)$. So the graphs are not P_5-equicoverable.

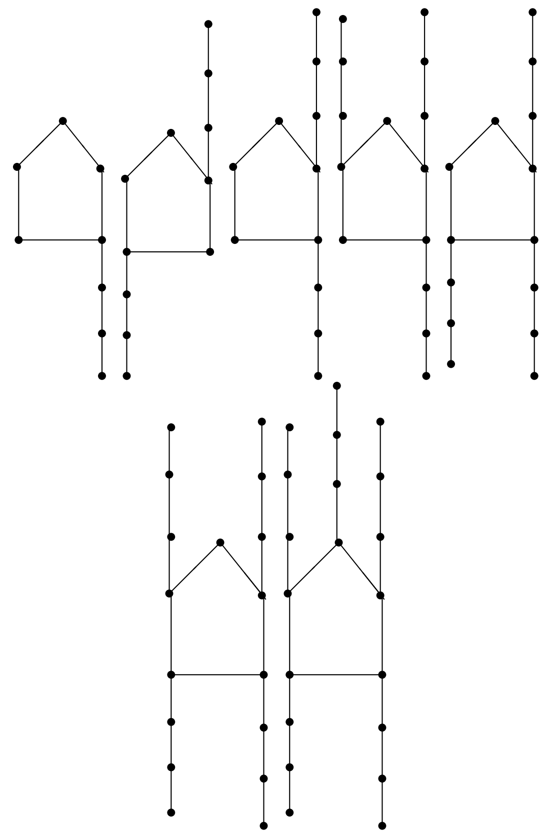


Figure 7: graphs obtained by adding at most one

P_4 to each vertex of C_5

(2)If each vertex of C_5 can be added to any copies of P_4. G is obtained by adding copies of P_4 to the vertices of the 5-cycle part of G_0, where G_0 is one of the graphs in Figure 7. If the number of the copies of P_4 added is n, we can get a minimal P_5-covering whose covering number is $C(G_0) + n$(using $C(G_0)$ copies of P_5 to cover the G_0 part and n copies of P_5 to cover other parts), while the number of the minimum P_5-covering is at most $c(G_0) + n$. By (1), each of G_0 is not P_5-equicoverable, then $C(G_0) > c(G_0)$. So G is not P_5-equicoverable.

Case 9: G is obtained by adding copies of P_2 and P_4 to the vertices of C_5.

If only copies of P_2 or only copies of P_4 are added, G has been discussed in Case 1 or Case 8. Otherwise, we have:

(1)If each vertex of C_5 can be added to only one P_2 or one P_4, G can only be one of 24 graphs similar to Figure 5. No matter which graph is, a minimal P_5-covering whose covering number $C(G)$ is greater than the number of the minimum P_5-covering $c(G)$. So the graphs are not P_5-equicoverable.

(2)If each vertex of C_5 can be added to any copies of P_2 or P_4. G is obtained by adding copies of P_2 and P_4 to the vertices of the 5-cycle part of G_0, where G_0 is one of the graphs in (1). If the number of the copies of P_2 and P_4 added is n, we can get a minimal P_5-covering whose covering number is $C(G_0) + n$(using $C(G_0)$ copies of P_5 to cover the G_0 part and n copies of P_5 to cover other parts), while the number of the minimum P_5-covering is at most $c(G_0) + n$. By (1), each of G_0 is not P_5-equicoverable, then $C(G_0) > c(G_0)$. So G is not P_5-equicoverable.

(3)If each vertex of C_5 can be added to at most one $P_2 \cdot P_4$, G can only be one of the seven graphs similar to Figure 6. No matter which graph is, a minimal P_5-covering whose covering number $C(G)$ is greater than the number of the minimum P_5-covering $c(G)$. So the graphs are not P_5-equicoverable. If each vertex of C_5 can be added to any copies of $P_2 \cdot P_4$, G can be obtained by adding copies of $P_2 \cdot P_4$ to the vertices of the 5-cycle part of G_0, where G_0 is one of the graphs above. If the sum of the number of the copies of $P_2 \cdot P_4$ added is n, we can get a minimal P_5-covering whose covering number is $C(G_0) + n$(using $C(G_0)$ copies of P_5 to cover the G_0 part and n copies of P_5 to cover other parts), while the number of the minimum P_5-covering is at most $c(G_0) + n$. Each of G_0 is not P_5-equicoverable, then $C(G_0) > c(G_0)$. So G is not P_5-equicoverable.

