Solution of Singularly Perturbed Delay Differential Equations with Dual Layer Behaviour using Numerical Integration

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Abstract: In this paper, we proposed a numerical integration method for the solution of singularly perturbed delay differential equation with dual layer behaviour. In this method, an asymptotically equivalent first order neutral type delay differential equation is obtained from the second order singularly perturbed delay differential equation and employed Trapezoidal rule on it. Then, linear interpolation is used to get three term recurrence relation which is solved by discrete invariant imbedding algorithm. Numerical illustrations for various values of the delay parameter and perturbation parameter are presented to validate the proposed method. Convergence of the proposed method is also analyzed.

Key–Words: Singularly perturbed delay differential equations, Dual layer, Exponential Integrating Factor, Numerical Integration

1 Introduction

Singularly perturbed differential-difference equations(SPDDEs) with negative shifts, popularly know as, delay differential equations are special cases of functional differential equations, where the evolution of a system at certain time, depends on the present state of the system as well as the state of the system at an earlier time. For example, in the predator-prey model [17], the birth of predators is affected by prior levels of predator prey model along with its recent levels. The manner in which these differential-difference equations analyze bio system dynamics [2, 3] and many more [4, 12, 13, 19], has been and remains an active area of research. In a series of papers published by Lange and Miura [14, 16] on singularly perturbed differential-difference equations, detailed studied is carried out on the solutions of SPDDEs exhibiting rapid oscillations, resonance behaviour, turning point behaviour and boundary and interior layer behaviour. Kadalbajoo and Kumar [8] and Kadalbajoo and Sharma [10, 11] proposed numerical methods based on finite differences for singularly perturbed delay differential difference equations. Amiraliyeva et al. [1] constructed an exponentially fitted finite difference scheme in an equidistant mesh, which gives first order uniform convergence, for singularly perturbed delay initial value problems. Ramesh and Kadalbajoo [9] proposed a numerical algorithm for singularly perturbed linear second order reaction-diffusion boundary value problems with small shifts. Pratima and Sharma [18] described a numerical method based on fitted operator finite difference, namely the Il'in Allen-Southwell fitting for the singularly perturbed delay differential equations with turning points. Soujanya et al. [20] proposed an exponentially Fitted non symmetric finite difference method to solve a class of singular perturbation problems. Chunfang Miao and Yunquan [21] investigated the global asymptotic stability of second-order neutral type Cohen-Grossberg neural networks with time-varying delays. Aleksey A. Kabanov, discussed the synthesis of composite control for nonlinear singularly perturbed system using feedback linearization method.

With this motivation, in the present paper we proposed a numerical integration method for the solution of singularly perturbed delay differential equations with dual layer behaviour. In section 2, the numerical scheme to solve the singularly perturbed delay differential equations with dual layer is discussed. Convergence of the numerical method is analyzed in section 3. Numerical illustrations and results to support the method are given in section 4. Finally, the conclusions are given in section 5.

2 Description of the method

To discuss our method, we considered singularly perturbed delay differential equation of the form:

$$\varepsilon y''(x) + a(x)y(x-\delta) + b(x)y(x) = f(x) \quad (1)$$

with boundary conditions

$$y(0) = \phi(x)\ ;\ \delta \leq x \leq 0, \quad y(1) = \beta \quad (2)$$

where ε is a small positive perturbation parameter, $0 < \varepsilon << 1$ and δ is also small positive shifting parameter, $0 < \delta < 1$; a(x), b(x), f(x) and $\phi(x)$ are bounded continuous functions in $(0, 1)$ and β is finite constant. If $a(x) + b(x) \leq 0$ then the solution of the (1)-(2) exhibits boundary layer on both ends of the interval $[0, 1]$ and for $a(x) + b(x) > 0$, the solution of (1)-(2) exhibits oscillatory behaviour. The boundary value problem considered here is of the reaction-diffusion type, so there will be two boundary layers at both the end points i.e., at $x = 0$ and $x = 1$.

Discretizing the interval $[0, 1]$ into N equal parts with mesh size h, let $0 = x_0, x_1, ..., x_N = 1$ be the mesh points. Then we have $x_i = ih$ for i = 0, 1, , N. Since the problem exhibits two boundary layers, we divide the interval $[0, 1]$ into two sub intervals $\left[0, \frac{1}{1}\right]$ and $\left[\frac{1}{2}, 1\right]$. We choose l such that $x_{l/2} = \frac{1}{2}$. In the interval $\left[0, \frac{1}{2}\right]$ the boundary layer will be on the left hand side i.e., at $x = 0$ and in the interval $\left[\frac{1}{2}, 1\right]$ the boundary layer will be on the right hand side i.e., $x = 1$. Hence, we derive the numerical method for both left-end and right-end layers.

