Potential Eventual Positivity of One New Tree Sign Pattern

BER-LIN YU, HONG-MEI BAO
Huaiyin Institute of Technology
Faculty of Mathematics and Physics
Huia'an, Jiangsu 223003
CHINA
berlinyu@163.com

JIE CUI
Huaiyin Institute of Technology
School of Economics and Management
Huia'an, Jiangsu 223003
CHINA
cuijie@hyit.edu.cn

Abstract: A sign pattern is a matrix whose entries belong to the set $\{+,-,0\}$. An n-by-n sign pattern $\mathcal{A}$ is said to allow an eventually positive matrix or be potentially eventually positive if there exist some real matrices A with the same sign pattern as $\mathcal{A}$ and a positive integer k_0 such that $A^k > 0$ for all $k \geq k_0$. Identifying the necessary and sufficient conditions for an n-by-n sign pattern to be potentially eventually positive, and classifying the n-by-n sign patterns that allow an eventually positive matrix were posed as two open problems by Berman, Catral, Dealba, et al. In this article, a new tree sign pattern $\mathcal{A}$ obtained from one tridiagonal sign pattern by adding one pendent edge are investigated. Some necessary conditions for the sign pattern $\mathcal{A}$ to allow an eventually positive matrix are established first. Then all the minimal tree sign patterns that allow an eventually positive matrix are identified. Finally, all the tree sign patterns that allow an eventually positive matrix are classified.

Key–Words: Sign pattern, Potential eventual positivity, Checkerboard block sign pattern

1 Introduction

A sign pattern is a matrix $\mathcal{A} = [\alpha_{ij}]$ with entries in the set $\{+,-,0\}$. An n-by-n real matrix A with the same sign pattern as $\mathcal{A}$ is called a realization of $\mathcal{A}$. The set of all realizations of sign pattern $\mathcal{A}$ is called the qualitative class of $\mathcal{A}$ and is denoted by $Q(\mathcal{A})$. A subpattern of $\mathcal{A} = [\alpha_{ij}]$ is an n-by-n sign pattern $\mathcal{B} = [\beta_{ij}]$ such that $\beta_{ij} = 0$ whenever $\alpha_{ij} = 0$. If $\mathcal{B} \neq \mathcal{A}$, then $\mathcal{B}$ is a proper subpattern of $\mathcal{A}$. If $\mathcal{B}$ is a subpattern of $\mathcal{A}$, then $\mathcal{A}$ is said to be a superpattern of $\mathcal{B}$. A permutation pattern is a sign pattern matrix with exactly one entry in each row and column equal to $+$, and the remaining entries equal to 0. A product of the form $\mathcal{S}^T\mathcal{A}\mathcal{S}$, where $\mathcal{S}$ is a permutation pattern and $\mathcal{A}$ is a sign pattern matrix of the same order as $\mathcal{S}$, is called a permutation similarity. A pattern $\mathcal{A}$ is reducible if there is a permutation matrix $\mathcal{P}$ such that

$$\mathcal{P}^T\mathcal{A}\mathcal{P} = \begin{bmatrix} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix},$$

where $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are square matrices of order at least one. A pattern is irreducible if it is not reducible; see [1] and [2] for more details.

A sign pattern matrix $\mathcal{A}$ is said to require a certain property P referring to real matrices if every real matrix $A \in Q(\mathcal{A})$ has the property P and allow P or be potentially P if there is some $A \in Q(\mathcal{A})$ that has property P.

Recall that an n-by-n real matrix A is said to be eventually positive if there exists a nonnegative integer k_0 such that $A^k > 0$ for all $k \geq k_0$; see, e.g., [3]. An n-by-n sign pattern $\mathcal{A}$ is said to allow an eventually positive matrix or be potentially eventually positive (PEP, for short), if there exists some $A \in Q(\mathcal{A})$ such that A is eventually positive; see, e.g., [4] and the references therein. An n-by-n sign pattern $\mathcal{A}$ is said to be a minimal potentially eventually positive sign pattern (MPEP sign pattern) if $\mathcal{A}$ is PEP and no proper subpattern of $\mathcal{A}$ is PEP; see, e.g. [5] and [6]. Sign patterns that allow an eventually positive matrix have been studied first in [4], where a sufficient condition and some necessary conditions for a sign pattern to be potentially eventually positive have been established. However, the identification of necessary and sufficient conditions for an n-by-n sign pattern ($n \geq 4$) to be potentially eventually positive remains open. Also open is the classification of sign patterns that are potentially eventually positive.

Recall that an n-by-n real matrix A is said to be power-positive if there exists a nonnegative integer k such that $A^k > 0$. An n-by-n sign pattern $\mathcal{A}$ is said to allow a power-positive matrix or be potentially power-positive (PPP), if there exists some $A \in Q(\mathcal{A})$ such that A is power-positive; see, e.g., [7]. A relation between potentially eventually positive sign patterns and potentially power-positive sign patterns has been established in [7]. An n-by-n sign pattern $\mathcal{A}$ is said to be a minimal potentially power-positive (MPPP) sign pattern, if $\mathcal{A}$ is PPP and no proper subpattern of $\mathcal{A}$ is PPP; see, e.g., [6]. A relation between minimal potentially eventually positive sign patterns and minimal potentially power-positive sign patterns has been established in [6]. At present, there are a few literatures on the potential eventual positivity of some specific sign pattern matrices; see e.g., [5], [6], [7], [8], [9] and [10]. Especially, the potentially eventually positive double star sign patterns have been identified and classified in [5]. More recently, the minimal potentially eventually positive tridiagonal sign patterns have been identified and all potentially eventually positive tridiagonal sign patterns have been classified in [8]. In [9], the eventual positivity of one specific tree sign pattern obtained from the double star sign pattern $S_{3,2}$ by adding a pendent edge has been investigated.