Case 10: G is obtained by adding copies of P_3 and P_4 to the vertices of C_5.

If only copies of P_3 or only copies of P_4 are added, G has been discussed in Case 2 or Case 8. Otherwise, we have:

(1)If each vertex of C_5 can be added to only one P_3 or one P_4, G can only be one of the 24 graphs similar to Figure 5. No matter which graph is, a minimal P_5-covering whose covering number $C(G)$ is greater than the number of the minimum P_5-covering $c(G)$. So the graphs are not P_5-equicoverable.

(2)If each vertex of C_5 can be added to any copies of P_3 or P_4. G is obtained by adding copies of P_3 and P_4 to the vertices of the 5-cycle part of G_0, where G_0 is one of the graphs above in (1). If the number of the copies of P_3 and P_4 added is n, we can get a minimal P_5-covering whose covering number is $C(G_0)+n$(using $C(G_0)$ copies of P_5 to cover the G_0 part and n copies of P_5 to cover other parts), while the number of the minimum P_5-covering is at most $c(G_0)+n$. By (1), each of G_0 is not P_5-equicoverable, then $C(G_0) > c(G_0)$. So G is not P_5-equicoverable.

(3)If each vertex of C_5 can be added to at most one $P_3 \cdot P_4$, G can only be one of the seven graphs similar to Figure 6. No matter which graph is, a minimal P_5-covering whose covering number $C(G)$ is greater than the number of the minimum P_5-covering $c(G)$. So the graphs are not P_5-equicoverable. If each vertex of C_5 can be added to any copies of $P_3 \cdot P_4$, G can be obtained by adding copies of $P_3 \cdot P_4$ to the vertices of the 5-cycle part of G_0, where G_0 is one of the graphs above. If the number of the copies of $P_3 \cdot P_4$ added is n, we can get a minimal P_5-covering whose covering number is $C(G_0) + 2n$(using $C(G_0)$ copies of P_5 to cover the G_0 part and $2n$ copies of P_5 to cover other parts), while the number of the minimum P_5-covering is at most $c(G_0) + 2n$. Each of G_0 is not P_5-equicoverable, then $C(G_0) > c(G_0)$. So G is not P_5-equicoverable.

Case 11: G is obtained by adding copies of P_2, P_3 and P_4 to the vertices of C_5.

P_2, P_3 and P_4 are all added to the vertices of C_5, otherwise the cases has been discussed.

First, G can be obtained by adding copies of P_2 and P_3 to the vertices of C_5 and we denote it by G_{23}. Next we add P_4 to G_{23}. If the number of the copies of P_4 added is n, we can get a minimal P_5-covering whose covering number is $C(G_{23})+n$(using $C(G_{23})$ copies of P_5 to cover the G_{23} part and n copies of P_5 to cover other parts), while the number of the minimum P_5-covering is at most $c(G_{23}) + n$. Each of G_{23} is not P_5-equicoverable by Case 4, then $C(G_{23}) > c(G_{23})$. So G is not P_5-equicoverable.

Case 12: G is obtained by adding copies of P_4 and $K_{1,t}(t \geq 3)$ to the vertices of C_5.

The case is similar to Case 10. G is not P_5-equicoverable.

Case 13: G is obtained by adding copies of P_2, P_4 and $K_{1,t}(t \geq 3)$ to the vertices of C_5.

The case is similar to Case 11. G is not P_5-equicoverable.

Case 14: G is obtained by adding copies of P_3, P_4 and $K_{1,t}$ to the vertices of C_5.