In the interval $\left[0, \frac{1}{2}\right]$, using Taylor series expansion in the neighbourhood of the point x, we have

$$y'(x-\varepsilon) \approx y'(x) - \varepsilon y''(x) \quad (3)$$

and consequently, Eq.(1) is replaced by the following approximate first order differential equation with a small deviating argument:

$$\begin{aligned} &y'(x) + b(x)y(x) \\ &= f(x) + y'(x-\varepsilon) - a(x)y(x-\delta) \quad (4) \end{aligned}$$

The transition from Eq.(1) to Eq.(4) is admitted, because of the condition that ε is small. This replacement is significant from the computational point of view. For details on the validity of this transition one can refer Elsgolts and Norkin [7]. Here, for consolidation of our ideas of the method we assume the a(x) and b(x) are constants. We now apply an integrating factor e^{bx} on Eq.(4) (as in Brian J. McCartin [5]):

$$\begin{aligned} &\frac{d}{dx}\left\{e^{bx}y(x)\right\} \\ &= e^{bx}\left\{f(x) + y'(x-\varepsilon) - ay(x-\delta)\right\} \quad (5) \end{aligned}$$

Now integrating Eq.(5) in the interval $[0, 1]$, with respect to x from x_i to x_{i+1} , we get:

$$\begin{aligned} &\int_{x_i}^{x_{i+1}} \frac{d}{dx}\left(e^{bx}y(x)\right) = \int_{x_i}^{x_{i+1}} e^{bx}y'(x-\varepsilon)dx \\ &- \int_{x_i}^{x_{i+1}} ae^{bx}y(x-\delta)dx + \int_{x_i}^{x_{i+1}} e^{bx}f(x)dx \\ &e^{bx_{i+1}}y_{i+1} - e^{bx_i}y_i = e^{bx_{i+1}}y(x_{i+1}-\varepsilon) \\ &-e^{bx_i}y(x_i-\varepsilon) - b\int_{x_i}^{x_{i+1}} e^{bx}y(x-\varepsilon)dx \\ &- \int_{x_i}^{x_{i+1}} ae^{bx}y(x-\delta)dx + \int_{x_i}^{x_{i+1}} e^{bx}f(x)dx \end{aligned}$$

By employing Trapezoidal rule to evaluate the integrals, we get

$$\begin{aligned} & e^{b_{i+1}x_{i+1}}y_{i+1} - e^{b_ix_i}y_i \\ = \ & e^{b_{i+1}x_{i+1}}y(x_{i+1}-\varepsilon) - e^{b_ix_i}y(x_i-\varepsilon) \\ - \ & \frac{bh}{2}\left(e^{b_ix_i}y(x_i-\varepsilon) + e^{b_{i+1}x_{i+1}}y(x_{i+1}-\varepsilon)\right) \\ - \ & \frac{h}{2}\left(e^{b_ix_i}a(x_i)y(x_i-\delta) + e^{b_{i+1}x_{i+1}}a(x_{i+1})y(x_{i+1}-\delta)\right) \\ + \ & \frac{h}{2}\left(e^{b_{i+1}x_{i+1}}f(x_{i+1}) + e^{b_ix_i}f(x_i)\right) \quad (6) \end{aligned}$$

By Taylor series expansion and then approximating $y'(x)$ by linear interpolation, we obtain:

$$\begin{aligned} &y(x_i-\delta) \approx y(x_i) - \delta y'(x_i) \\ &= \left(1 - \frac{\delta}{h}\right)y_i + \frac{\delta}{h}y_{i-1} \quad (7) \end{aligned}$$

$$\begin{aligned} &y(x_{i+1}-\delta) \approx y(x_{i+1}) - \delta y'(x_{i+1}) \\ &= \left(1 - \frac{\delta}{h}\right)y_{i+1} + \frac{\delta}{h}y_i \quad (8) \end{aligned}$$

$$\begin{aligned} &y(x_i-\varepsilon) \approx y(x_i) - \varepsilon y'(x_i) \\ &= \left(1 - \frac{\varepsilon}{h}\right)y_i + \frac{\varepsilon}{h}y_{i-1} \quad (9) \end{aligned}$$

$$\begin{aligned} &y(x_{i+1}-\varepsilon) \approx y(x_{i+1}) - \varepsilon y'(x_{i+1}) \\ &= \left(1 - \frac{\varepsilon}{h}\right)y_{i+1} + \frac{\varepsilon}{h}y_i \quad (10) \end{aligned}$$

Using Eqs.(7)-(10) in Eq.(6) and rearranging the terms, we get

$$\begin{aligned} &\left(e^{b_{i+1}h} - e^{b_{i+1}h}\left(1-\frac{\varepsilon}{h}\right) + e^{b_{i+1}h}\frac{b_{i+1}h}{2}\left(1-\frac{\varepsilon}{h}\right)\right)y_{i+1} \\ &+ \left(e^{b_{i+1}h}\frac{ha_{i+1}}{2}\left(1 - \frac{\delta}{h}\right)\right)y_{i+1} \\ &\left(1 + e^{b_{i+1}h}\frac{\varepsilon}{h} - \frac{b_{i+1}h}{2}\left(1 - \frac{\varepsilon}{h}\right) - \frac{b_{i+1}\varepsilon}{2}\right)y_i \end{aligned}$$

$$-\left(\frac{ha_i}{2}\left(1-\frac{\delta}{h}\right)-e^{b_{i+1}h}\frac{\delta a_{i+1}}{2}\right)y_i$$
$$-\left(\frac{\varepsilon}{h}+\frac{b_{i+1}\varepsilon}{2}+\frac{\delta a_i}{2}\right)y_{i-1}$$
$$=\frac{h}{2}(f_i+e^{b_{i+1}h}f_{i+1}) \quad (11)$$

