# Minimum Principle Sufficiency for a Variational Inequality with Pseudomonotone Mapping

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*Abstract:* For a variational inequality problem (VIP) with a psudomonotone mapping $F$ on its solution set $C^*$, we give equivalent statements for $C^*$ to be determined by the zeroes $\Gamma(c^*)$ of the primal gap function of VIP, where $c^* \in C^*$. One sufficient condition is also presented in terms of weaker sharpness of $C^*$. With the psudomonotonicity$_*$ of $F$ on $C^*$ being characterized, $C^*$ turns out to coincide with the zeroes $\Lambda(c^*)$ of the dual gap function of VIP. If also $F$ has the same direction on $\Gamma(c^*)$, then $\Gamma(c^*)$ coincides with $C^*$, $\Lambda(c^*)$, and the solution set $C_*$ of the dual variational inequality problem. This has further been shown to be equivalent to saying that $F$ is constant on $\Gamma(c^*)$ when $F$ is psudomonotonone$^+$ on $C^*$.

*Key–Words:* Variational inequality, minimum principle sufficiency, weaker sharpness, pseudomonotonicity$_*$, gap functions

## 1 Introduction

Let $H$ be a Hilbert space and $F : H \to H$ a mapping. For a nonempty convex closed subset $C$ in $H$, the *variational inequality problem* (VIP(C, F)) is to find $c^* \in C$ such that

$$\langle F(c^*), c - c^*\rangle \geq 0 \quad \text{for all } c \in C$$

while the *dual variational inequality problem* (DVIP(C, F)) is to solve the following inequality for $c_* \in C$ such that

$$\langle F(c), c - c_*\rangle \geq 0 \quad \text{for all } c \in C.$$

We denote their solution sets by $C^*$ and $C_*$ respectively and suppose that the solution sets are nonempty.

Variational inequality problems receive our attention because of their great number of applications for which the reader can refer to [4, 5] and references therein.

To study $C^*$ and $C_*$, we define the *primal gap function* associated with VIP(C, F) by

$$g(x) := \sup\{\langle F(x), x - c\rangle : c \in C\} \quad \text{for } x \in H$$

and the *dual gap function* $G(x)$ associated with DVIP(C, F) by

$$G(x) := \sup\{\langle F(c), x - c\rangle : c \in C\} \quad \text{for } x \in H.$$

Their evaluation is relevant to the following two sets:

$$\begin{aligned} \Gamma(x) &:= \{c \in C : \langle F(x), x - c\rangle = g(x)\}; \\ \Lambda(x) &:= \{c \in C : \langle F(c), x - c\rangle = G(x)\}. \end{aligned}$$

It is easy to see that the functions $g$ and $G$ are nonnegative on $C$. Using the above concepts and the following relations, we can determine whether a point $c \in C$ lies in $C^* \cup C_*$ or not:

$$\begin{aligned} c \in C^* \Leftrightarrow g(c) = 0 \quad &\Leftrightarrow \quad c \in \Gamma(c); \\ c \in C_* \Leftrightarrow G(c) = 0 \quad &\Leftrightarrow \quad c \in \Lambda(c) \end{aligned}$$

(see [10, Proposition 2.1]). In addition, there hold the following inclusions

$$\begin{aligned} C^* \subseteq \Lambda(c_*) \quad &\text{for} \quad c_* \in C_*, \\ C_* \subseteq \Gamma(c^*) \quad &\text{for} \quad c^* \in C^* \end{aligned}$$

([10, Proposition 2.3]). Hence, if $\Gamma(c^*) \subseteq C_*$ for some $c^* \in C^*$, then $\Gamma(c^*) = C_*$. In particular, if $\Gamma(c^*) \subseteq C^* \subseteq C_*$ for some $c^* \in C^*$, then $C^* = C_* = \Gamma(c^*)$. In such case, $C^*$ (and $C_*$) coincides with $\Gamma(c^*)$ and the VIP is said to possess *minimum principle sufficiency* (MPS). As a solution set of a linear programming, $\Gamma(x)$ is easier to be found than others. So it makes sense to study sufficient conditions for $\Gamma(x) \subseteq C^*$ to hold. In [3], Ferris and Mangasarian studied a convex quadratic programming with nonempty solution set $\overline{S}$. They have proved that the MPS property is equivalent to the span of the Hassian of the objective function being contained in the normal cone to the feasible region at any solution point, plus the cone generated by the gradient of the objective function at any solution point. This is in turn equivalent to the quadratic program having a weak sharp minimum, which has been extended in [1]. As one sufficient condition for $\Gamma(c^*) \subseteq C^*$, the concept of weak sharpness of $C^*$ has similarly been introduced in [8] and extensively studied in terms of gap functions for several special cases of $C^* \subseteq C_*$ (see [6, 7, 10, 11, 13]).

For the general case $C^* \subseteq C_*$, it is easy to see that $\Gamma(c^*) \subseteq C^*$ iff $C^* = \Gamma(c^*) = C_*$. So it is of certain significance to characterize $\Gamma(c^*) \subseteq C^*$ without other assumptions. In this paper we first present such a characterization in Section 2. In Section 3, we apply the characterization to the case where $C^*$ is weaker sharp (a more general case than weakly sharp) and weaker sharpness of $C^*$ will be further characterized. Section 4 is devoted to characterizing a pseudomonotone$^*$ mapping on $C^*$ in which the same direction of $F$ on $\Gamma(c^*)$ implies the minimum principle sufficiency of VIP. In Section 5 we study a pseudomonotone$_+$ mapping on $C^*$ and show that constancy of $F$ on $\Gamma(c^*)$ is equivalent to minimum principle sufficiency of VIP.

For further discussion, we recall some notions for this paper as below.