The case is similar to Case 10. G is not P_5-equicoverable.

Case 15: G is obtained by adding copies of P_2, P_3, P_4 and $K_{1,t}(t \geq 3)$ to the vertices of C_5.

The case is similar to Case 11. G is not P_5-equicoverable.

Case 16: G is obtained by adding copies of P_5 to the vertices of C_5.

(1)If we add n copies of P_5 to only one vertex of C_5, both the minimal P_5-covering number and the minimum P_5-covering number are $n + 2$. So it is P_5-equicoverable. We denote the graph by $C_5 \cdot S_n^{3*}$.

(2)If we add n copies of P_5 to at least two vertices of C_5, there exists a minimal P_5-covering number is

$n+3$ and the minimum P_5-covering number is $n+2$. Obviously,$c(G) \neq C(G)$, G is not P_5-equicoverable.

Case 17: G is obtained by adding copies of $P_2 \cdot K_{1,t}(t \geq 3)$ to the vertices of C_5.

The case is similar to Case 8. G is not P_5-equicoverable.

Case 18: G is obtained by adding copies of $P_3 \cdot K_{1,t}(t \geq 3)$ to the vertices of C_5.

We identify one endpoint of P_3 with one of the vertices of C_5.

(1)If we add n copies of $P_3 \cdot K_{1,t}(t \geq 3)$ to only one vertex of C_5, both the minimal P_5-covering number and the minimum P_5-covering number are $n(t-1)+2$. So it is P_5-equicoverable.

(2)If we add n copies of $P_3 \cdot K_{1,t}(t \geq 3)$ to at least two vertices of C_5, there exists a minimal P_5-covering number is $n(t-1)+3$ and the minimum P_5-covering number is $n(t-1)+2$. Obviously, $c(G) \neq C(G)$, G is not P_5-equicoverable.

Case 19: G is a graph not contained in Case 1-18.

Each G can be decomposed into two connected components: a graph G_0 which is not P_5-equicoverable contained in Case 1-18 and a graph which is P_5-coverable. By Lemma 3, G is not P_5-equicoverable. □

In summary, G is not P_5-equicoverable unless G is $C_5 \cdot S_n^{3*}$ or G is obtained by adding n copies of $P_3 \cdot K_{1,t}(t \geq 3)$ to only one vertex of C_5.

Next we consider graphs that contains a cycle with length larger than 5.

Lemma 10 *$C_n \cdot P_2 (n \geq 6)$ is P_5-equicoverable if and only if $n = 8$.*

Proof: (1)If C_n is P_5-equicoverable, we have $n = 6, 7, 9$. Because $C(C_n \cdot P_2; P_5) > c(C_n \cdot P_2; P_5)(n = 6, 7, 9)$, $C_6 \cdot P_2$ and $C_7 \cdot P_2$ and $C_9 \cdot P_2$ are not P_5-equicoverable.

(2)If C_n is not P_5-equicoverable, we have $n \neq 6, 7, 9$. It is easy to find that $C(C_8 \cdot P_2; P_5) = c(C_8 \cdot P_2; P_5) = 3$. $C_8 \cdot P_2$ is P_5-equicoverable. For $n \geq 10$, C_n is not P_5-equicoverable. We can use $C(C_n)$ copies of P_5 to cover the C_n part and one copy of P_5 to cover the else. Also, we can use $c(C_n)$ copies of P_5 to cover the C_n part and one copy of P_5 to cover the else. While $c(C_n \cdot P_2) \leq c(C_n) + 1 < C(C_n) + 1$, G is not P_5-equicoverable. □

Lemma 11 *$C_n \cdot P_3 (n \geq 6)$ is P_5-equicoverable if and only if $n = 7$.*

Proof: (1)If C_n is P_5-equicoverable, we have $n = 6, 7, 9$. Because $C(C_n \cdot P_3; P_5) > c(C_n \cdot P_3; P_5)(n = 6, 9)$, $C_6 \cdot P_3$ and $C_9 \cdot P_3$ are not P_5-equicoverable. While $C(C_7 \cdot P_3; P_5) = c(C_7 \cdot P_3; P_5) = 3$. $C_7 \cdot P_3$ is P_5-equicoverable.