Eq. (11) can be written as a three term recurrence relation of the form

$$E_i y_{i-1}-F_i y_i+G_i y_{i+1}=H_i \;;\; i=1,2,..,l/2-1.$$

where

$$E_i=\frac{\varepsilon}{h}+\frac{b_{i+1}\varepsilon}{2}+\frac{\delta a_i}{2}$$
$$F_i=1+e^{b_{i+1}h}\frac{\varepsilon}{h}-\frac{b_{i+1}h}{2}\left(1-\frac{\varepsilon}{h}\right)-\frac{b_{i+1}\varepsilon}{2}$$
$$-\frac{ha_i}{2}\left(1-\frac{\delta}{h}\right)-e^{b_{i+1}h}\frac{\delta a_{i+1}}{2}$$
$$G_i=e^{b_{i+1}h}-e^{b_{i+1}h}\left(1-\frac{\varepsilon}{h}\right)$$
$$+e^{b_{i+1}h}\frac{b_{i+1}h}{2}\left(1-\frac{\varepsilon}{h}\right)+e^{b_{i+1}h}\frac{ha_{i+1}}{2}\left(1-\frac{\delta}{h}\right)$$
$$H_i=f_i+e^{b_{i+1}h}f_{i+1}$$

Now, in the interval $\left[\frac{1}{2},1\right]$, by Taylor series expansion in the neighbourhood of the point x, we have

$$y'\left(x+\varepsilon\right)\approx y'(x)+\varepsilon y''(x) \quad (12)$$

and consequently, Eq.(1) is replaced by the following approximate first order differential equation with a small deviating argument:

$$y'(x)-b(x)y(x)$$
$$=y'(x+\varepsilon)+a(x)y(x-\delta)-f(x) \quad (13)$$

We now apply an integrating factor e^{-bx} on Eq.(13):

$$\frac{d}{dx}\left\{e^{-bx}y(x)\right\}=e^{-bx}\left(y'(x+\varepsilon)\right)$$
$$+e^{-bx}\left(a(x)y(x-\delta)-f(x)\right) \quad (14)$$

In the interval $\left[\frac{1}{2},\,1\right]$, integrating Eq.(14) with respect to x from x_{i-1} to x_i , we get

$$\int_{x_{i-1}}^{x_i}\frac{d}{dx}\left(e^{-bx}y(x)\right)=\int_{x_{i-1}}^{x_i}e^{-bx}y'(x+\varepsilon)dx$$
$$+\int_{x_{i-1}}^{x_i}ae^{-bx}y(x-\delta)dx-\int_{x_{i-1}}^{x_i}e^{-bx}f(x)dx$$
$$e^{-b_ix_i}y(x_i)-e^{-b_{i-1}x_{i-1}}y(x_{i-1})=e^{-b_ix_i}y(x_i+\varepsilon)$$
$$-e^{-b_{i-1}x_{i-1}}y(x_{i-1}+\varepsilon)+b\int_{x_{i-1}}^{x_i}e^{-bx}y(x+\varepsilon)dx$$
$$+a\int_{x_{i-1}}^{x_i}e^{-bx}y(x-\delta)dx-\int_{x_{i-1}}^{x_i}e^{-bx}f(x)dx$$

By employing Trapezoidal rule to evaluate the integrals, we get

$$e^{-bx_i}y(x_i)-e^{-bx_{i-1}}y(x_{i-1})=e^{-bx_i}y(x_i+\varepsilon)$$
$$-e^{-bx_{i-1}}y(x_{i-1}+\varepsilon)+\frac{b_{i-1}h}{2}e^{-bx_{i-1}}y(x_{i-1}+\varepsilon)$$
$$+\frac{b_ih}{2}e^{-bx_i}y(x_i+\varepsilon)+a_ie^{-bx_i}y(x_i-\delta)$$
$$-a_{i-1}e^{-bx_{i-1}}y(x_{i-1}-\delta)+\frac{a_ib_ih}{2}e^{-bx_i}y(x_i-\delta)$$
$$+\frac{a_{i-1}b_{i-1}h}{2}e^{-bx_{i-1}}y(x_{i-1}-\delta)$$
$$-\frac{h}{2}e^{-bx_i}f(x_i)-e^{-bx_{i-1}}f(x_{i-1}) \quad (15)$$

By Taylor series expansion and then approximating $y'(x)$ by linear interpolation, we obtain:

$$y(x_i-\delta)\approx y(x_i)-\delta y'(x_i)$$
$$=\left(1+\frac{\delta}{h}\right)y_i-\frac{\delta}{h}y_{i+1} \quad (16)$$
$$y(x_{i-1}-\delta)\approx y(x_{i-1})-\delta y'(x_{i-1})$$
$$=\left(1+\frac{\delta}{h}\right)y_{i-1}-\frac{\delta}{h}y_i \quad (17)$$
$$y(x_i+\varepsilon)\approx y(x_i)+\varepsilon y'(x_i)$$
$$=\left(1-\frac{\varepsilon}{h}\right)y_i+\frac{\varepsilon}{h}y_{i+1} \quad (18)$$
$$y(x_{i-1}+\varepsilon)\approx y(x_{i-1})+\varepsilon y'(x_{i-1})$$
$$=\left(1-\frac{\varepsilon}{h}\right)y_{i-1}+\frac{\varepsilon}{h}y_i \quad (19)$$