In this article, we focus on the eventual positivity of one new specific tree sign pattern obtained from a tridiagonal sign pattern by adding an pendent edge. Our work is organized as follows. In Section 2, some preliminary results are established. In Section 3, the minimal potentially eventually positive sign patterns are identified and consequently, the tree sign patterns that allow an eventually positive matrix are classified. Some remarks and future work are concluded in Section 4.

2 Preliminary Results for Tree Sign Pattern $\mathcal{A}$ to be Potentially Eventually Positive

We begin this section with introducing some necessary graph theoretical concepts which can be seen from [2], [11], [12], [13] and the references therein.

A square sign pattern $\mathcal{A} = [\alpha_{ij}]$ is combinatorially symmetric if $\alpha_{ij} \neq 0$ whenever $\alpha_{ji} \neq 0$. Let $G(\mathcal{A})$ be the graph of order n with vertices $1, 2, \cdots, n$ and an edge $\{i, j\}$ joining vertices i and j if and only if $i \neq j$ and $\alpha_{ij} \neq 0$. We call $G(\mathcal{A})$ the graph of the pattern $\mathcal{A}$. A combinatorially symmetric sign pattern matrix $\mathcal{A}$ is called a tree sign pattern if $G(\mathcal{A})$ is a tree. Similarly, path (or tridiagonal) and double star sign patterns can be defined.

A sign pattern $\mathcal{A} = [\alpha_{ij}]$ has signed digraph $\Gamma(\mathcal{A})$ with vertex set $\{1, 2, \cdots, n\}$ and a positive (respectively, negative) arc from i to j if and only if α_{ij} is positive (respectively, negative). A (directed) simple cycle of length k is a sequence of k arcs $(i_1, i_2), (i_2, i_3), \cdots, (i_k, i_1)$ such that the vertices $i_1, \cdots, i_k$ are distinct. Recall that a digraph $D = (V, E)$ is primitive if it is strongly connected and the greatest common divisor of the lengths of its cycles is 1. It is well known that a digraph D is primitive if and only if there exists a natural number k such that for all $V_i \in V$, $V_j \in V$, there is a walk of length k from V_i to V_j. A nonnegative sign pattern $\mathcal{A}$ is primitive if its signed digraph $\Gamma(\mathcal{A})$ is primitive; see, e.g. [4] for more details.

For a sign pattern $\mathcal{A} = [\alpha_{ij}]$, we define the positive part of $\mathcal{A}$ to be $\mathcal{A}^+ = [\alpha_{ij}^+]$, where $\alpha_{ij}^+ = +$ for $\alpha_{ij} = +$, otherwise $\alpha_{ij}^+ = 0$. The negative part of $\mathcal{A}$ can be defined similarly. In [4], it has been shown that if sign pattern $\mathcal{A}^+$ is primitive, then $\mathcal{A}$ is PEP. Here, we cite some necessary conditions for an n-by-n sign pattern to be potentially eventually positive in [4] as Lemmas 1 to 2 in order to state our work clearly.

Lemma 1 *If an $n \times n$ sign pattern $\mathcal{A}$ is PEP, then*

(1) Every superpattern of $\mathcal{A}$ is PEP.

(2) Every row and column of $\mathcal{A}$ has at least one $+$ and the minimal number of $+$ entries in $\mathcal{A}$ is $n+1$.

(3) If $\hat{\mathcal{A}}$ is the sign pattern obtained from sign pattern $\mathcal{A}$ by changing all 0 and $-$ diagonal entries to $+$, then $\hat{\mathcal{A}}$ is PEP.

(4) There is an eventually positive matrix $A \in Q(\mathcal{A})$ such that

(a) $\rho(A) = 1$.

(b) $A\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the $n \times 1$ all ones vector.

If $n \geq 2$, the sum of all the off-diagonal entries of A is positive.

We denote a sign pattern consisting entirely of positive (respectively, negative) entries by $[+]$ (respectively, $[-]$). Let $[+]_{i\times i}$ be a square block sign pattern of order i consisting entirely of positive entries. For block sign patterns, we have the following lemma.

Lemma 2 *Let $\mathcal{A}$ be the checkerboard block sign pattern*

$$\begin{bmatrix} [+] & [-] & [+] & \cdots \\ [-] & [+] & [-] & \cdots \\ [+] & [-] & [+] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with square diagonal blocks. Then $-\mathcal{A}$ is not PEP, and if $\mathcal{A}$ has a negative entry, then $\mathcal{A}$ is not PEP.

Recall that a vertex with degree 1 in a graph is called a leaf vertex or end vertex, and the edge incident with that vertex is called a pendant edge; see [2] for example. Now, we turn our attention to the specific sign pattern $\mathcal{A}$ which are obtained from the tridiagonal sign pattern P_5 by adding a pendent edge as shown in the Figure 1.

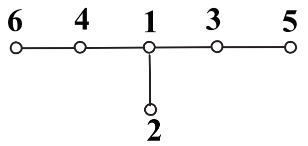

Figure 1: The graph $G(\mathcal{A})$ of sign pattern $\mathcal{A}$.

Since sign pattern $\mathcal{A}$ is potentially eventually positive if and only if $\mathcal{A}^T$ or $\mathcal{P}^T\mathcal{A}\mathcal{P}$ is potentially eventually positive, for any permutation pattern $\mathcal{P}$. Thus, without loss of generality, let the sign pattern

$$\mathcal{A} = \begin{bmatrix} ? & * & * & * & & \\ & ? & & & & \\ & & ? & & * & \\ & & & ? & & * \\ & & * & & ? & \\ & & & * & & ? \end{bmatrix},$$

where ? denotes an entry from $\{+,-,0\}$, $*$ denotes a nonzero entry, and the entries not specified in the sign pattern are all zeros. Note that sign pattern $\mathcal{A}$ is a combinatorially symmetric sign pattern and its graph $G(\mathcal{A})$ is a tree. It is clear that $\mathcal{A}$ is not a doublet star sign pattern which has been investigated in [5] and is not a tridiagonal sign pattern which has been studied in [8]. Furthermore, sign pattern $\mathcal{A}$ is different from the tree sign pattern investigated in [9].