(2)If C_n is not P_5-equicoverable, we have $n \neq 6, 7, 9$. It is easy to find that $C(C_8 \cdot P_3; P_5) > c(C_8 \cdot P_3; P_5)$. $C_8 \cdot P_3$ is not P_5-equicoverable. For $n \geq 10$, C_n is not P_5-equicoverable. We can use $C(C_n)$ copies of P_5 to cover the C_n part and one copy of P_5 to cover the else. Also, we can use $c(C_n)$ copies of P_5 to cover the C_n part and one copy of P_5 to cover the else. While $c(C_n \cdot P_3) \leq c(C_n) + 1 < C(C_n) + 1$, G is not P_5-equicoverable. □

Lemma 12 *$C_n \cdot P_4 (n \geq 6)$ is P_5-equicoverable if and only if $n = 6$.*

Proof: (1)If C_n is P_5-equicoverable, we have $n = 6, 7, 9$. Because $C(C_n \cdot P_4; P_5) > c(C_n \cdot P_4; P_5)(n = 7, 9)$, $C_7 \cdot P_4$ and $C_9 \cdot P_4$ are not P_5-equicoverable. While $C(C_6 \cdot P_4; P_5) = c(C_6 \cdot P_4; P_5) = 3$. $C_6 \cdot P_4$ is P_5-equicoverable.

(2)If C_n is not P_5-equicoverable, we have $n \neq 6, 7, 9$. It is easy to find that $C(C_8 \cdot P_4; P_5) > c(C_8 \cdot P_4; P_5)$. $C_8 \cdot P_4$ is not P_5-equicoverable. For $n \geq 10$, C_n is not P_5-equicoverable. We can use $C(C_n)$ copies of P_5 to cover the C_n part and one copy of P_5 to cover the else. Also, we can use $c(C_n)$ copies of P_5 to cover the C_n part and one copy of P_5 to cover the else. While $c(C_n \cdot P_4) \leq c(C_n) + 1 < C(C_n) + 1$, G is not P_5-equicoverable. □

Lemma 13 *$C_n \cdot P_5 (n \geq 6)$ is not P_5-equicoverable.*

Lemma 14 *$C_n \cdot K_{1,t}(n \geq 4, t \geq 3)$ is not P_5-equicoverable.*

Lemma 15 *$C_n \cdot P_2 \cdot K_{1,t}(n \geq 4)$ is not P_5-equicoverable.*

Lemma 16 *$C_n \cdot P_3 \cdot K_{1,t}(n \geq 6)$ is not P_5-equicoverable*

Lemma 17 *G is a connected graph that is not a cycle. If G doesn't contain cycles with length smaller than 6 and contains a 6-cycle,G is P_5-equicoverable if and only if G is $C_6 \cdot P_4$.*

Proof: Case 1: G is obtained by adding copies of P_2 to the vertices of C_6.

(1)If we add one P_2 to only one vertex of C_6, by Lemma 10, it is not P_5-equicoverable.

(2)If we add $n(n \geq 2)$ copies of P_2 to only one vertex of C_6, there will be a minimal P_5-covering whose covering number is $n+2$. While the number of the minimum P_5-covering number is less than or equal to $n+1$.

(3)If we add $n(n \geq 2)$ copies of P_2 to at least two vertices of C_6 and each vertex of C_6 can be added to at most one P_2, G must be one of the eleven graphs shown in Figure 7. For each graph which contains a 6-cycle, we can blow up a vertex that no P_2 is added to of C_6 to two vertices. As a consequence, the original graph with a 6-cycle turns out to be a tree. A blowing up that makes the result tree not P_5-equicoverable must exist. So G is not P_5-equicoverable. For example, we blow up v_1 of the left graph to two vertices v_2 and v_3 of the right graph in Figure 8. Obviously, it's not P_5-equicoverable.