Using Eqs.(16)-(19) in Eq.(15) and rearranging the terms, we get

$$\left(-\frac{\varepsilon}{h}-\frac{b_{i-1}h}{2}\left(1-\frac{\varepsilon}{h}\right)-\frac{ha_{i-1}}{2}-\frac{\delta a_{i-1}}{2}\right)y_{i-1}$$
$$-\left(-\frac{\varepsilon}{h}e^{-b_ih}+\frac{\varepsilon}{h}+\frac{b_{i-1}\varepsilon}{2}+\frac{b_ih}{2}\left(1-\frac{\varepsilon}{h}\right)e^{-b_ih}\right)y_i$$
$$-\left(-\frac{\delta a_{i-1}}{2}+\frac{ha_{i-1}}{2}e^{-b_ih}+\frac{\delta a_{i-1}}{2}e^{-b_ih}\right)y_i$$
$$+\left(-\frac{\varepsilon}{h}e^{-b_ih}-\frac{\varepsilon b_i}{h}e^{-b_ih}+\frac{\delta a_i}{h}e^{-b_ih}\right)y_{i+1}$$
$$=-\frac{h}{2}(f_{i-1}+e^{-b_ih}f_i) \quad (20)$$

Eq. (20) can be written as a three term recurrence relation of the form:

$$E_iy_{i-1}-F_iy_i+G_iy_{i+1}=H_i \;;\; i=\frac{1}{2}+1,\frac{1}{2}+2,,N-1$$

where

$$E_i = -\frac{\varepsilon}{h} - \frac{b_{i-1}h}{2}\left(1-\frac{\varepsilon}{h}\right) - \frac{ha_{i-1}}{2} - \frac{\delta a_{i-1}}{2}$$
$$F_i = -\frac{\varepsilon}{h}e^{-b_ih}+\frac{\varepsilon}{h}+\frac{b_{i-1}\varepsilon}{2}+\frac{b_ih}{2}\left(1-\frac{\varepsilon}{h}\right)e^{-b_ih}$$
$$-\frac{\delta a_{i-1}}{2}+\frac{ha_{i-1}}{2}e^{-b_ih}+\frac{\delta a_{i-1}}{2}e^{-b_ih}$$
$$G_i = -\frac{\varepsilon}{h}e^{-b_ih} - \frac{\varepsilon b_i}{h}e^{-b_ih} + \frac{\delta a_i}{h}e^{-b_ih}$$
$$H_i = -\frac{h}{2}\left[f_{i-1}+e^{-b_ih}f_i\right]$$

We now have from Eq.(11) in $\left[0,\frac{1}{2}\right]$ for $i = 1,2,,l/2-1$; and Eq.(20) in $\left[\frac{1}{2},1\right]$ for $i = l/2+1, l/2+2,,N-1$; a system of $(N-2)$ equations with $(N+1)$ unknowns. From the given boundary conditions (2), we get two more equations.

We need one more equation to solve for the unknowns $y_0, y_1, ...y_N$. To get this equation we considered the reduced problem of Eq.(1) by setting $\varepsilon = 0$ i.e.,

$$a(x)y(x-\delta)+b(x)y(x) = f(x) \tag{21}$$

which does not satisfy both the boundary conditions.

At $x = x_{l/2} = \frac{1}{2}$, Eq.(21) becomes

$$a(x_{l/2})y(x_{l/2}-\delta)+b(x_{l/2})y(x_{l/2}) = f(x_{l/2}) \tag{22}$$

By Taylor series expansion, we have

$$y(x_{l/2}-\delta) \approx y(x_{l/2}) - \delta y'(x_{l/2})$$
$$= y_{l/2} - \delta\left(\frac{y_{l/2+1}-y_{l/2-1}}{2h}\right) \tag{23}$$

Substituting (23) in Eq.(22) and by simplifying, we get

$$\frac{a_{l/2}\delta}{2h}y_{l/2-1} + \left(a_{l/2}+b_{l/2}\right)y_{l/2}$$
$$-\frac{a_{l/2}\delta}{2h}y_{l/2+1} = f_{l/2} \tag{24}$$

With this equation, we now have $(N+1)$ equations to solve for the $(N+1)$ unknowns $y_0, y_1, ...y_N$. Now we solve this tri diagonal system using discrete invariant imbedding algorithm.

3 Error analysis

Writing the tri-diagonal system (11) in matrix-vector form, we get

$$AY = C \tag{25}$$

in which $A = (m_{ij}), 1 \le i,j \le l/2-1$ is a tri diagonal matrix of order $N-1$, with

$$m_{ii+1} = -\varepsilon e^{b_{i+1}h} - e^{b_{i+1}h}\frac{b_{i+1}h^2}{2} + e^{b_{i+1}h}\frac{b_{i+1}h\varepsilon}{2}$$
$$-e^{b_{i+1}h}\frac{h^2a_{i+1}}{2} + e^{b_{i+1}h}\frac{\delta ha_{i+1}}{2},$$
$$m_{ii} = e^{b_{i+1}h}\varepsilon + \varepsilon - \frac{b_{i+1}h^2}{2} - \frac{h^2a_i}{2} + \frac{h\delta a_i}{2}$$
$$-e^{b_{i+1}h}\frac{\delta ha_{i+1}}{2},$$
$$m_{ii-1} = -\varepsilon - \frac{\varepsilon hb_{i+1}}{2} - \frac{\delta ha_i}{2}$$

and $C = (d_i)$ is a column vector with

$$d_i = -\frac{h^2}{2}\left(f_i + e^{b_{i+1}h}f_{i+1}\right), i = 1,2,...l/2-1$$

with local truncation error

$$T_i(h_i) = h^2\left[\frac{1}{2}\left((2\varepsilon+\delta)a_i+\varepsilon b_i\right)y''_i - \frac{\varepsilon}{2}y'''_i\right] + O(h^3) \tag{26}$$