Next, we address on the general properties of potentially eventually positive sign pattern $\mathcal{A}$. Following [4] and [5], we use the notation $\ominus$ to denote one of $0, -$, $\oplus$ to denote one of $0, +$. The following two results are necessary for tree sign pattern $\mathcal{A}$ to be potentially eventually positive.

Proposition 3 *If sign pattern $\mathcal{A}$ is potentially eventually positive, then $\mathcal{A}$ is symmetric.*

Proof: Since sign pattern $\mathcal{A}$ is potentially eventually positive, let the real matrix $A = [a_{ij}] \in Q(\mathcal{A})$ be eventually positive. By Lemma 1, let $a_{22} = 1 - a_{21}$, $a_{33} = 1 - a_{31} - a_{35}$, $a_{44} = 1 - a_{41} - a_{46}$, $a_{55} = 1 - a_{53}$ and $a_{66} = 1 - a_{64}$. To complete the proof, it suffices to show that $a_{21}a_{12} > 0$, $a_{31}a_{13} > 0$, $a_{41}a_{14} > 0$, $a_{35}a_{53} > 0$ and $a_{46}a_{64} > 0$. Suppose the positive left eigenvector of A is $w = (w_1, w_2, \ldots, w_6)^T$. Then by $w^T A = w^T$, we have the following equalities:

$$w_4 a_{46} + w_6(1 - a_{64}) = w_6, \tag{1}$$

$$w_3 a_{35} + w_5(1 - a_{53}) = w_5, \tag{2}$$

$$w_1 a_{14} + w_4(1 - a_{41} - a_{46}) + w_6 a_{64} = w_4, \tag{3}$$

$$w_1 a_{13} + w_3(1 - a_{31} - a_{35}) + w_5 a_{53} = w_3, \tag{4}$$

and

$$w_1 a_{12} + w_2(1 - a_{21}) = w_2. \tag{5}$$

By Equalities 1, 2 and 5, we have $a_{21}a_{12} > 0$, $a_{35}a_{53} > 0$ and $a_{46}a_{64} > 0$. By Equalities 1 and 3, we have $a_{41}a_{14} > 0$. By equalities 2 and 4, we have $a_{13}a_{31} > 0$. It follows that $\mathcal{A}$ is symmetric. □

Theorem 4 *If sign pattern $\mathcal{A}$ is potentially eventually positive, then all nonzero off-diagonal entries of $\mathcal{A}$ are positive.*

Proof: By Proposition 3, the potentially eventually positive sign pattern $\mathcal{A}$ is symmetric. To complete the proof, it suffices to show that $\alpha_{12} = +$, $\alpha_{13} = +$, $\alpha_{14} = +$, $\alpha_{35} = +$ and $\alpha_{46} = +$. By a way of contradiction, assume that there are some off-diagonal entries that are $-$ in the set $\{\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{35}, \alpha_{46}\}$. Let k be the number of off-diagonal entries $-$ in the set $\{\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{35}, \alpha_{46}\}$. For the potentially eventually positive sign patter $\mathcal{A}$ has at least 7 positive entries, $1 \leq k \leq 4$.

Case 1. $k = 1$.

Suppose that $\alpha_{12} = -$. Then the potentially eventually positive sign pattern

$$\mathcal{A} = \begin{bmatrix} ? & - & + & + & & \\ - & ? & & & & \\ + & & ? & & + & \\ + & & & ? & & + \\ & & + & & ? & \\ & & & + & & ? \end{bmatrix}.$$

By changing all diagonal entries to be $+$, we have a potentially eventually positive sign pattern

$$\hat{\mathcal{A}} = \begin{bmatrix} + & - & + & + & & \\ - & + & & & & \\ + & & + & & + & \\ + & & & + & & + \\ & & + & & + & \\ & & & + & & + \end{bmatrix}$$

with a proper checkerboard superpattern

$$\begin{bmatrix} [+]_{1\times1} & [-] & [+] \\ [-] & [+]_{1\times1} & [-] \\ [+] & [-] & [+]_{4\times4} \end{bmatrix}.$$

It follows from Lemma 2 that sign pattern $\hat{\mathcal{A}}$ (and hence $\mathcal{A}$) is not potentially eventually positive. It is a contradiction.

In general, the other Subcases $\alpha_{13} = -$, $\alpha_{14} = -$, $\alpha_{35} = -$ and $\alpha_{46} = -$ can be discussed similarly. We list the corresponding sign patterns and their checkerboard block sign patterns as following:

$$\begin{bmatrix} ? & + & - & + & & \\ + & ? & & & & \\ - & & ? & & + & \\ + & & & ? & & + \\ & & + & & ? & \\ & & & + & & ? \end{bmatrix},$$

and its checkerboard block sign pattern

$$\begin{bmatrix} [+]_{2\times2} & [-] & [+] & [-] & [+] \\ [-] & [+]_{1\times1} & [-] & [+] & [-] \\ [+] & [-] & [+]_{1\times1} & [-] & [+] \\ [-] & [+] & [-] & [+]_{1\times1} & [-] \\ [+] & [-] & [+] & [-] & [+]_{1\times1} \end{bmatrix};$$

$$\begin{bmatrix} ? & + & + & - & & \\ + & ? & & & & \\ + & & ? & & + & \\ - & & & ? & & + \\ & & + & & ? & \\ & & & + & & ? \end{bmatrix},$$

and its checkerboard block sign pattern

$$\begin{bmatrix} [+]_{3\times3} & [-] & [+] & [-] \\ [-] & [+]_{1\times1} & [-] & [+] \\ [+] & [-] & [+]_{1\times1} & [-] \\ [-] & [+] & [-] & [+]_{1\times1} \end{bmatrix};$$