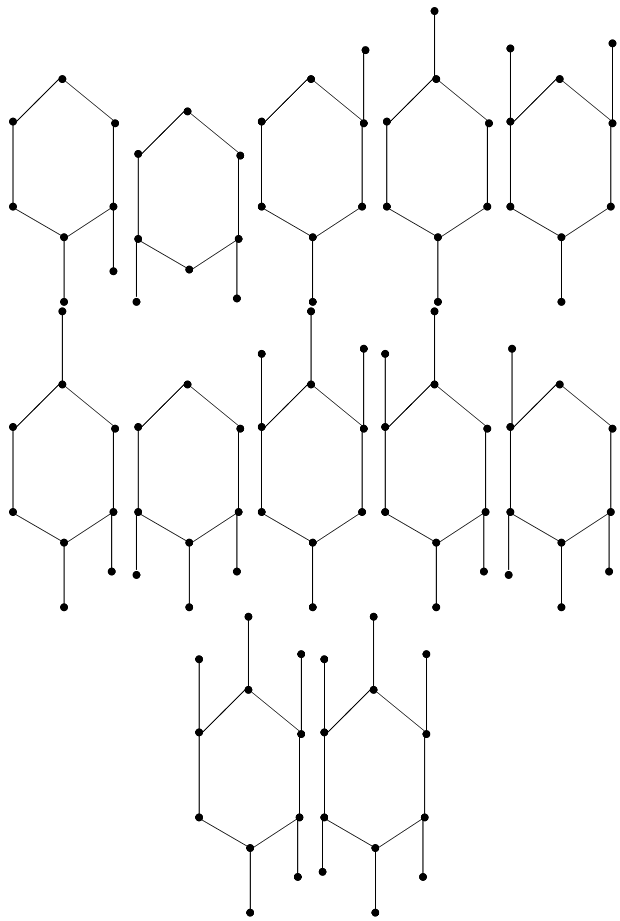


Figure 7: graphs obtained by adding $n(n \geq 2)$ copies of P_2 to at least two vertices of C_6 can be added to at most one P_2

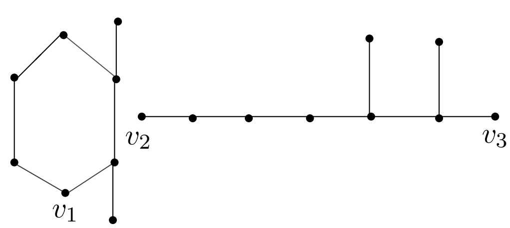


Figure 8: v_1 blown up to two vertices v_2 and v_3

(4)If we add $n(n \geq 2)$ copies of P_2 to at least two vertices of C_6 and each vertex of C_6 can be added to any copies of P_2. Without loss of generality, suppose G is obtained by adding m copies of P_2 to G_0, where G_0 is one of graphs above in (3). Then there exists a minimal P_5-covering whose covering number is $C(G_0)+m$. We can use $C(G_0)$ copies of P_5 to cover the G_0 part and use m copies of P_5 to cover other parts. While the number of the minimum P_5-covering number is at most $c(G_0) + m$. As we all know, for each G_0, there exists a minimal P_5-covering whose $C(G_0) > c(G_0)$, then it is not P_5-equicoverable.

Case 2: G is obtained by adding copies of P_3 to the vertices of C_6.

(1)If we add one P_3 to only one vertex of C_6, by Lemma 11, it is not P_5-equicoverable.

(2)If we add $n(n \geq 2)$ copies of P_3 to only one vertex of C_6, there will be a minimal P_5-covering whose covering number is $n + 2$. While the number of the minimum P_5-covering number is less than or equal to $n + 1$.

(3)If we add $n(n \geq 2)$ copies of P_3 to at least two vertices of C_6 and each vertex of C_6 can be added to at most one P_3, G must be one of the eleven graphs similar to Figure 7. For each graph which contains a 6-cycle, we can blow up a vertex that no P_3 is added to of C_6 to two vertices. As a consequence, the original graph with a 6-cycle turns out to be a tree. A blowing up that makes the result tree not P_5-equicoverable must exist. So G is not P_5-equicoverable.