For a nonempty convex set $C$, the *normal cone* $N_C(x)$ to $C$ at $x \in H$ is defined by

$$\begin{cases} \{\, \xi \in H : \langle \xi, c - x\rangle \le 0 \text{ for all } c \in C\} & \text{if } x \in C; \\ \emptyset & \text{if } x \notin C. \end{cases}$$

The *tangent cone* to $C$ at $x$ is given by

$$\begin{aligned} T_C(x) &:= \{v \in H : \langle v, \xi\rangle \le 0 \text{ for all } \xi \in N_C(x)\} \\ &= \{v \in H : d'_C(x; v) = 0\} \end{aligned}$$

(see [2]), where $d_C$ stands for the distance function associated with $C$ given by

$$d_C(x) := \inf\{\|c - x\| : c \in C\} \quad \text{for } x \in H$$

and $d'_C(x; v)$ is the directional derivative of $d_C$ at $x$ in the direction $v \in H$:

$$d'_C(x; v) := \lim_{t \to 0^+} \frac{d_C(x + tv) - d_C(x)}{t}.$$

For $c^* \in C^*$, we have

$$-F(c^*) \in N_C(c^*) = [T_C(c^*)]^\circ,$$

where $A^\circ$ is the polar set of $A$. The set $C^*$ is said to be *weakly sharp* (according to Patriksson [9]) provided that

$$-F(c^*) \in int \bigcap_{c \in C^*} [T_C(c) \cap N_{C^*}(c)]^\circ \text{ for all } c^* \in C^*.$$

This is equivalent to saying that for each $c^* \in C^*$ there exists $\alpha > 0$ such that

$$\alpha B \subseteq F(c^*) + \bigcap_{c \in C^*} [T_C(c) \cap N_{C^*}(c)]^\circ,$$

where $B$ denotes the open unit ball in $H$ with $\overline{B}$ being its closure.

A mapping $F : H \to H$ is said to be

$(i)$ *pseudomonotone* at $x \in C$ if for each $y \in C$ there holds

$$\langle F(x), y - x\rangle \ge 0 \Rightarrow \langle F(y), y - x\rangle \ge 0;$$

$(ii)$ *pseudomonotone*$_*$ at $x \in C$ if $F$ is pseudomonotone at $x$ and, for each $y \in C$,

$$\begin{aligned} &\langle F(x), y - x\rangle \ge 0 \text{ and } \langle F(y), y - x\rangle = 0 \\ \Rightarrow\ & F(y) = k(y)F(x) \text{ for some } k(y) > 0; \end{aligned}$$

$(iii)$ *pseudomonotone*$^+$ at $x \in C$ if $F$ is pseudomonotone at $x$ and, for each $y \in C$,

$$\begin{aligned} &\langle F(x), y - x\rangle \ge 0 \text{ and } \langle F(y), y - x\rangle = 0 \\ \Rightarrow\ & F(y) = F(x); \end{aligned}$$

$(iv)$ *pseudomonotone* (*pseudomonotone*$_*$, *pseudomonotone*$^+$) on a set $A \subseteq C$ if it is pseudomonotone (*pseudomonotone*$_*$, *pseudomonotone*$^+$) at each $x \in A$.

## 2 Characterization of $\Gamma(\overline{x}) \subseteq C_1$

For $\overline{x} \in C$, let $\Gamma(\overline{x}) \neq \emptyset$. By the definitions of $C^*$ and $\Gamma(\overline{x})$, $\overline{x} \in C^*$ iff $\langle F(\overline{x}), c - \overline{x}\rangle \ge 0$ for all $c \in \Gamma(\overline{x})$ iff there exists $\overline{c} \in \Gamma(\overline{x})$ such that $\langle F(\overline{x}), \overline{c} - \overline{x}\rangle \ge 0$. So if $\Gamma(\overline{x}) \subseteq C_1 \subseteq C$ and $\langle F(\overline{x}), c - \overline{x}\rangle \ge 0$ for all $c \in C_1$, then $\overline{x} \in C^*$. In addition, for each $c \in C \backslash \Gamma(\overline{x})$ there exists $\overline{c} \in \Gamma(\overline{x})$ such that

$$\langle F(\overline{x}), \overline{x} - c\rangle < \langle F(\overline{x}), \overline{x} - \overline{c}\rangle,$$

from which we have $\langle F(\overline{x}), c - \overline{c}\rangle > 0$. The following proposition states that it is the inequality that completely characterizes the inclusion $\Gamma(\overline{x}) \subseteq C_1$.

**Proposition 1** *Let $\emptyset \neq C_1 \subseteq C$. Then, for $\overline{x} \in H$, the following are equivalent:*

$(i)$ $\Gamma(\overline{x}) \subseteq C_1$.

$(ii)$ *For each $c \in C \backslash C_1$ there exists $\overline{c} \in C$ such that*

$$\langle F(\overline{x}), c - \overline{c}\rangle > 0.$$

*Hence, for $\overline{x} \in H$, if $\emptyset \neq C_1 \subseteq \Gamma(\overline{x})$, then $\Gamma(\overline{x}) = C_1$ iff $(ii)$ holds.*

**Proof:** $(i) \Rightarrow (ii)$: Let $\Gamma(\overline{x}) \subseteq C_1$. Then, since $C_1 \subseteq C$, $C\backslash C_1 \subseteq C\backslash\Gamma(\overline{x})$. So, for each $c \in C\backslash C_1$, $c \notin \Gamma(\overline{x})$, that is, there exists $\overline{c} \in C$ such that

$$\langle F(\overline{x}), \overline{x} - c\rangle < \langle F(\overline{x}), \overline{x} - \overline{c}\rangle.$$

Thus $(ii)$ follows.

$(ii) \Rightarrow (i)$: Suppose that for any $c \in C\backslash C_1$ there exists $\overline{c} \in C$ such that $\langle F(\overline{x}), c - \overline{c}\rangle > 0$. Then

$$\langle F(\overline{x}), c\rangle \neq \inf\{\langle F(\overline{x}), x\rangle : x \in C\},$$

that is, $c \notin \Gamma(\overline{x})$. Hence $\Gamma(\overline{x}) \subseteq C_1$. □

**Remark 2** Statement $(ii)$ in Proposition 1 is equivalent to saying that for each $c \in C\backslash C_1$ the set

$$\{\overline{c} \in C : \langle F(\overline{x}), c - \overline{c}\rangle \leq 0\}$$

is a proper subset of $C$. If also $C_1 \subseteq \Gamma(\overline{x})$, then $C_1 = \Gamma(\overline{x})$. A natural question is whether there exists $\overline{x} \in H$ such that $C_1 = \Gamma(\overline{x})$. Recall that $C_* \subseteq \Gamma(c^*)$ for all $c^* \in C^*$. Upon taking $C_1 = C_*$ and $\overline{x} = c^* \in C^*$ in Proposition 1, we obtain

**Corollary 3** *Let $C_* \neq \emptyset$. Then, for $c^* \in C^*$, the following are equivalent:*

$(i)$ $C_* = \Gamma(c^*)$.