Writing the tri-diagonal system (20) in matrix-vector form, we get

$$AY = C \tag{27}$$

in which $A = (m_{ij}), l/2+1 \le i,j \le N-1$ is a tri-diagonal matrix of order $N-1$, with

$$m_{ii+1} = \varepsilon e^{-b_ih} + \frac{\varepsilon hb_i}{2}e^{-b_ih} - \frac{\delta ha_i}{2}e^{-b_ih}$$
$$m_{ii} = -\varepsilon e^{-b_ih} - \varepsilon + \frac{\varepsilon hb_{i-1}}{2} + \frac{b_ih^2}{2}e^{-b_ih}$$
$$-\frac{\varepsilon hb_i}{2}e^{-b_ih} - \frac{\delta ha_{i-1}}{2} + \frac{h^2a_i}{2}e^{-b_ih} + \frac{h\delta a_i}{2}e^{-b_ih},$$
$$m_{ii-1} = \varepsilon + \frac{h^2b_{i-1}}{2} - \frac{h\varepsilon b_{i-1}}{2} + \frac{h^2a_{i-1}}{2} + \frac{h\delta a_{i-1}}{2}$$

and $C = (d_i)$ is a column vector with

$$d_i = \frac{h^2}{2}\left(f_{i-1} + e^{-b_ih}f_i\right),\ i = l/2+1,...,N-1$$

with local truncation error

$$T_i(h) = \frac{h^2}{2}\left[\left((-2\varepsilon-\delta)a_i - \varepsilon b_i\right)y''_i + \varepsilon y'''_i\right] + O(h^3) \tag{28}$$

and $Y = (y_0, y_1, y_2, ..., y_N)^t$.

Also we have

$$A\bar{\bar{Y}} - T(h) = C \tag{29}$$

where $\bar{\bar{Y}} = (\bar{\bar{y_0}}, \bar{\bar{y_1}}, \bar{\bar{y_2}}, ..., \bar{\bar{y_N}})^t$ denotes the actual solution and $T(h) = (T_1(h), T_2(h), ..., T_N(h))^t$ is the local truncation error.

From (27) and (29), we get

$$A\left(\bar{\bar{Y}} - Y\right) = T(h) \tag{30}$$

Thus the error equation is

$$AE = T(h) \tag{31}$$

where $E = \bar{\bar{Y}} - Y = (e_0, e_1, e_2, ..., e_N)^t$.

Clearly, we have

$$S_i = \sum_{j=1}^{N-1} m_{ij} = h\frac{1}{2} - b_{i+1}(1 - e^{b_{i+1}h}) + O(h^2)$$
$$= hB_i{}', \text{ for } i = 1(1)l/2 - 1$$

where $B'_i = \left[\frac{1}{2}\left(-b_{i+1}(1 - e^{b_{i+1}h})\right)\right]$;

$$S_i = h\left(a_n + b_n\right) + O(h^2) = hB''_i, \text{ for } i = l/2$$

where $B''_i = (a_n + b_n)$;

$$S_i = \sum_{j=1}^{N-1} m_{ij}$$
$$= h\left[\frac{1}{2}\left(2a_i e^{-b_i h} + 2a_{i-1} + b_i - b_{i-1}\right)\right] + O(h^2)$$
$$= hB'''_i \text{ for } i = l/2 + 1(1)N - 1$$

where $B'''_i = \left[\frac{1}{2}\left(2a_i e^{-b_i h} + 2a_{i-1} + b_i - b_{i-1}\right)\right]$.

We can choose h such that the matrix A is irreducible and monotone [19]. It follows that A^{-1} exists and its elements are non negative. Hence from Eq. (31), we get

$$E = A^{-1}T(h) \tag{32}$$

Also from the theory of matrices we have

$$\sum_{i=1}^{N-1} \bar{\bar{m}}_{k,i} S_i = 1, k = 1(1)N - 1 \tag{33}$$

where $\bar{\bar{m}}_{k,i}$ is (k, i) element of the matrix A^{-1}, therefore

$$\sum_{i=1}^{N-1} \bar{m}_{k,i} \leq \frac{1}{\min\limits_{1\leq i\leq N-1} S_i} = \frac{1}{hB_{i_0}} \leq \frac{1}{h\left|B_{i_0}\right|} \tag{34}$$

for some i_0 between 1 and $N - 1$ and

$$B_{i0} = \begin{bmatrix} B'_i, & i = 1(1)l/2 - 1 \\ B''_i, & i = l/2 \\ B'''_i, & i = l/2 + 1(1)N - 1 \end{bmatrix}$$

From (31), (32), (33) and (34), we get

$$e_j = \sum_{i=1}^{N-1} \bar{m}_{k.i}\, T_i(h), \quad j = 1(1)N - 1 \tag{35}$$

which implies

$$e_j \leq \frac{kh_i}{|B_{i0}|}, \quad j = 1(1)N - 1 \tag{36}$$

where k is a constant independent of h. Therefore from (36) we have,

$$\|E\| = O\left(h\right)$$

Hence our method gives first order convergence for uniform mesh.