$$\begin{bmatrix} ? & + & + & + & & \\ + & ? & & & & \\ + & & ? & & - & \\ + & & & ? & & + \\ & & - & & ? & \\ & & & + & & ? \end{bmatrix},$$

and its checkerboard block sign pattern

$$\begin{bmatrix} [+]_{4\times4} & [-] & [+] \\ [-] & [+]_{1\times1} & [-] \\ [+] & [-] & [+]_{1\times1} \end{bmatrix};$$

$$\begin{bmatrix} ? & + & + & + & & \\ + & ? & & & & \\ + & & ? & & + & \\ + & & & ? & & - \\ & & + & & ? & \\ & & & - & & ? \end{bmatrix},$$

and its checkerboard block sign pattern

$$\begin{bmatrix} [+]_{5\times5} & [-] \\ [-] & [+]_{1\times1} \end{bmatrix}.$$

Case 2. $k = 2$.

Suppose that $\alpha_{12} = \alpha_{13} = -$. Then the potentially eventually positive sign pattern

$$\mathcal{A} = \begin{bmatrix} ? & - & - & + & & \\ - & ? & & & & \\ - & & ? & & + & \\ + & & & ? & & + \\ & & + & & ? & \\ & & & + & & ? \end{bmatrix}.$$

By changing all diagonal entries to be $+$, we have a potentially eventually positive sign pattern

$$\hat{\mathcal{A}} = \begin{bmatrix} + & - & - & + & & \\ - & + & & & & \\ - & & + & & + & \\ + & & & + & & + \\ & & + & & + & \\ & & & + & & + \end{bmatrix}$$

with a proper checkerboard superpattern

$$\begin{bmatrix} [+]_{1\times1} & [-] & [+] & [-] & [+] \\ [-] & [+]_{2\times2} & [-] & [+] & [-] \\ [+] & [-] & [+]_{1\times1} & [-] & [+] \\ [-] & [+] & [-] & [+]_{1\times1} & [-] \\ [+] & [-] & [+] & [-] & [+]_{1\times1} \end{bmatrix}.$$

It follows from Lemma 2 that sign pattern $\hat{\mathcal{A}}$ (and hence $\mathcal{A}$) is not potentially eventually positive; this is a contradiction.

In general, the other 9 Subcases $\alpha_{12} = \alpha_{14} = -$, $\alpha_{12} = \alpha_{35} = -$, $\alpha_{12} = \alpha_{46} = -$, $\alpha_{13} = \alpha_{14} = -$, $\alpha_{13} = \alpha_{35} = -$, $\alpha_{13} = \alpha_{46} = -$, $\alpha_{14} = \alpha_{35} = -$, $\alpha_{14} = \alpha_{46} = -$ and $\alpha_{35} = \alpha_{46} = -$ can be discussed in a similar way. We list the corresponding sign patterns and their respective checkerboard block sign patterns as following:

$$\begin{bmatrix} ? & - & + & - & & \\ - & ? & & & & \\ + & & ? & & + & \\ - & & & ? & & + \\ & & + & & ? & \\ & & & + & & ? \end{bmatrix},$$

and its checkerboard block sign pattern

$$\begin{bmatrix} [+] & [-] & [+] & [-] & [+] & [-] \\ [-] & [+] & [-] & [+] & [-] & [+] \\ [+] & [-] & [+] & [-] & [+] & [-] \\ [-] & [+] & [-] & [+] & [-] & [+] \\ [+] & [-] & [+] & [-] & [+] & [-] \\ [-] & [+] & [-] & [+] & [-] & [+] \end{bmatrix};$$

$$\left[\begin{array}{cccccc} ? & - & + & + & & \\ - & ? & & & & \\ + & & ? & & - & \\ + & & & ? & & + \\ & & - & & ? & \\ & & & + & & ? \end{array}\right],$$

and its checkerboard block sign pattern

$$\left[\begin{array}{ccccc} [+]_{1\times 1} & [-] & [+] & [-] & [+] \\ [-] & [+]_{1\times 1} & [-] & [+] & [-] \\ [+] & [-] & [+]_{2\times 2} & [-] & [+] \\ [-] & [+] & [-] & [+]_{1\times 1} & [-] \\ [+] & [-] & [+] & [-] & [+]_{1\times 1} \end{array}\right];$$

$$\left[\begin{array}{cccccc} ? & - & + & + & & \\ - & ? & & & & \\ + & & ? & & + & \\ + & & & ? & & - \\ & & + & & ? & \\ & & & - & & ? \end{array}\right],$$

and its checkerboard block sign pattern

$$\left[\begin{array}{cccc} [+]_{1\times 1} & [-] & [+] & [-] \\ [-] & [+]_{1\times 1} & [-] & [+] \\ [+] & [-] & [+]_{3\times 3} & [-] \\ [-] & [+] & [-] & [+]_{1\times 1} \end{array}\right];$$

$$\left[\begin{array}{cccccc} ? & + & - & - & & \\ + & ? & & & & \\ - & & ? & & + & \\ - & & & ? & & + \\ & & + & & ? & \\ & & & + & & ? \end{array}\right],$$

and its checkerboard block sign pattern

$$\left[\begin{array}{ccc} [+]_{2\times 2} & [-] & [+] \\ [-] & [+]_{2\times 2} & [-] \\ [+] & [-] & [+]_{2\times 2} \end{array}\right];$$

$$\left[\begin{array}{cccccc} ? & + & - & + & & \\ + & ? & & & & \\ - & & ? & & - & \\ + & & & ? & & + \\ & & - & & ? & \\ & & & + & & ? \end{array}\right],$$

and its checkerboard block sign pattern

$$\left[\begin{array}{ccc} [+]_{2\times 2} & [-] & [+] \\ [-] & [+]_{1\times 1} & [-] \\ [+] & [-] & [+]_{3\times 3} \end{array}\right];$$