$(ii)$ *For each $c \in C\backslash C_*$ there exists $\overline{c} \in C$ such that*

$$\langle F(c^*), c - \overline{c}\rangle > 0.$$

For a pseudomonotone mapping on $C^*$, the inclusion $\Gamma(c^*) \subseteq C^*$ implies that both $C^*$ and $C_*$ can be represented by $\Gamma(c^*)$. This can be characterized by Proposition 1 as below.

**Theorem 4** *Let $c^* \in C^*$. Then the following are equivalent:*

$(i)$ $C^* = \Gamma(c^*) = C_*$.

$(ii)$ *$C^* \subseteq C_*$ and for each $c \in C\backslash C^*$ there exists $\overline{c} \in C$ such that $\langle F(c^*), c - \overline{c}\rangle > 0$.*

$(iii)$ *$C^* \subseteq C_*$ and for each $c \in C\backslash C^*$ there holds*

$$\langle F(c^*), c - c^*\rangle > 0.$$

$(iv)$ *$C^* \subseteq C_*$ and for each $c \in C\backslash C^*$ there holds $\langle F(c^*), c - \overline{c}\rangle > 0$ for all $\overline{c} \in C^*$.*

*If also $C^*$ is closed, then*

$$(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v),\ \textit{where}$$

$(v)$ *$C^* \subseteq C_*$ and for each $c \in C\backslash C^*$ there exists $\alpha := \alpha(c^*, c) > 0$ such that*

$$\alpha d_{C^*}(c) \leq \langle F(c^*), c - c^*\rangle.$$

**Proof:** $(i) \Leftrightarrow (ii)$: Since $C_* \subseteq \Gamma(c^*)$, $(i)$ is valid iff $C^* \subseteq C_*$ and $\Gamma(c^*) \subseteq C^*$, which, by Proposition 1, is equivalent to $(ii)$.

$(i) \Leftrightarrow (iii)$: If $(i)$ is true, then for each $c \in C\backslash C^*$ there holds $c \notin \Gamma(c^*)$ and hence

$$\langle F(c^*), c^* - c\rangle \neq 0,$$

that is, $(iii)$ is valid since $c^* \in C^*$.

Now, for any $c \in C\backslash C^*$, suppose that

$$\langle F(c^*), c - c^*\rangle > 0.$$

Then $(ii)$ is true for $\overline{c} = c^*$. Thus $(i)$ follows.

$(iii) \Leftrightarrow (iv)$: If $C^* \subseteq C_*$, then

$$\langle F(c^*), c^* - \overline{c}\rangle = 0 \text{ for all } \overline{c} \in C^*.$$

It follows that

$$\begin{aligned} \langle F(c^*), c - \overline{c}\rangle &= \langle F(c^*), c^* - \overline{c}\rangle + \langle F(c^*), c - c^*\rangle \\ &= \langle F(c^*), c - c^*\rangle. \end{aligned}$$

Thus $(iii) \Leftrightarrow (iv)$.

Finally, if $C^*$ is closed and $C^* \subseteq C_*$, then for each $c \in C\backslash C^*$ there holds $\|c - c^*\| \geq d_{C^*}(c) > 0$. Taking

$$\alpha := \frac{\langle F(c^*), c - c^*\rangle}{\|c - c^*\|},$$

we obtain $(iii) \Leftrightarrow (v)$. The proof is complete. □

**Remark 5** The equivalence $(i) \Leftrightarrow (iii)$ stated in Theorem 4 shows that, for a pseudomonotone mapping $F$ on $C^*$, $C^* = \Gamma(c^*)$ for $c^* \in C^*$ iff

$$\langle F(c^*), c - c^*\rangle > 0 \text{ for all } c \in C\backslash C^*.$$

When $C_1$ is closed and convex, the following result provides a sufficient condition for $(ii)$ of Proposition 1 to hold, and hence for $\Gamma(\overline{x}) \subseteq C_1$ to be valid.

**Theorem 6** *Let $C_1$ be a nonempty closed and convex subset of $C$. For $\overline{x} \in H$, if each $x \in C_1$ satisfies*

$$-F(\overline{x}) \in \text{int}\,[T_C(x) \cap N_{C_1}(x)]^\circ, \tag{1}$$

*then $\Gamma(\overline{x}) \subseteq C_1$. In particular, if each $x \in C_1$ satisfies*

$$-F(\overline{x}) \in \text{int}\, N_C(x) \cup T_{C_1}(x),$$

*then* $\Gamma(\overline{x}) \subseteq C_1$.

*If* $\overline{x} \in C^*$ *satisfies* (1) *with* $C_1 = C_*$, *then* $\overline{x} \in \Gamma(\overline{x}) = C_*$. *Hence if each* $c^* \in C^*$ *satisfies* (1) *with* $\overline{x} = c^*$ *and* $C_1 = C_*$, *then*

$$C^* \subseteq \Gamma(c^*) = C_* \quad \textit{for all } c^* \in C^*.$$

*If* $C^*$ *is closed and convex and* $\overline{x} \in C^*$ *satisfies* (1) *with* $C_1 = C^*$, *then* $C_* \subseteq \Gamma(\overline{x}) \subseteq C^*$. *Hence, if also each* $c^* \in C^*$ *satisfies* (1) *with* $\overline{x} = c^*$ *and* $C_1 = C_*$, *then* $C^* = \Gamma(c^*) = C_*$ *for all* $c^* \in C^*$.