4 Numerical Experiments

To validate the computational efficiency of the scheme, we have applied it to four cases of the problems of the form:

$$\varepsilon y''(x) + a(x)y(x-\delta) + b(x)y(x) = f(x), \quad \forall x \in (0, 1)$$

subject to the interval and boundary conditions

$$y(x) = \phi(x), \text{ on } -\delta \leq x \leq 0$$
$$y(x) = \gamma(x), \text{ on } 1 \leq x \leq 1 + \eta$$

The exact solution of such boundary value problems having constant coefficients (i.e. $a(x) = a$, $b(x) = b$, $c(x) = c$, $\phi(x) = \phi$ and $\gamma(x) = \gamma$ are constants) is given by

$$y(x) = \frac{\begin{bmatrix} ((1-a-b)\exp(m_2) - 1)\exp(m_1 x) \\ -((1-a-b)\exp(m_1) - 1)\exp(m_2 x) \end{bmatrix}}{(a+b)(\exp(m_1) - \exp(m_2))}$$

where

$$m_1 = \frac{\left(a\delta + \sqrt{a^2\delta^2 - 4\varepsilon^2(a+b)}\right)}{2\varepsilon^2},$$
$$m_2 = \frac{\left(a\delta - \sqrt{a^2\delta^2 - 4\varepsilon^2(a+b)}\right)}{2\varepsilon^2}.$$

The maximum absolute errors for the examples with variable coefficients are calculated using the double mesh principle [6], $E^N = \max\limits_{0\leq i\leq N} \left|y_i^N - y_{2i}^{2N}\right|$.

Example 1. *Consider singularly perturbed delay differential equation with layer behaviour:*

$$\varepsilon^2 y''(x) - 2y(x-\delta) - y(x) = 1$$

under the interval with boundary conditions y(0) = 1, $-\delta \leq x \leq 0$ *and* $y(1) = 0$.

The maximum absolute errors are presented in Table 1(a) and Table 1(b) for different values of ε and for different values of δ. Also, the computed solutions for $\varepsilon = 0.1, 0.01$ and for different values of δ are plotted in Figures 1 and 2.

Table 1a: Maximum absolute errors of Example 1 for different values of δ and $\varepsilon = 0.1$

δ/h	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$
0.0ε	0.0199215	0.0020621	0.0005333	0.0005366
0.3ε	0.0198840	0.0020683	0.0005328	0.0005362
0.6ε	0.0198326	0.0020739	0.0005324	0.0005358
0.9ε	0.0197671	0.0020789	0.0005320	0.0005355

Table 1b: Maximum absolute errors of Example 1 for different values of δ and $\varepsilon = 0.01$

δ/h	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$
0.0ε	0.1301282	0.0200896	0.0021110	0.0002117
0.3ε	0.1298531	0.0200858	0.0021110	0.0002117
0.6ε	0.1295761	0.0200818	0.0021109	0.0002117
0.9ε	0.1292971	0.0200777	0.0021109	0.0002117

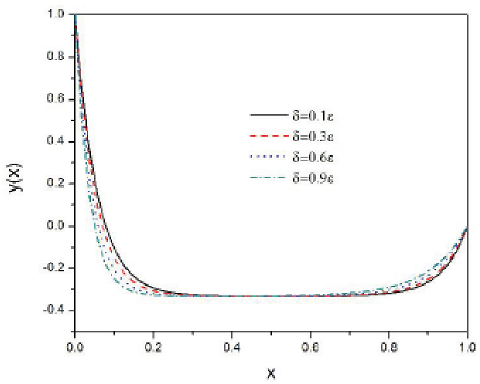

Figure 1: Numerical solution of Example 1 for $\varepsilon = 0.1$ with different values of δ

Example 2. *Consider singularly perturbed delay differential equation with layer behaviour:*

$$\varepsilon^2 y''(x) + 0.25y(x-\delta) - y(x) = 1$$

with $y(0) = 1, -\delta \le x \le 0, \ y(1) = 0.$

The maximum absolute errors are presented in Table 2 for different values of δ . Also, the computed solutions for ε=0.1, 0.01 and for different values of δ are plotted in Figures 3 and 4.

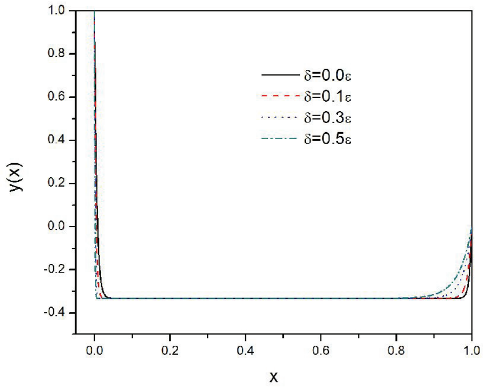

Figure 2: Numerical solution of Example 1 for $\varepsilon = 0.01$ with different values of δ