(4)If we add $n(n \geq 2)$ copies of P_3 to at least two vertices of C_6 and each vertex of C_6 can be added to any copies of P_3. Without loss of generality, suppose G is obtained by adding m copies of P_3 to G_0, where G_0 is one of graphs above in (3). Then there exists a minimal P_5-covering whose covering number is $C(G_0)+m$. We can use $C(G_0)$ copies of P_5 to cover the G_0 part and use m copies of P_5 to cover other parts. While the number of the minimum P_5-covering number is at most $c(G_0) + m$. As we all know, for each G_0, there exists a minimal P_5-covering whose $C(G_0) > c(G_0)$, then it is not P_5-equicoverable.

Case 3: G is obtained by adding copies of $K_{1,t}(t \geq 3)$ to the vertices of C_6.

Similar to Case 2, G is not P_5-equicoverable.

Case 4: G is obtained by adding copies of P_4 to the vertices of C_6.

(1)If we add one P_4 to only one vertex of C_6, by Lemma 12, it is P_5-equicoverable.

(2)The following proof is similar to (2),(3),(4) in Case 2, G is not P_5-equicoverable.

Case 5: G is obtained by adding copies of $P_2, P_3, P_4, K_{1,t}(t \geq 3)$ to the vertices of C_6.

There are eleven subcases: G is obtained by adding copies of at least two of $P_2, P_3, P_4, K_{1,t}(t \geq 3)$. Similar to the proof process of Case 2, G is not P_5-equicoverable.

Case 6: G is obtained by adding copies of P_5 to the vertices of C_6.

(1)If we add one P_5 to only one vertex of C_6, by Lemma 13, it is not P_5-equicoverable.

(2)If G is not the graph in (1), G can be decomposed into two connected components: a graph which is not P_5-equicoverable and a P_5-coverable graph. By Lemma 3, G is not P_5-equicoverable.

Case 7: G is obtained by adding copies of P_4 and P_5 to the vertices of C_6.

If only copies of P_4 or only copies of P_5 are added, G has been discussed in previous. Otherwise, similar to Case 4 of Lemma 9, G is not P_5-equicoverable.

Case 8: G is a graph not contained in Case 1-7.

We decompose G into two connected components: a graph G_0 contained in Case 1-7 and a graph which is P_5-coverable. G_0 is not P_5-equicoverable, by Lemma 3, G is not P_5-equicoverable.

In summary, G is not P_5-equicoverable unless it is $C_6 \cdot P_4$. □

Lemma 18 *G is a connected graph that is not a cycle. If G doesn't contain cycles with length smaller than 7 and contains a 7-cycle,G is P_5-equicoverable if and only if G is $C_7 \cdot P_3$.*

Lemma 19 *G is a connected graph that is not a cycle. If G doesn't contain cycles with length smaller than 8 and contains a 8-cycle,G is P_5-equicoverable if and only if G is $C_8 \cdot P_2$.*

Lemma 20 *G is a connected graph that is not a cycle. If G doesn't contain cycles with length smaller than 9, G is not P_5-equicoverable.*

Proof: Case 1: If G is one of the graphs in Lemma 10-Lemma 16, G is not P_5-equicoverable.

Case 2: If G is not a graph in Case 1, according to the proof process of Lemma 17, G can be decomposed into connected components: a tree which is not P_5-equicoverable and P_5-coverable graphs. □

In the end, we conclude the main results: A connected graph G is P_5-equicoverable if and only if G satisfies one of the following:

Theorem 21 *Let G be a connected graph that doesn't contain 3-cycles or 4-cycles and contains a cycle with length at least 5. Then G is P_5-equicoverable if and only if either of the following holds:*

(1)G is a cycle $C_n (n = 5, 6, 7, 9)$;

(2)G is $C_5 \cdot S_n^{3*} (n \geq 1)$;

(3)G is obtained by adding n copies of $P_3 \cdot K_{1,t} (t \geq 3)$ to only one vertex of C_5.