$$\left[\begin{array}{cccccc} ? & + & - & + & & \\ + & ? & & & & \\ - & & ? & & + & \\ + & & & ? & & - \\ & & + & & ? & \\ & & & - & & ? \end{array}\right],$$

and its checkerboard block sign pattern

$$\left[\begin{array}{cccc} [+]_{2\times 2} & [-] & [+] & [-] \\ [-] & [+]_{1\times 1} & [-] & [+] \\ [+] & [-] & [+]_{1\times 1} & [-] \\ [-] & [+] & [-] & [+]_{2\times 2} \end{array}\right];$$

$$\left[\begin{array}{cccccc} ? & + & + & - & & \\ + & ? & & & & \\ + & & ? & & - & \\ - & & & ? & & + \\ & & - & & ? & \\ & & & + & & ? \end{array}\right],$$

and its checkerboard block sign pattern

$$\left[\begin{array}{cc} [+]_{3\times 3} & [-] \\ [-] & [+]_{3\times 3} \end{array}\right];$$

$$\left[\begin{array}{cccccc} ? & + & + & - & & \\ + & ? & & & & \\ + & & ? & & + & \\ - & & & ? & & - \\ & & + & & ? & \\ & & & - & & ? \end{array}\right],$$

and its checkerboard block sign pattern

$$\left[\begin{array}{ccc} [+]_{3\times 3} & [-] & [+] \\ [-] & [+]_{1\times 1} & [-] \\ [+] & [-] & [+]_{2\times 2} \end{array}\right];$$

$$\left[\begin{array}{cccccc} ? & + & + & + & & \\ + & ? & & & & \\ + & & ? & & - & \\ + & & & ? & & - \\ & & - & & ? & \\ & & & - & & ? \end{array}\right],$$

and its checkerboard block sign pattern

$$\left[\begin{array}{cc} [+]_{4\times 4} & [-] \\ [-] & [+]_{2\times 2} \end{array}\right].$$

Case 3. $k = 3$.

Assume that $\alpha_{12} = \alpha_{13} = \alpha_{14} = -$. Then the potentially eventually positive sign pattern

$$\mathcal{A} = \left[\begin{array}{cccccc} ? & - & - & - & & \\ - & ? & & & & \\ - & & ? & & + & \\ - & & & ? & & + \\ & & + & & ? & \\ & & & + & & ? \end{array}\right].$$

By changing all diagonal entries to be $+$, we have a potentially eventually positive sign pattern

$$\hat{\mathcal{A}} = \left[\begin{array}{cccccc} + & - & - & - & & \\ - & + & & & & \\ - & & + & & + & \\ - & & & + & & + \\ & & + & & + & \\ & & & + & & + \end{array}\right]$$

with a proper checkerboard superpattern

$$\left[\begin{array}{cc} [+]_{1\times 1} & [-] \\ [-] & [+]_{5\times 5} \end{array}\right].$$

It follows from Lemma 2 that sign pattern $\hat{\mathcal{A}}$ (and hence $\mathcal{A}$) is not potentially eventually positive; a contradiction.

In general, the other 9 Subcases $\alpha_{12} = \alpha_{13} = \alpha_{35} = -$, $\alpha_{12} = \alpha_{13} = \alpha_{46} = -$, $\alpha_{12} = \alpha_{14} = \alpha_{35} = -$, $\alpha_{12} = \alpha_{14} = \alpha_{46} = -$, $\alpha_{12} = \alpha_{35} = \alpha_{46} = -$, $\alpha_{13} = \alpha_{14} = \alpha_{35} = -$, $\alpha_{13} = \alpha_{14} = \alpha_{46} = -$, $\alpha_{13} = \alpha_{35} = \alpha_{46} = -$ and $\alpha_{14} = \alpha_{35} = \alpha_{46} = -$ can be discussed in a similar way. We list the corresponding sign patterns and their respective checkerboard block sign patterns as following:

$$\left[\begin{array}{cccccc} ? & - & - & + & & \\ - & ? & & & & \\ - & & ? & & - & \\ + & & & ? & & + \\ & & - & & ? & \\ & & & + & & ? \end{array}\right],$$

and its checkerboard block sign pattern

$$\left[\begin{array}{ccc} [+]_{1\times 1} & [-] & [+] \\ [-] & [+]_{2\times 2} & [-] \\ [+] & [-] & [+]_{3\times 3} \end{array}\right];$$

$$\left[\begin{array}{cccccc} ? & - & - & + & & \\ - & ? & & & & \\ - & & ? & & + & \\ + & & & ? & & - \\ & & + & & ? & \\ & & & - & & ? \end{array}\right],$$

and its checkerboard block sign pattern

$$\left[\begin{array}{cccc} [+]_{1\times 1} & [-] & [+] & [-] \\ [-] & [+]_{2\times 2} & [-] & [+] \\ [+] & [-] & [+]_{1\times 1} & [-] \\ [-] & [+] & [-] & [+]_{2\times 2} \end{array}\right];$$

$$\left[\begin{array}{cccccc} ? & - & + & - & & \\ - & ? & & & & \\ + & & ? & & - & \\ - & & & ? & & + \\ & & - & & ? & \\ & & & + & & ? \end{array}\right],$$

and its checkerboard block sign pattern

$$\left[\begin{array}{cccc} [+]_{1\times 1} & [-] & [+] & [-] \\ [-] & [+]_{1\times 1} & [-] & [+] \\ [+] & [-] & [+]_{1\times 1} & [-] \\ [-] & [+] & [-] & [+]_{3\times 3} \end{array}\right];$$

$$\left[\begin{array}{cccccc} ? & - & + & - & & \\ - & ? & & & & \\ + & & ? & & + & \\ - & & & ? & & - \\ & & + & & ? & \\ & & & - & & ? \end{array}\right],$$