**Proof:** Let $\overline{x} \in H$ be such that each $x$ in $C_1$ satisfies (1). Suppose that $c \in C \backslash C_1$. Then, since $C_1$ is closed and convex, there exists a unique $c_1 \in C_1$ such that $\|c - c_1\| = d_{C_1}(c)$. This with (1) implies that $c_1$ satisfies

$$c - c_1 \in T_C(c_1) \cap N_{C_1}(c_1)$$

and there exists $\delta > 0$ such that

$$\langle -F(\overline{x}) + u, c - c_1 \rangle \leq 0 \quad \text{for all } u \in \delta B,$$

from which, taking $u = \frac{\delta}{2\|c-c_1\|}(c - c_1)$, we obtain

$$0 < \frac{\delta}{2}\|c - c_1\| \leq \langle F(\overline{x}), c - c_1 \rangle.$$

Thus it follows from Proposition 1 that $\Gamma(\overline{x}) \subseteq C_1$.

Now if for each $x \in C_1$ there holds

$$-F(\overline{x}) \in \text{ int } N_C(x) \cup T_{C_1}(x),$$

then

$$\begin{aligned} -F(\overline{x}) &\in \text{int } [T_C(x)]^\circ \cup [N_{C_1}(x)]^\circ \\ &\subseteq \text{int } [T_C(x) \cap N_{C_1}(x)]^\circ. \end{aligned}$$

So $\Gamma(\overline{x}) \subseteq C_1$.

Next, if $\overline{x} \in C^*$ satisfies (1) with $C_1 = C_*$, then $\Gamma(\overline{x}) \subseteq C_*$. This implies $\overline{x} \in \Gamma(\overline{x}) = C_*$ based on [10, Propositions 2.1 and 2.3].

Finally, if $C^*$ is closed and convex and $\overline{x} \in C^*$ satisfies (1) with $C_1 = C^*$, then, by [10, Proposition 2.3] and the first conclusion of the theorem, $C_* \subseteq \Gamma(\overline{x}) \subseteq C^*$. ☐

**Remark 7** For a closed and convex $C^*$, as stated in Theorem 6, if $\overline{x} \in C^*$ satisfies (1) with $C_1 = C^*$, then $C_* \subseteq \Gamma(\overline{x}) \subseteq C^*$. This implies that $C_* = \Gamma(\overline{x}) = C^*$ if $F$ is pseudomonotone. Hence Theorem 6 extends [8, Theorem 4.2] in which $F$ is continuous and pseudomonotone and $C^*$ is weakly sharp.

Based on Theorem 6, the following convergence result is immediate.

**Theorem 8** *Let* $C^*$ *be a closed and convex set in* $H$. *Suppose that* $\{x_n\}$ *is a compact sequence in* $H$. *If each convergent subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *and each* $c^* \in C^*$ *satisfy*

$$-F(x_{n_k}) \in \text{int } [T_C(c^*) \cap N_{C^*}(c^*)]^\circ$$

*for sufficiently large* $k$, *then* $\Gamma(x_n) \subseteq C^*$ *for sufficiently large* $n$. *In particular, if each convergent subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *and each* $c^* \in C^*$ *satisfy*

$$-F(x_{n_k}) \in \text{int } N_C(c^*) \cup T_{C^*}(c^*)$$

*for sufficiently large* $k$, *then* $\Gamma(x_n) \subseteq C^*$ *holds for sufficiently large* $n$.

**Proof:** Suppose that $\Gamma(x_n) \subseteq C^*$ is not true for all sufficiently large $n$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Gamma(x_{n_k}) \not\subseteq C^*$. By assumption, there exists a convergent subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that for each $c^* \in C^*$ the inclusion

$$-F(x_{n_{k_i}}) \in \textit{int } [T_C(c^*) \cap N_{C^*}(c^*)]^\circ$$

is satisfied. This with Theorem 6 gives $\Gamma(x_{n_{k_i}}) \subseteq C^*$, a contradiction. ☐

Theorem 8 extends some existing algorithm results such as [10, Theorem 3.2] and [7, Theorem 2].

## 3 Weaker sharpness of $C^*$

To get more results than the relation

$$C_* = \Gamma(\overline{x}) = C^*,$$

we need the following concept.

**Definition 9** Let $C_1$ be a closed and convex subset of $C$. The set $C_1$ is said to be *weaker sharp* provided that each $c_1 \in C_1$ satisfies

$$-F(c_1) \in \textit{int } [T_C(x) \cap N_{C_1}(x)]^\circ \quad \text{for all } x \in C_1.$$

The set $C_1$ is said to be *weakly sharp* provided that each $c_1 \in C_1$ satisfies

$$-F(c_1) \in \textit{int } \cap_{x \in C_1} [T_C(x) \cap N_{C_1}(x)]^\circ.$$

It is easy to see that the weak sharpness of $C_1$ implies the weaker sharpness of $C_1$ since

$$\begin{aligned} &\textit{int } \cap_{c \in C_1} [T_C(c) \cap N_{C_1}(c)]^\circ \\ \subseteq\ &\cap_{c \in C_1} \textit{int } [T_C(c) \cap N_{C_1}(c)]^\circ \\ \subseteq\ &\textit{int } [T_C(x) \cap N_{C_1}(x)]^\circ \quad \text{for all } x \in C_1. \end{aligned}$$

As a result of Theorem 6, next theorem shows that both $C^*$ and $C_*$ can be determined by $\Gamma(c^*)$ for $c^* \in C^*$ in the case where $C^* \cap C_* \neq \emptyset$ and both $C^*$ and $C_*$ are weaker sharp.

**Theorem 10** *Let $C^*$ be closed and convex. If $C^* \cap C_* \neq \emptyset$ and both $C^*$ and $C_*$ are weaker sharp, then $C^* = \Gamma(c^*) = C_*$ for all $c^* \in C^*$.*

Next result shows that both $C^*$ and $C_*$ can be determined by $\Gamma(c^*)$ when $C^* \subseteq C_*$ and $C^*$ is weaker sharp.