(4)G is $C_6 \cdot P_4$.

(5)G is $C_7 \cdot P_3$.

(6)G is $C_8 \cdot P_2$.

For disconnected graphs ,we have:

Theorem 22 *A graph G that doesn't contain 3-cycles or 4-cycles and contains at least one cycle with length larger than 4 is P_5-equicoverable if and only if each component of G is P_5-equicoverable.*

3 Results of P_k-equicoverable graphs

Theorem 23 *$C_n \cdot P_2$ is P_k-equicoverable if and only if $n = k - 1$ or $n = 2k - 2$.*

Proof:

(1)When $n \leq k - 2$, $C_n \cdot P_2$ doesn't contain the subgraph of P_k. Then it is not P_k-equicoverable.

(2)When $n = k - 1$, $C_n \cdot P_2$ is P_k-equicoverable and $C(C_n \cdot P_2; P_k) = c(C_n \cdot P_2; P_k) = 2$.

(3)When $k \leq n \leq 2k - 3$, it is easy to find $c(C_n \cdot P_2; P_k) = 2$. Conveniently, denote the edges of $C_n \cdot P_2$ by $e_0, e_1, \cdots e_n$. There exits a minimal P_k-covering as following: we denote it by $H = \{H_1, H_2, H_3\}$,

$$\begin{cases} H_1 = \{e_0, e_1, e_2, \cdots, e_{k-2}\}, \\ H_2 = \{e_n, e_1, e_2, \cdots, e_{k-2}\}, \\ H_3 = \{e_{k-1}, e_k, e_{k+1}, \cdots, e_{n-1}\}. \end{cases}$$

Then H is a minimal P_k-covering instead of the minimum P_k-covering of C_n. It is not P_k-equicoverable.

(4)When $n = 2k-2$, $C_n \cdot P_2$ is P_k-equicoverable. It is clear that $c(C_n \cdot P_2; P_k) = 3$. We denote the vertices of $C_n \cdot P_2$ by $v_0, v_1, v_2, \cdots, v_{2k-2}$. Generally speaking, suppose that there exists a copy of P_k covering the edge v_1v_2, which is denoted by $H_0 = \{v_1v_2, v_2v_3, \cdots, v_{k-1}v_k\}$. Then there also exists a copy of P_k covering the edge v_kv_{k+1}, which is denoted by $H_i = \{v_iv_{i+1}, v_{i+1}v_{i+2}, \cdots, v_{i+k-2}v_{i+k-1}\} (2 \leq i \leq k - 1)$. Similarly, there must be a copy of P_k covering the edge v_1v_0, which is denoted by $H_1 = \{v_{k+1}v_{k+2}, v_{k+2}v_{k+3}, \cdots, v_{2k-3}v_{2k-2}, v_{2k-2}v_1, v_1v_0\}$. And by the definition of the equicoverable, $\{H_0, H_i, H_1 | 2 \leq i \leq k - 1\}$ is the family of the minimal P_k-covering of $C_n \cdot P_2$.(or

$$\begin{cases} H_0 = \{v_0v_1, v_1v_2, v_2v_3, \cdots, v_{k-2}v_{k-1}\}, \\ H_i = \{v_iv_{i+1}, v_{i+1}v_{i+2}, \cdots, v_{i+k-2}v_{i+k-1}\} \\ \qquad 2 \leq i \leq k - 1, \\ H_1 = \{v_{k+1}v_{k+2}, v_{k+2}v_{k+3} \cdots, v_{2k-3}v_{2k-2}, \\ \qquad v_{2k-2}v_1, v_1v_0\}. \end{cases}$$

As a result, the number of minimal P_k-covering of $C_n \cdot P_2$ is only 3. $C_n \cdot P_2$ is P_k-equicoverable.