and its checkerboard block sign pattern

$$\left[\begin{array}{ccccc} [+]_{1\times 1} & [-] & [+] & [-] & [+] \\ [-] & [+]_{1\times 1} & [-] & [+] & [-] \\ [+] & [-] & [+]_{1\times 1} & [-] & [+] \\ [-] & [+] & [-] & [+]_{1\times 1} & [-] \\ [+] & [-] & [+]_{1\times 1} & [-] & [+]_{2\times 2} \end{array}\right];$$

$$\left[\begin{array}{cccccc} ? & - & + & + & & \\ - & ? & & & & \\ + & & ? & & - & \\ + & & & ? & & - \\ & & - & & ? & \\ & & & - & & ? \end{array}\right],$$

and its checkerboard block sign pattern

$$\left[\begin{array}{cccc} [+]_{1\times 1} & [-] & [+] & [-] \\ [-] & [+]_{1\times 1} & [-] & [+] \\ [+] & [-] & [+]_{2\times 2} & [-] \\ [-] & [+] & [-] & [+]_{2\times 2} \end{array}\right];$$

$$\left[\begin{array}{cccccc} ? & + & - & - & & \\ + & ? & & & & \\ - & & ? & & - & \\ - & & & ? & & + \\ & & - & & ? & \\ & & & + & & ? \end{array}\right],$$

and its checkerboard block sign pattern

$$\begin{bmatrix} [+]_{2\times2} & [-] & [+] & [-] \\ [-] & [+]_{2\times2} & [-] & [+] \\ [+] & [-] & [+]_{1\times1} & [-] \\ [-] & [+] & [-] & [+]_{1\times1} \end{bmatrix};$$

$$\begin{bmatrix} ? & + & - & - & & \\ + & ? & & & & \\ - & & ? & & + & \\ - & & & ? & & - \\ & & + & & ? & \\ & & & - & & ? \end{bmatrix},$$

and its checkerboard block sign pattern

$$\begin{bmatrix} [+]_{2\times2} & [-] & [+]] \\ [-] & [+]_{3\times3} & [-] \\ [+] & [-] & [+]_{1\times1} \end{bmatrix};$$

$$\begin{bmatrix} ? & + & - & + & & \\ + & ? & & & & \\ - & & ? & & - & \\ + & & & ? & & - \\ & & - & & ? & \\ & & & - & & ? \end{bmatrix},$$

and its checkerboard block sign pattern

$$\begin{bmatrix} [+]_{2\times2} & [-] & [+] & [-] \\ [-] & [+]_{1\times1} & [-] & [+] \\ [+] & [-] & [+]_{2\times2} & [-] \\ [-] & [+] & [-] & [+]_{1\times1} \end{bmatrix};$$

$$\begin{bmatrix} ? & + & + & - & & \\ + & ? & & & & \\ + & & ? & & - & \\ - & & & ? & & - \\ & & - & & ? & \\ & & & - & & ? \end{bmatrix},$$

and its checkerboard block sign pattern

$$\begin{bmatrix} [+]_{3\times3} & [-] & [+] \\ [-] & [+]_{2\times2} & [-] \\ [+] & [-] & [+]_{1\times1} \end{bmatrix}.$$

Case 4. $k = 4$.

If $\alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{35} = -$. Then the potentially eventually positive sign pattern

$$\mathcal{A} = \begin{bmatrix} ? & - & - & - & & \\ - & ? & & & & \\ - & & ? & & - & \\ - & & & ? & & + \\ & & - & & ? & \\ & & & + & & ? \end{bmatrix}.$$

By changing all diagonal entries to be $+$, we have a potentially eventually positive sign pattern

$$\hat{\mathcal{A}} = \begin{bmatrix} + & - & - & - & & \\ - & + & & & & \\ - & & + & & - & \\ - & & & + & & + \\ & & - & & + & \\ & & & + & & + \end{bmatrix}$$

with a proper checkerboard superpattern

$$\begin{bmatrix} [+]_{1\times1} & [-] & [+] & [-] \\ [-] & [+]_{3\times3} & [-] & [+] \\ [+] & [-] & [+]_{1\times1} & [-] \\ [-] & [+] & [-] & [+]_{1\times1} \end{bmatrix}.$$

It follows from Lemma 2 that sign pattern $\hat{\mathcal{A}}$ (and hence $\mathcal{A}$) is not potentially eventually positive; a contradiction.

In general, the other 4 Subcases $\alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{46} = -$, $\alpha_{12} = \alpha_{13} = \alpha_{35} = \alpha_{46} = -$ and $\alpha_{13} = \alpha_{14} = \alpha_{35} = \alpha_{46} = -$ can be discussed similarly. We list the corresponding sign patterns and their respective checkerboard block sign patterns as following:

$$\begin{bmatrix} ? & - & - & - & & \\ - & ? & & & & \\ - & & ? & & + & \\ - & & & ? & & - \\ & & + & & ? & \\ & & & - & & ? \end{bmatrix},$$

and its checkerboard block sign pattern

$$\begin{bmatrix} [+]_{1\times1} & [-] & [+] \\ [-] & [+]_{4\times4} & [-] \\ [+] & [-] & [+]_{1\times1} \end{bmatrix};$$

$$\begin{bmatrix} ? & - & - & + & & \\ - & ? & & & & \\ - & & ? & & - & \\ + & & & ? & & - \\ & & - & & ? & \\ & & & - & & ? \end{bmatrix},$$

and its checkerboard block sign pattern

$$\begin{bmatrix} [+]_{1\times1} & [-] & [+] & [-] \\ [-] & [+]_{2\times2} & [-] & [+] \\ [+] & [-] & [+]_{2\times2} & [-] \\ [-] & [+] & [-] & [+]_{1\times1} \end{bmatrix};$$