**Theorem 11** *Let $C^*$ be nonempty, closed, and convex. Then*

$$(i) \Leftrightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \text{ and } (iv')$$

*for the following statements:*

$(i)$ *$C^*$ is weaker sharp.*

$(ii)$ *For each $(c_1, c^*) \in C^* \times C^*$ there exists $\alpha := \alpha(c_1, c^*) > 0$ such that*

$$\alpha\overline{B} \subseteq F(c_1) + [T_C(c^*) \cap N_{C^*}(c^*)]^\circ.$$

$(iii)$ *For each $(c_1, c^*) \in C^* \times C^*$ there exists $\alpha := \alpha(c_1, c^*) > 0$ such that*

$$\alpha\|v\| \leq \langle F(c_1), v\rangle \text{ for all } v \in T_C(c^*) \cap N_{C^*}(c^*).$$

$(iv)$ *For each $(c_1, c) \in C^* \times C$ there exist $c^* \in C^*$ and $\alpha := \alpha(c_1, c) > 0$ such that*

$$\alpha\|c - c^*\| = \alpha d_{C^*}(c) \leq \langle F(c_1), c - c^*\rangle. \quad (2)$$

$(iv')$ *For each $c \in C$ there exist $c^* \in C^*$ and $\alpha := \alpha(c^*, c) > 0$ such that*

$$\alpha\|c - c^*\| = \alpha d_{C^*}(c) \leq \langle F(c^*), c - c^*\rangle. \quad (3)$$

*Furthermore, if $F$ is constant on $C^*$, then $(iv') \Leftrightarrow (iv'')$, where*

$(iv'')$ *For each $c \in C \setminus C^*$ there exists $\alpha := \alpha(c) > 0$ such that*

$$\alpha d_{C^*}(c) \leq \langle F(c^*), c - c^*\rangle \text{ for all } c^* \in C^*.$$

*If also $C^* \subseteq C_*$, then $(iv) \Leftrightarrow (v) \Leftrightarrow (vi)$, where*

$(v)$ *$C^* = \Gamma(c_1) = C_*$ for all $c_1 \in C^*$;*

$(vi)$ *For each $(c_1, c) \in C^* \times C$ there exists $\alpha := \alpha(c_1, c) > 0$ such that*

$$\alpha d_{C^*}(c) \leq \langle F(c_1), c - c^*\rangle \text{ for all } c^* \in C^*. \quad (4)$$

*If $(vi)$ holds with*

$$\alpha(c_1, c^*) := \liminf_{C \ni c \to c^*} \alpha(c_1, c) > 0,$$

*then $C^*$ is weaker sharp.*

**Proof:** $(i) \Leftrightarrow (ii)$ is obvious from the definition of weaker sharpness of $C^*$.

$(ii) \Rightarrow (iii)$: Suppose that for each $(c_1, c^*) \in C^* \times C^*$ there exists $\alpha > 0$ such that

$$\alpha\overline{B} \subseteq F(c_1) + [T_C(c^*) \cap N_{C^*}(c^*)]^\circ.$$

Then for every $y \in \overline{B}$ we have

$$\alpha y - F(c_1) \in [T_C(c^*) \cap N_{C^*}(c^*)]^\circ.$$

It follows that for each $0 \neq v \in T_C(c^*) \cap N_{C^*}(c^*)$ there holds

$$\left\langle \alpha\frac{v}{\|v\|} - F(c_1), v\right\rangle \leq 0,$$

from which we obtain

$$\alpha\|v\| \leq \langle F(c_1), v\rangle \text{ for all } v \in T_C(c^*) \cap N_{C^*}(c^*).$$

$(iii) \Rightarrow (iv)$: The conclusion is obviously true if $c \in C^*$. For each $c \in C\backslash C^*$, since $C^*$ is closed and convex, there exists $c^* \in C^*$ such that

$$\|c - c^*\| = d_{C^*}(c),$$

which implies that $c - c^* \in T_C(c^*) \cap N_{C^*}(c^*)$. For each $c_1 \in C^*$, by $(iii)$, there exists $\alpha > 0$ such that

$$\alpha\|v\| \leq \langle F(c_1), v\rangle \text{ for all } v \in T_C(c^*) \cap N_{C^*}(c^*).$$

Taking $v = c - c^*$ gives (2).

$(iii) \Rightarrow (iv')$: Let $(iii)$ be true. Then, by taking $c_1 = c^*$ in the proof of $(iii) \Rightarrow (iv)$, we obtain $(iv')$.

$(iv') \Leftrightarrow (iv'')$: If $F$ is constant on $C^*$, then $(iv') \Leftarrow (iv'')$ is immediate. Now for each $c \in C\backslash C^*$ and $c^*$ in $(iv')$, $\langle F(c^*), c^*\rangle = \langle F(\overline{c}), \overline{c}\rangle$ for all $\overline{c} \in C^*$. Thus $(iv'')$ follows from $(iv')$.

Next, to prove $(iv) \Leftrightarrow (v) \Leftrightarrow (vi)$, we suppose that $C^* \subseteq C_*$.

$(iv) \Rightarrow (v)$: Let $(iv)$ be true. Then for each $(c_1, c) \in C^* \times (C \setminus C^*)$ we have

$$0 < \langle F(c_1), c - c^*\rangle \text{ for all } c^* \in C^*.$$

By Theorem 4, $(v)$ follows.

$(v) \Rightarrow (vi)$: Note that $C^* \subseteq C_*$. We have

$$\langle F(c_1), c_1 - c^*\rangle = 0 \text{ for all } (c_1, c^*) \in C^* \times C^*.$$

This implies that for each $c_1 \in C^*$ there holds $\langle F(c_1), c^*\rangle = \langle F(c_1), c_1\rangle$ for all $c^* \in C^*$. Thus $(vi)$ follows from $(v)$ and Theorem 4.

$(vi) \Rightarrow (iv)$ is immediate by taking $c^* \in C^*$ such that $\|c - c^*\| = d_{C^*}(c)$ for each $c \in C$.

Finally, suppose that for each $(c_1, c) \in C^* \times C$ there exists $\alpha(c_1, c) > 0$ such that each $c^* \in C^*$ satisfies (4) and $\alpha(c_1, c^*) := \liminf_{C \ni c \to c^*} \alpha(c_1, c) > 0$.

To show that $C^*$ is weaker sharp, it suffices to claim that for each $c^* \in C^*$ there holds

$$\alpha(c_1, c^*)B \subseteq F(c_1) + [T_C(c^*) \cap N_{C^*}(c^*)]^\circ, \quad (5)$$

where $\alpha(c_1, c^*) := \liminf_{C \ni c \to c^*} \alpha(c_1, c) > 0$. This is obvious for each $c^* \in C^*$ satisfying $T_C(c^*) \cap N_{C^*}(c^*) = \{0\}$. It remains to show that (5) still holds for any $c^*$ in $C^*$ with $T_C(c^*) \cap N_{C^*}(c^*) \neq \{0\}$.