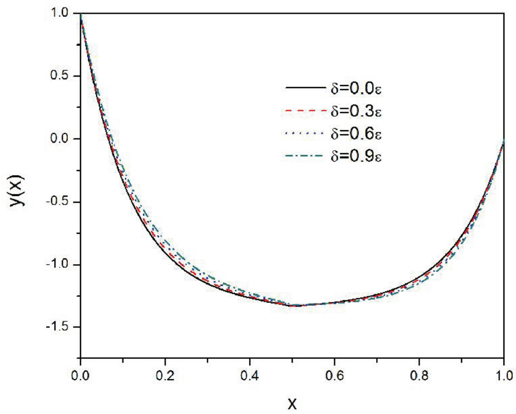

Figure 3: Numerical solution of Example 2 for $\varepsilon = 0.1$ with different values of δ

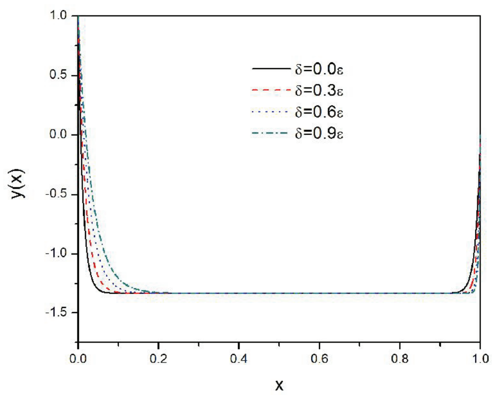

Figure 4: Numerical solution of Example 2 for $\varepsilon = 0.01$ with different values of δ

Table 2: Maximum absolute errors of Example 2 for $\varepsilon = 0.01$

δ/h	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$
0.0ε	0.1432048	0.0180800	0.0018807	0.000216
0.3ε	0.1432163	0.0180806	0.0018807	0.000216
0.6ε	0.1432277	0.0180811	0.0018807	0.000216
0.9ε	0.1432390	0.0180816	0.0018807	0.000216

Example 3. *Consider singularly perturbed delay differential equation with layer behaviour*

$$\varepsilon^2 y''(x) - y(x-\delta) + 0.5y(x) = 0$$

with $y(0) = 1, -\delta \le x \le 0, y(1) = 1$.

The maximum absolute errors are presented in Table 3 for different values of δ. Also, the computed solutions for ε=0.01 and for different values of δ are plotted in Figure 5.

Table 3: Maximum absolute errors of Example 3 for $\varepsilon = 0.01$

δ/h	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$
0.0ε	0.0513942	0.0063602	0.0006534	0.0000696
0.3ε	0.0514638	0.0063604	0.0006534	0.0000696
0.6ε	0.0515325	0.0063606	0.0006534	0.0000696
0.9ε	0.0516006	0.0063607	0.0006534	0.0000696

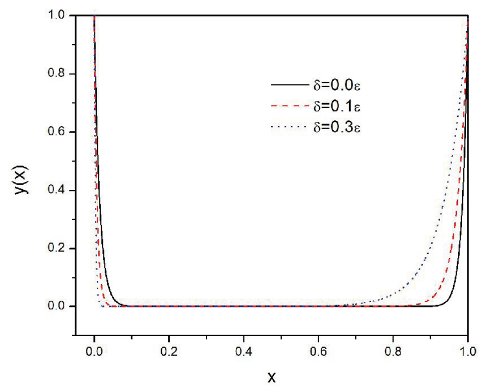

Figure 5: Numerical solution of Example 3 for $\varepsilon = 0.01$ with different values of δ

Example 4. *Consider singularly perturbed delay differential equation with variable coefficient and layer behaviour*

$$\varepsilon^2 y''(x) - e^x y(x-\delta) - y(x) = 0$$

with $y(0) = 1, -\delta \le x \le 0, y(1) = 1$.

The maximum absolute errors are presented in Tables 4 and 5 for different values of δ. Also, the computed solutions for ε=0.1 and for different values of δ are plotted in Figure 6.

Table 4: Maximum absolute errors of Example 4 for $\varepsilon = 0.01$

δ/h	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$
0.0ε	0.0079181	0.0008643	0.0000872	0.0000087
0.3ε	0.0079374	0.0008641	0.0000871	0.0000087
0.6ε	0.0079644	0.0008635	0.0000870	0.0000087
0.9ε	0.0079832	0.0008626	0.0000869	0.0000087

Table 5: Maximum absolute errors of Example 4 for $\varepsilon = 0.01$

δ/h	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$
0.0ε	0.043379	0.0080557	0.0008766	0.0000884
0.3ε	0.043535	0.0080570	0.0008766	0.0000884
0.6ε	0.043689	0.0080582	0.0008766	0.0000884
0.9ε	0.043842	0.0080593	0.0008766	0.0000884

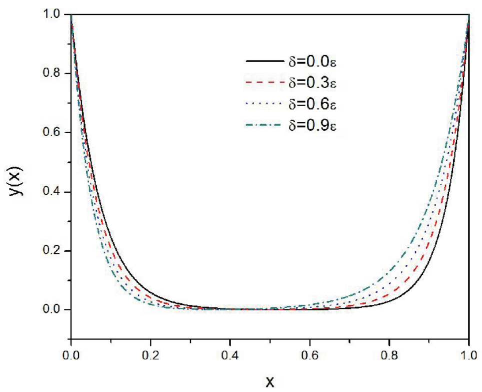

Figure 6: Numerical solution of Example 4 for $\varepsilon = 0.1$ with different values of δ

5 Discussions and Conclusion

We have proposed a numerical integration method to solve singularly perturbed delay differential equations with dual layer behaviour. In general, numerical solution of second order differential equation will be more difficult than numerical solution of first order differential equation. Hence, in this method, we have replaced the second order singularly perturbed delay differential equation to first order neutral type delay differential equation and employed the Trapezoidal rule of numerical integration. Then, linear interpolation is used to get three term recurrence relation which is solved easily by discrete invariant imbedding algorithm. The method is demonstrated by implementing it on model examples by taking various values for the delay parameter and perturbation parameter. The maximum absolute errors in the solution are presented in tables to support the method. From the graphs 1-7, it is observed that as the delay parameter increases, the thickness of the left end boundary layer decreases while that of the right end boundary layer increases.