$$\begin{bmatrix} ? & - & + & - & & \\ - & ? & & & & \\ + & & ? & & - & \\ - & & & ? & & - \\ & & - & & ? & \\ & & & - & & ? \end{bmatrix},$$

and its checkerboard block sign pattern

$$\begin{bmatrix} [+]_{1\times 1} & [-] & [+] & [-] & [+] \\ [-] & [+]_{1\times 1} & [-] & [+] & [-] \\ [+] & [-] & [+]_{1\times 1} & [-] & [+] \\ [-] & [+] & [-] & [+]_{2\times 2} & [-] \\ [+] & [-] & [+] & [-] & [+]_{1\times 1} \end{bmatrix};$$

$$\begin{bmatrix} ? & + & - & - & & \\ + & ? & & & & \\ - & & ? & & - & \\ - & & & ? & & - \\ & & - & & ? & \\ & & & - & & ? \end{bmatrix},$$

and its checkerboard block sign pattern

$$\begin{bmatrix} [+]_{2\times 2} & [-] & [+] \\ [-] & [+]_{2\times 2} & [-] \\ [+] & [-] & [+]_{2\times 2} \end{bmatrix}.$$

As discussed in above, If $k = 1, 2, 3$ or 4, then the corresponding sign pattern $\mathcal{A}$ is not potentially eventually positive. It follows that if the sign pattern $\mathcal{A}$ is potentially eventually positive, then $k = 0$. Thus, all nonzero off-diagonal entries of $\mathcal{A}$ are positive. □

3 The Minimality of Potentially Eventually Positive Sign Pattern $\mathcal{A}$

Recall that an n-by-n sign pattern $\mathcal{A}$ is said to be a minimal potentially eventually positive sign pattern (MPEP sign pattern) if $\mathcal{A}$ is potentially eventually positive and no proper subpattern of $\mathcal{A}$ is potentially eventually positive. To consider the minimality of potentially eventually positive sign pattern $\mathcal{A}$, the follow proposition is necessary.

Proposition 5 *If sign pattern $\mathcal{A}$ is potentially eventually positive, then there exists some $i \in \{1, 2, 3, 4, 5, 6\}$ such that $\alpha_{ii} = +$.*

Proof: By a way of contradiction, assume that $\alpha_{ii} = -$ or 0 for all $i = 1, 2, \ldots, 6$. Since sign pattern $\mathcal{A}$ is potentially eventually positive, it follows that all nonzero off-diagonal entries are $+$ by Theorem 4. For every superpattern of potentially eventually positive sign patterns is also potentially eventually positive, it suffices to consider the sign pattern whose diagonal entries are all $-$. That is ,

$$\mathcal{A} = \begin{bmatrix} - & + & + & + & & \\ + & - & & & & \\ + & & - & & + & \\ + & & & - & & + \\ & & + & & - & \\ & & & + & & - \end{bmatrix}.$$

It is clear that sign pattern $\mathcal{A}$ is a proper subpattern of the checkerboard block sign pattern

$$\begin{bmatrix} [-]_{1\times 1} & [+] & [-] \\ [+] & [-]_{3\times 3} & [+] \\ [-] & [+] & [-]_{2\times 2} \end{bmatrix}.$$

Consequently, $\mathcal{A}$ is not potentially eventually positive by Lemmas 2 and 1. It is a contradiction. It follows that there exists some $i \in \{1, 2, 3, 4, 5, 6\}$ such that $\alpha_{ii} = +$. □

To state our main results clearly, let

$$\mathcal{A}_1 = \begin{bmatrix} + & + & + & + & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & + & 0 \\ + & 0 & 0 & 0 & 0 & + \\ 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 \end{bmatrix},$$

$$\mathcal{A}_2 = \begin{bmatrix} 0 & + & + & + & 0 & 0 \\ + & + & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & + & 0 \\ + & 0 & 0 & 0 & 0 & + \\ 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 \end{bmatrix},$$

$$\mathcal{A}_3 = \begin{bmatrix} 0 & + & + & + & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & + & 0 & + & 0 \\ + & 0 & 0 & 0 & 0 & + \\ 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 \end{bmatrix},$$

and

$$\mathcal{A}_5 = \begin{bmatrix} 0 & + & + & + & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & + & 0 \\ + & 0 & 0 & 0 & 0 & + \\ 0 & 0 & + & 0 & + & 0 \\ 0 & 0 & 0 & + & 0 & 0 \end{bmatrix}.$$

The following theorem indicates that $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_5$ are minimal potentially eventually positive sign patterns.

Theorem 6 *Sign patterns $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_5$ are minimal potentially eventually positive sign patterns.*

Proof: Sign patterns $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_5$ are potentially eventually positive for their positive parts are primitive, respectively. If some nonzero off-diagonal entries are changed to be 0, then the corresponding subpatterns are not potentially eventually positive by Theorem 4. By Proposition 5, the diagonal entries of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ can not be changed to be 0. Therefore, no proper subpatterns of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_5$ are potentially eventually positive. It follows that sign patterns $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_5$ are minimal potentially eventually positive sign patterns. □

Proposition 7 *If $\mathcal{A}$ is a minimal potentially eventually positive sign pattern, then $\mathcal{A}$ has exactly one positive diagonal entry.*

Proof: By a way of contradiction, assume that $\mathcal{A}$ has at least two nonzero diagonal entries. Then by Theorem 4, all nonzero off-diagonal entries of $\mathcal{A}$ are positive, and $\mathcal{A}$ has at least one positive diagonal entry by Proposition 5. Consequently, $\mathcal{A}$ must be a proper superpattern of one of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_5$. By Theorem 6, $\mathcal{A}$ is not a minimal potentially eventually positive sign pattern; a contradiction. Hence, $\mathcal{A}$ has exactly one positive diagonal entry. □

Note that if $\mathcal{A}$ is a minimal potentially eventually positive sign pattern, then $\mathcal{A}$ has exactly one positive diagonal entry by Proposition 7. Moreover, the other diagonal entries of $\mathcal{A}$ must be zero. Thus, all minimal potentially eventually positive sign patterns are identified in the following Theorem 8.