Let $c^* \in C^*$ and $0 \neq v \in T_C(c^*) \cap N_{C^*}(c^*)$. Then

$$\langle v, v\rangle > 0 \text{ and } \langle v, y^* - c^*\rangle \leq 0 \text{ for all } y^* \in C^*.$$

So $C^*$ and $c^* + v$ are separated by the hyperplane

$$H_v := \{x \in H : \langle v, x - c^*\rangle = 0\}.$$

In addition, for each positive sequence $\{t_k\}$ decreasing to 0, by [2, Theorem 2.4.5], there exists a sequence $\{v_k\}$ such that $v_k \to v$ and $c^* + t_k v_k \in C$ for sufficiently large $k$. Hence, for sufficiently large $k$, $c^* + t_k v_k$ lies in the open set

$$\{x \in H : \langle v, x - c^*\rangle > 0\}$$

which is separated by $H_v$ from $C^*$. Thus

$$d_{C^*}(c^* + t_k v_k) \geq d_{H_v}(c^* + t_k v_k) = \frac{t_k \langle v_k, v\rangle}{\|v\|}.$$

It follows that

$$\begin{aligned}\langle F(c_1), v\rangle &= \lim_{k\to+\infty} \frac{\langle F(c_1), (c^* + t_k v_k) - c^*\rangle}{t_k} \\ &\geq \liminf_{k\to+\infty} \frac{\alpha(c_1, c^* + t_k v_k) d_{C^*}(c^* + t_k v_k)}{t_k} \\ &\geq \alpha(c_1, c^*)\|v\|.\end{aligned}$$

For each $u \in B$ we have

$$\begin{aligned}\langle \alpha(c_1, c^*)u &- F(c_1), v\rangle \\ &= \langle \alpha(c_1, c^*)u, v\rangle - \langle F(c_1), v\rangle \\ &\leq \alpha(c_1, c^*)\|v\| - \alpha(c_1, c^*)\|v\| = 0.\end{aligned}$$

This implies that (5) is valid. Thus $C^*$ is weaker sharp. □

**Remark 12** When $C^*$ is closed and convex and satisfies $C^* \subseteq C_*$, [12, Theorem 2.1] states that $C^*$ is weakly sharp iff for each $c_1 \in C^*$ there exists $\alpha > 0$ such that

$$\alpha d_{C^*}(c) \leq \langle F(c_1), c - c^*\rangle \text{ for all } (c^*, c) \in C^* \times C.$$

In this case, $(vi)$ of Theorem 11 is satisfied with $\alpha(c_1, c) = \alpha$. Such a condition is stronger than weaker sharpness of $C^*$ for $C^* = \Gamma(c^*)$.

# 4 Pseudomonotone$_*$ mappings on $C^*$

It is easy to see that $F$ is pseudomonotone on $C^*$ iff $C^* \subseteq C_*$. For the special case where $F$ is pseudomonotone$_*$ on $C^*$, we will show that $C^* = \Gamma(c^*)$ for $c^* \in C^*$ iff $F(c)$ and $F(c^*)$ have the same direction for all $c \in \Gamma(c^*)$. We begin with a characterization of a pseudomonotone$_*$ mapping on $C^*$.

**Proposition 13** *Let $C^* \neq \emptyset$. Then $F$ is pseudomonotone$_*$ on $C^*$ iff $C^* \subseteq C_*$, for each $c^* \in C^*$, $C^* = \Lambda(c^*)$, and for each $c \in \Lambda(c^*)$ there exists $k(c) > 0$ such that $F(c) = k(c)F(c^*)$.*

**Proof:** We first prove the necessity. Let $F$ be pseudomonotone$_*$ on $C^*$. For each $c^* \in C^*$ and all $c \in C$, we have $\langle F(c^*), c - c^*\rangle \geq 0$. By the pseudomonotonicity$_*$ of $F$ on $C^*$, $\langle F(c), c - c^*\rangle \geq 0$. So $C^* \subseteq C_*$ and $G(c^*) = 0$, from which it follows that for $c \in \Lambda(c^*)$ we have

$$\langle F(c), c - c^*\rangle = -G(c^*) = 0$$

and hence $F(c) = k(c)F(c^*)$ for some $k(c) > 0$. Since $c^* \in C^* \subseteq C_*$, it follows from [10, Proposition 2.3 and Theorem 2.6] that $C^* = \Lambda(c^*)$.

To show the sufficiency, we suppose that $C^* \subseteq C_*$ and for each $c^* \in C^*$, $C^* = \Lambda(c^*)$, and for each $c \in \Lambda(c^*)$ there exists $k(c) > 0$ such that $F(c) = k(c)F(c^*)$. Then for each $c^* \in C^*$ and all $c \in C$,

$$\langle F(c^*), c - c^*\rangle \geq 0 \Rightarrow \langle F(c), c - c^*\rangle \geq 0.$$

This implies that $F$ is pseudomonotone on $C^*$ and $G(c^*) = 0$, which is equivalent to saying that $c^* \in \Lambda(c^*)$ (see [10, Proposition 2.1]).

Now, if $\langle F(c), c - c^*\rangle = 0$, then $c \in \Lambda(c^*)$ and hence, by assumption, $F(c) = k(c)F(c^*)$ for some $k(c) > 0$. Therefore $F$ is pseudomonotone$_*$ on $C^*$. □

**Remark 14** If $F$ is pseudomonotone$_*$ on $C^*$, then from Proposition 13 we see that, for any $c^* \in C^*$, $C^*$ can be determined by $\Lambda(c^*)$ and, for any $c \in \Lambda(c^*)$, $F(c)$ and $F(c^*)$ have the same direction. Note that $k(c)$ in Proposition 13 may also depend on $c^*$ but we write it in the simple way here and in what follows.

For a pseudomonotone$_*$ mapping $F$ on $C^*$, we have $\Lambda(c^*) = C^* \subseteq C_*$ and hence $F$ is pseudomonotone on $\Lambda(c^*)$. If also $\Gamma(c^*) \subseteq C^*$, then, by Proposition 13, for each $c \in \Gamma(c^*)$ there exists $k(c) > 0$ such that $F(c) = k(c)F(c^*)$. This with pseudomonotonicity of $F$ on $\Lambda(c^*)$ in turn supplies a sufficient condition for $F$ to be pseudomonotone$_*$ on $C^*$ and for $C^* = \Gamma(c^*)$.