(5)When $n = 3k-3$, it is easy to find $c(C_n \cdot P_2; P_k) = 4$. We denote the edges of $C_n \cdot P_2$ by $e_0, e_1, \cdots, e_{3k-3}$. There exits a minimal P_k-covering as following: we denote it by $H = \{H_1, H_2, H_3, H_4, H_5\}$,

$$\begin{cases} H_1 = \{e_0, e_1, e_2, \cdots, e_{k-2}\}, \\ H_2 = \{e_1, e_2, \cdots, e_{k-1}\}, \\ H_3 = \{e_k, e_{k+1}, \cdots, e_{2k-2}\}, \\ H_4 = \{e_{k+1}, e_{k+2}, \cdots, e_{2k-1}\}, \\ H_5 = \{e_{2k}, e_{2k+1}, \cdots, e_{3k-3}, e_1\}. \end{cases}$$

So it is not P_k-equicoverable.

(6)When $2k-1 \leq n \leq 3k-4$ and $n \geq 3k-2$, $C_n \cdot P_2$ is not P_k-equicoverable by Theorem 4. □

Corollary 24 *$C_n \cdot P_3 (n \geq k+1)$ is P_k-equicoverable if and only if $n = 2k-3$.*

Corollary 25 *$C_n \cdot P_4 (n \geq k+1)$ is P_k-equicoverable if and only if $n = 2k-4$.*

Corollary 26 *$C_n \cdot P_5 (n \geq k+1)$ is P_k-equicoverable if and only if $n = 2k-5$.*

Theorem 27 *$C_n \cdot P_k (n \geq k+1, k \geq 6)$ is not P_k-equicoverable.*

Proof:

(1)When $k+1 \leq n \leq 2k-2$ and $n \geq 2k$, it is easy to come to the conclusion according to Theorem 5.

(2)When $n = 2k-1$, $c(C_n \cdot P_k; P_k) = 4$. We denote its edges by $e_{p1}, e_{p2}, \cdots, e_{p(k-1)}, e_{c1}, e_{c2}, \cdots, e_{c(2k-1)}$. There exits a minimal P_k-covering as following: we denote it by $H = \{H_1, H_2, H_3, H_4, H_5\}$,

$$\begin{cases} H_1 = \{e_{c1}, e_{p1}, e_{p2}, \cdots, e_{p(k-2)}\}, \\ H_2 = \{e_{c(2k-1)}, e_{p1}, \cdots, e_{p(k-2)}\}, \\ H_3 = \{e_{p1}, e_{p2}, \cdots, e_{p(k-1)}\}, \\ H_4 = \{e_{c2}, e_{c3}, \cdots, e_{ck}\}, \\ H_5 = \{e_{ck}, e_{c(k+1)}, \cdots, e_{c(2k-2)}\}. \end{cases}$$

So it is also not P_k-equicoverable. □

Corollary 28 *$C_n \cdot K_{1,t} (n \geq k-1, t \geq 3)$ is not P_k-equicoverable.*

Theorem 29 *$C_n \cdot S_m^{(k-2)*}$ is P_k-equicoverable if and only if $3 \leq n \leq k$ and $c(G) = C(G) = m+2$.*

Proof: (1)When $n \geq k+1$, it is not P_k-equicoverable by Theorem 27.

(2)When $3 \leq n \leq k-1$, the subgraph C_n doesn't contain P_k. There must be m copies of P_k covering the part of $S_m^{(k-2)*}$; The else can be covered by using only two copies of P_k. It is P_k-equicoverable and $c(G) = C(G) = m+2$.

(3)When $n = k$, the $S_m^{(k-2)*}$ part must be covered by m copies of P_k. We can only use two copies of P_k to cover the else C_n part. Then the $C_n \cdot S_m^{(k-2)*}$ is P_k-equicoverable. □

The next comment follows immediately from Theorem 29.

Corollary 30 *$C_n \cdot P_{k-2} \cdot K_{1,t}$ is P_k-equicoverable if and only if $3 \leq n \leq k$.*

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