Theorem 8 *Tree sign pattern $\mathcal{A}$ is a minimal potentially eventually positive sign pattern if and only if $\mathcal{A}$ is equivalent to one of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_5$.*

Proof: The sufficiency follows from Theorem 6. For the necessity, if sign pattern $\mathcal{A}$ is a minimal potentially eventually positive sign pattern, then all nonzero off-diagonal entries are positive by Theorem 4, and $\mathcal{A}$ has exactly one positive diagonal entry by Proposition 7. Thus, up to equivalence, $\mathcal{A}$ is one of sign patterns $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$. □

In the following Theorem 9, all potentially eventually positive tree sign patterns are classified.

Theorem 9 *Tree sign pattern $\mathcal{A}$ is potentially eventually positive if and only if $\mathcal{A}$ is equivalent to a superpattern of one of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_5$.*

Proof: Theorem 9 follows readily from Theorem 8 and the fact that every superpattern of potentially eventually positive sign patterns is also potentially eventually positive. □

Recall that an n-by-n sign pattern $\mathcal{A}$ is said to require an eventually positive matrix (REP, for short), if every matrix $A \in Q(\mathcal{A})$ is eventually positive; see e.g., [14]. It is obvious that sign pattern $\mathcal{A}$ requires an eventually positive matrix, then $\mathcal{A}$ is potentially eventually positive. But the converse is not true in general. We end this section with an interesting corollary.

Corollary 10 *For the tree sign pattern $\mathcal{A}$, if $\mathcal{A}$ has exactly 11 nonzero entries, then following statements are equivalent:*

(1) $\mathcal{A}$ is a minimal potentially eventually positive sign pattern;

(2) $\mathcal{A}$ requires an eventually positive matrix;

(3) $\mathcal{A}$ is nonnegative and primitive, and has exactly one positive diagonal entry.

Proof: Corollary 10 follows readily from Theorem 4 and Theorem 2.3 in [14]. □

4 Concluding Remarks

In this article, we have identified all the minimal potentially eventually positive tree sign patterns as four specific tree sign patterns $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_5$. Consequently, we have classified all the potentially eventually positive tree sign patterns as the superpatterns of the previous specific tree sign patterns.

However, it seems that the difficulty in identifying and classifying the (minimal) potentially eventually positive (tree)sign patterns is great increasing, when the order of (tree) sign patterns to be discussed is increasing. In a follow-up paper, we will consider the potential eventual positivity of tree sign patterns with bigger orders.

Acknowledgements: The authors' work was supported by NSFC (grant no. 71301060) and a grant from Huaiyin Institute of Technology (grant no. 491713317). The authors would like to thank the two anonymous referees and Prof. Gen Qi Xu for their helpful comments and suggestions that improved the manuscript.

References:

[1] F. Hall, Z. Li, *Sign pattern matrices, in: L. Hogben(Ed.), Handbook of Linear Algebra*, Boca Ration: Chapman & Hall/CRC Press, 2007.

[2] R. A. Brualdi, H. J. Ryser, *Combinatorial Matrix Theory,* Cambridge: Cambridge University Press, 1991.

[3] R. A. Horn, C. R. Johnson, *Matrix Analysis,* Cambridge: Cambridge University Press, 1995.

[4] A. Berman, M. Catral, L. M. Dealba, A. Elhashash, F. Hall, L. Hogben, I. J. Kim, D. D. Olesky, P. Tarazaga, M. J. Tsatsomeros, P. van den Driessche, Sign patterns that allow eventual positivity, *Electronic Journal of Linear Algebra* 19, 2010, pp. 108–120.

[5] B. -L. Yu, T. -Z. Huang, J. Luo, H. B. Hua, Potentially eventually positive double star sign patterns, *Applied Mathematics Letters* 25, 2012, pp. 1619–1624.

[6] B. -L. Yu, T. -Z. Huang, On minimal potentially power-positive sign patterns, *Operators and Matrices* 6, 2012, pp. 159–167.

[7] M. Catral, L. Hogben, D. D. Olesky, P. van den Driessche, Sign patterns that require or allow power-positivity, *Electronic Journal of Linear Algebra* 19, 2010, pp. 121–128.

[8] B. -L. Yu, T. -Z. Huang, C. Hong, D. D. Wang, Eventual positivity of tridiagonal sign patterns, *Linear and Multilinear Algebra*, 62, 2014, pp. 853–859.

[9] B. -L. Yu, J. Cui, Potential eventual positivity of one specific tree sign pattern, *WSEAS Transactions on Mathematics*, 15,2016, pp. 261–270.

[10] M. Archer, M. Catral, C. Erickson, R. Haber, L. Hogben, X. Martinez-Rivera, A. Ochoa, Constructions of potentially eventually positive sign patterns with reducible positive part, *Involve* 4, 2011, pp. 405–410.

[11] Y. Hu, X. Zhou, On the Harmonic index of the unicyclic and bicyclic graphs , *WSEAS Transactions on Mathematics* 12, 2013, pp. 716–726.

[12] T. Liu, Y. Hu, The 3-Rainbow index of graph operations, *WSEAS Transactions on Mathematics* 13, 2014, pp. 161–170.

[13] L. Zuo, F. Wu, C. Hong, S. Zhang, Equitable colorings of Cartesian product graphs of wheels with complete bipartite graphs, *WSEAS Transactions on Mathematics* 13, 2014, pp. 236–245.

[14] E. M. Ellison, L. Hogben, M. J. Tsatsomeros, Sign patterns that require eventual positivity or require eventual nonnegativity, *Electronic Journal of Linear Algebra* 19, 2010, pp. 98–107.