**Proposition 15** *Let $c^* \in C^* \cap C_*$ and let $F$ be pseudomonotone on $\Lambda(c^*)$. If for each $c \in \Gamma(c^*)$ there exists $k(c) > 0$ such that $F(c) = k(c)F(c^*)$, then $F$ is pseudomonotone$_*$ on $C^*$ and*

$$C^* = C_* = \Gamma(c^*) = \Lambda(c^*).$$

**Proof:** By assumption and Proposition 13 we only need to prove

$$C^* = C_* = \Gamma(c^*) = \Lambda(c^*).$$

By assumption and [10, Proposition 2.3], we have $C^* \subseteq C_* \subseteq \Gamma(c^*)$. In addition, by [10, Proposition 3.1], $\Gamma(c^*) \subseteq C^*$. So $C^* = \Gamma(c^*) = C_*$.

Now, for $c \in \Gamma(c^*)$, we have $c \in C^* \subseteq \Lambda(c^*)$ (by [10, Proposition 2.3]). Thus $\Gamma(c^*) \subseteq \Lambda(c^*)$.

Next, to show $\Lambda(c^*) \subseteq \Gamma(c^*)$, let $c \in \Lambda(c^*)$. Then

$$\langle F(c), c^* - c\rangle = G(c^*) = 0.$$

The pseudomonotonicity of $F$ on $\Lambda(c^*)$ implies that $\langle F(c^*), c^* - c\rangle \geq 0$. Since $c^* \in C^*$,

$$\langle F(c^*), c^* - c\rangle = g(c^*) = 0.$$

Thus $c \in \Gamma(c^*)$ and hence $\Lambda(c^*) \subseteq \Gamma(c^*)$. The proof is complete. □

**Remark 16** As we know, for $c^* \in C^*$, $\Gamma(c^*)$ is the solutions to minimize

$$f(x) := \langle F(c^*), x - c^*\rangle \text{ subject to } x \in C.$$

Under the conditions of Proposition 15, the solution set $C^*$ to VIP and $C_*$ to DVIP can be determined by $\Gamma(c^*)$ as well as $\Lambda(c^*)$.

For a pseudomonotone$_*$ mapping $F$ on $C^*$ and each $c^* \in C^*$, by Propositions 13 and 15, the statement that $F$ has the same direction on $\Gamma(c^*)$ as $F(c^*)$ is equivalent to saying that there holds the relation

$$C^* = C_* = \Gamma(c^*) = \Lambda(c^*).$$

**Proposition 17** *Let $F$ be pseudomonotone$_*$ on $C^*$. Then, for each $c^* \in C^*$, the following are equivalent:*

$(i)$ *For each $c \in \Gamma(c^*)$ there exists $k(c) > 0$ such that $F(c) = k(c)F(c^*)$.*

$(ii)$ $C^* = C_* = \Gamma(c^*) = \Lambda(c^*)$.

**Proof:** For each $c^* \in C^*$, by Proposition 13, $F$ is pseudomonotone on $\Lambda(c^*)$. The implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ follow from Propositions 15 and 13, respectively. □

Furthermore the following result states that for a pseudomonotone$_*$ mapping $F$ on $C^*$ the equality $C^* = \Gamma(c^*)$ for each $c^* \in C^*$ implies $(ii)$ in Proposition 17 for all $c^* \in C^*$.

**Theorem 18** *Let $F$ be pseudomonotone$_*$ on $C^*$. Then the following are equivalent:*

$(i)$ *For each $c^* \in C^*$ and each $c \in \Gamma(c^*)$ there exists $k(c) > 0$ such that $F(c) = k(c)F(c^*)$.*

$(ii)$ $C^* = C_* = \Gamma(c^*) = \Lambda(c^*)$ *for each* $c^* \in C^*$.

$(iii)$ $C^* = \Gamma(c^*) = \Lambda(c^*)$ *for each* $c^* \in C^*$.

$(iv)$ $C^* = \Gamma(c^*)$ *for each* $c^* \in C^*$.

**Proof:** The implication $(i) \Rightarrow (ii)$ is from Proposition 17 while the implications $(ii) \Rightarrow (iii) \Rightarrow (iv)$ are obvious. Finally $(iv) \Rightarrow (i)$ is direct from Proposition 13. □

Based on the above two results, we obtain the well-known simple result: If $F$ is pseudomonotone$_*$ on $C$, then

$$C^* = C_* = \Gamma(c^*) = \Lambda(c^*) \text{ for each } c^* \in C^*.$$

# 5 A pseudomonotone$^+$ mapping and its properties

In this section, we first study a pseudomonotone$^+$ mapping on $C^*$. With its characterizations being obtained, we will see that a pseudomonotone mapping $F$ on $C$ is constant on $\Gamma(c^*)$ iff $C^* = \Gamma(c^*)$ and $F$ is pseudomonotone$^+$ on $C^*$ and that a pseudomonotone$^+$ mapping $F$ on $C^*$ is constant on $\Gamma(c^*)$ iff $C^* = \Gamma(c^*)$.

For a pseudomonotone$_*$ mapping $F$ on $C^*$ and $c^* \in C^*$, if it is constant on $\Gamma(c^*)$, then, by Proposition 17, $C^* = C_* = \Gamma(c^*) = \Lambda(c^*)$. In this case, by Proposition 13, the mapping $F$ must be pseudomonotone$^+$ on $C^*$. Conversely, [8, Theorem 3.1] states that $F$ is constant on $C^*$ if $F$ is continuous and pseudomonotone$^+$ on $C$. Based on [11, Theorem 2.3], the Gâteaux differentiability of $G$ on $C_*$ with $C^* \subseteq C_*$ also implies the constancy of $F$ on $C^*$. Indeed in the second case the mapping $F$ must also be pseudomonotone$^+$ on $C^*$ which turns out to be completely characterized by the constancy of $F$ on $C^*$ with $\Lambda(c^*) = C^* \subseteq C_*$ for all $c^* \in C^*$ as we see from Proposition 13. The following result gives more exact characterizations.