References:

[1] I.–G. Amiraliyeva, F. Erdogan, G.–M. Amiraliyev, A uniform numerical method for dealing with a singularly perturbed delay initial value problem, *Applied Mathematics Letters.*, 23, 2010, pp. 1221C1225.

[2] O. Arino, M.–L. Hbid, E. Ait Dads, *Delay Differential Equations and Applications*, Springer, The Netherlands, 2006.

[3] A. Asachenkov, G. Marchuk, R. Mohler, S. Zuew, *Disease Dynamics*, Birkhauser, Boston, 1994.

[4] R. Bellman, K.–L. Cooke, *Differential Difference Equations*, Academic Press, New York, 1963.

[5] Brian J. McCartin, Exponentially fitting of the delayed recruitment/renewal equation, *Journal of Computational and Applied Mathematics*, 136, 2001, pp. 343C356.

[6] E.–P. Doolan, J.–J.–H. Miller, W.–H.–A. Schilders, *Uniform Numerical Methods for Problems with Initial and Boundary Layers*, Boole Press, Dublin, 1980.

[7] L.–E. Els'golts, S.–B. Norkin, *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*, Academic Press, Mathematics in Science and Engineering, 1973.

[8] M.–K. Kadalbajoo, D. Kumar, Numerical approximations for singularly perturbed differential-difference BVPs with layer and oscillatory behavior, *J. Numer. Math.*, 20, 2012, pp. 33C53.

[9] M.–K. Kadalbajoo, V.–P. Ramesh, Hybrid method for numerical solution of singularly perturbed delay differential equations, *Appl. Math. Comput.*, 187, 2007, pp. 797C814.

[10] M.–K. Kadalbajoo, K.–K. Sharma, Numerical Analysis of Boundary-Value of Problems for Singularly-Perturbed DifferentialCDifference Equations with Small Shifts Mixed Type, *Journal Optimization Theory and Applications*, 115, 2002, pp. 145-63.

[11] M.–K. Kadalbajoo, K.–K. Sharma, An Exponentially Fitted Finite Difference Scheme for solving Boundary-Value Problems for Singularly Perturbed Differential Difference Equations: Small Shifts of Mixed Type with Layer Behavior, *Journal of Computational Analysis and Applications*, 8, 2006, pp. 151-171.

[12] V. Kolmanovskii, A. Myshkis, *Applied Theory of Functional Differential Equations*, 85, Mathematics and its Applications, Kluwer Academic Publishers Group, Dordrecht, 1992.

[13] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Vol.191, *Mathematics in Science and Engineering*, Academic Press Inc., Boston, MA, 1993.

[14] C.–G. Lange, R.–M. Miura, Singular Perturbation Analysis of Boundary-Value Problems for Differential-Difference Equations. v. small shifts with layer behavior, *SIAM Journal on Applied Mathematics*, 54, 1994, pp. 249C272.

[15] C.–G. Lange, R.–M. Miura, Singular Perturbation Analysis of Boundary-Value Problems for Differential-Difference Equations II. Rapid Oscillations and Resonances, *SIAM Journal on Applied Mathematics*, 45, 1985, pp. 687-707.

[16] C.–G. Lange, R.–M. Miura, Singular Perturbation Analysis of Boundary-Value Problems for Differential-Difference Equations III. Turning Point Problems, *SIAM Journal on Applied Mathematics*, 45, 1985, pp. 708-734.

[17] A. Martin, S. Raun, Predetor-prey models with delay and prey harvesting, *J. Math. Bio.*, 43, 2001, pp. 247-267.

[18] R. Pratima, K.–K. Sharma, Numerical analysis of singularly perturbed delay differential turning point problem, *Applied Mathematics and Computation*, 218, 2011, pp. 3483C3498.

[19] R.–B. Stein, A theoretical analysis of neuronal variability, *Biophys. J.*, 5, 1965, pp. 173C194.

[20] GBSL. Soujanya, K. Phaneendra, Y.–N. Reddy, An Exponentially Fitted Non Symmetric Finite Difference Method for Singular Perturbation Problems, *WSEAS Transactions on Mathematics*, 12(7), 2013, pp. 767-776.

[21] K.E. Yunquan, Chunfang Miao, Global asymptotic stability for second-order neutral type Cohen-Grossberg neural networks with time delays, *WSEAS Transactions on Mathematics*, 13, 2014, pp. 973-982.

[22] Aleksey A. Kabanov, Composite Control for Nonlinear Singularly Perturbed Systems Based on Feedback Linearization Method, *WSEAS Transactions on Systems*, 14, 2015, pp. 215-221.