**Proposition 19** *Let $C^* \neq \emptyset$. Then the following are equivalent:*

$(i)$ *$F$ is pseudomonotone$^+$ on $C^*$.*

$(ii)$ *$C^* \subseteq C_*$, $F$ is constant on $\Lambda(c^*)$ for each $c^* \in C^*$.*

$(iii)$ *$C^* \subseteq C_*$, $F$ is constant on $C^*$, and $C^* = \Lambda(c^*)$ for each $c^* \in C^*$.*

**Proof:** The implications $(iii) \Rightarrow (i) \Rightarrow (ii)$ can easily be obtained from Proposition 13, so it suffices to show $C^* = \Lambda(c^*)$ for $(ii) \Rightarrow (iii)$. Since $C^* \subseteq C_*$, by [10, Proposition 2.3], we have $C^* \subseteq \Lambda(c^*)$. It remains to show $\Lambda(c^*) \subseteq C^*$.

Now, since $F(\bar{c}) = F(c^*)$ for all $\bar{c} \in \Lambda(c^*)$ and the inclusion $C^* \subseteq \Lambda(c^*)$ implies

$$\begin{aligned} \langle F(\bar{c}), c^* - \bar{c}\rangle &= \langle F(c^*), c^* - c^*\rangle = 0, \\ \langle F(\bar{c}), c - \bar{c}\rangle &= \langle F(c^*), c - c^*\rangle \geq 0 \text{ for all } c \in C, \end{aligned}$$

from which it follows that $\Lambda(c^*) \subseteq C^*$. The proof is complete. □

**Remark 20** The statement $(iii)$ in Proposition 19 has been presented in [7, Proposition 3] if $F$ is pseudomonotone$^+$ on $C$. Such a condition is sufficient for $(iii)$ but not necessary, as we see from Proposition 19.

Based on Propositions 17 and 19, the pseudomonotonicity$_*$ of $F$ on $C^*$ with its constancy on $\Gamma(c^*)$ indeed implies the pseudomonotonicity$^+$ of it on $C^*$ as the following result states.

**Proposition 21** *Let $c^* \in C^*$. Then the following are equivalent:*

$(i)$ *$F$ is pseudomonotone$_*$ on $C^*$ and constant on $\Gamma(c^*)$.*

$(ii)$ *$F$ is pseudomonotone$^+$ on $C^*$ and $\Gamma(c^*) = C^*$.*

Obviously for $c^* \in C^* \subseteq C_*$ the constancy of $F$ on $\Gamma(c^*)$ implies that of it on $C^*$. In addition, according to Proposition 19, if $F$ is pseudomonotone$^+$ on $C^*$, then so is it on $\Lambda(c^*)$ for $c^* \in C^*$. The following result shows that its converse is valid if $F$ is constant on $\Gamma(c^*)$.

**Proposition 22** *For $c^* \in C^* \cap C_*$, the following are equivalent:*

$(i)$ *$F$ is pseudomonotone on $\Lambda(c^*)$ and constant on $\Gamma(c^*)$.*

$(ii)$ *$F$ is pseudomonotone$^+$ on $C^*$ and $\Gamma(c^*) = C^*$.*

**Proof:** The implication $(ii) \Rightarrow (i)$ is immediate from Proposition 19 while $(i) \Rightarrow (ii)$ follows directly from Propositions 15 and 19. □

If $F$ is pseudomonotone on $C$ and constant on $\Gamma(c^*)$ for $c^* \in C^*$, then, by Proposition 22, $F$ must be pseudomonotone$^+$ on $C^*$. From Propositions 15, 19, and 22 and the relation $C_* \subseteq \Gamma(c^*)$ for $c^* \in C^*$, next result is immediate.

**Theorem 23** *Let $c^* \in C^* \cap C_*$ and $F$ pseudomonotone on $\Lambda(c^*)$. Then the following are equivalent:*

$(i)$ *$F$ is constant on $\Gamma(c^*)$.*

$(ii)$ *$C^* = C_* = \Gamma(c^*) = \Lambda(c^*)$ and $F$ is constant on $C^*$.*

$(iii)$ *$C^* = \Gamma(c^*) = \Lambda(c^*)$ and $F$ is constant on $C^*$.*

$(iv)$ *$\Gamma(c^*) = \Lambda(c^*)$ and $F$ is pseudomonotone$^+$ on $C^*$.*

$(v)$ *$C^* = \Gamma(c^*)$ and $F$ is pseudomonotone$^+$ on $C^*$.*

For a differentiable convex function $f : R^n \to R$, its gradient $\nabla f$ is pseudomonotone on $R^n$, so if $\nabla f$ is constant on $\Gamma(c^*)$ for $c^* \in C^*$, then, by Theorem 23, $\nabla f$ is pseudomonotone$^+$ on $C^*$ and $C^* = \Gamma(c^*)$ for $c^* \in C^*$. Usually, for a pseudomonotone$^+$ mapping $F$ on $C^*$, we have the following equivalent statements for $C^* = \Gamma(c^*)$ for $c^* \in C^*$.

**Theorem 24** *Let $F$ be pseudomonotone$^+$ on $C^*$. Then for each $c^* \in C^*$ the following are equivalent:*

$(i)$ *$F$ is constant on $\Gamma(c^*)$.*

$(ii)$ $C^* = C_* = \Gamma(c^*) = \Lambda(c^*)$.

$(iii)$ $C^* = \Gamma(c^*) = \Lambda(c^*)$.

$(iv)$ $\Gamma(c^*) = \Lambda(c^*)$.

$(v)$ $C^* = \Gamma(c^*)$.

**Remark 25** When $F$ is pseudomonotone$^+$ on $C$ (instead of $C^*$), the equivalence $(i) \Leftrightarrow (ii)$ in Theorem 24 has been obtained in [7, Proposition 5]. The equivalence $(iii) \Leftrightarrow (v)$ in Theorem 24 has been stated in [8, Theorem 4.3] under the condition that $F$ is pseudomonotone$^+$ and continuous on a compact polyhedral $C$. Theorem 24 has presented more equivalent statements under a weaker condition.

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