Integro-differential polynomial and trigonometrical splines and quadrature formulae

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Abstract: This work is one of many that are devoted to the further investigation of local interpolating polynomial splines of the fifth order approximation. Here, new polynomial and trigonometrical basic splines are presented. The main features of these splines are the following; the approximation is constructed separately for each grid interval (or elementary rectangular), the approximation constructed as the sum of products of the basic splines and the values of function in nodes and/or the values of its derivatives and/or the values of integrals of this function over subintervals. Basic splines are determined by using a solving system of equations which are provided by the set of functions. It is known that when integrals of the function over the intervals is equal to the integrals of the approximation of the function over the intervals then the approximation has some physical parallel. The splines which are constructed here satisfy the property of the fifth order approximation. Here, the one-dimensional polynomial and trigonometrical basic splines of the fifth order approximation are constructed when the values of the function are known in each point of interpolation. For the construction of the spline, we use the discrete analogues of the first derivative and quadrature with the appropriate order of approximation. We compare the properties of these splines with splines which are constructed when the values of the first derivative of the function are known in each point of interpolation and the values of integral over each grid interval are given. The one-dimensional case can be extended to multiple dimensions through the use of tensor product spline constructs. Numerical examples are represented.

Key–Words: Polynomial splines, Trigonometrical splines, Integro-Differential Splines, Interpolation.

1 Introduction

The idea of spline interpolation was born in England at the end of the 19th century when British engineers designed the first railroad tracks. These splines are now known as B-splines or in other words as splines with maximum smoothness. The polynomial spline interpolation was then considered as a more appropriate alternative to polynomial interpolation. Now there are a variety of different types of splines that are used for solving different mathematical, mechanical, physical and engineering problems.

This method of approximation using polynomial splines is widely used for the interpolation and approximation of discrete data. A lot of research has been devoted to the application of various splines with different properties for approximation and estimation of data. Special attention is given to methods of constructing images, splines can be used in signal processing [1–11].

As is well known, the one-dimensional case can be extended to multiple dimensions through the use of tensor product spline constructs [12–14].

Kireev V.I. became the first to use values of one-variable integrals of a function over subintervals for the construction of approximations.

Polynomial and trigonometrical basic splines of the fifth order approximation were constructed in [15, 16] when both the values of the function and its first derivative are known at the ends of each subinterval. In addition, values of the integrals over the subintervals are known.

Here, the one-dimensional polynomial and trigonometrical basic splines of the fifth order approximation are constructed when the values of the function are known in each point of interpolation. For the construction of the spline, we use the discrete analogues of the first derivative and quadrature with the appropriate order of approximation.

Suppose that n is a natural number, while a, b are real numbers, $h = (b-a)/n$. Let us build the grid of interpolation nodes $x_j = a + jh$, $j = 0, 1, \dots, n$.

2 Quadrature formula construction

Let the function $u(x)$ be such that $u \in C^6([x_{j-1}, x_{j+1}])$. Suppose that the values of the function $u(x)$ and its first derivative are known in x_{j-1}, x_j, x_{j+1}. We construct an approximation for $u(x)$ in $[x_{j-1}, x_{j+1}]$ in the following form:

$$\tilde{u}(x) = u_{j-1}\omega_{j-1,0}(x) + u_j\omega_{j,0}(x) + u_{j+1}\omega_{j+1,0}(x)$$

$$+u'_{j-1}\omega_{j-1,1}(x) + u'_j\omega_{j,1}(x) + u'_{j+1}\omega_{j+1,1}(x), \quad (1)$$

where $u_k = u(x_k)$, $u'_k = u'(x_k)$. Basic splines $\omega_{k,i}(x)$ we determine from the system

$$\tilde{u}(x) - u(x) = 0, \ u(x) = 1, x, x^2, x^3, x^4, x^5. \quad (2)$$

We have $\int\limits_{x_{j-1}}^{x_{j+1}} \omega_{j,1}(x)dx = 0$. Now we obtain the following formula:

$$\int\limits_{x_{j-1}}^{x_{j+1}} \tilde{u}(x)dx = u_{j-1} \int\limits_{x_{j-1}}^{x_{j+1}} \omega_{j-1,0}(x)dx +$$

$$+u_j \int\limits_{x_{j-1}}^{x_{j+1}} \omega_{j,0}(x)dx + u_{j+1} \int\limits_{x_{j-1}}^{x_{j+1}} \omega_{j+1,0}(x)dx +$$

$$+u'_{j-1} \int\limits_{x_{j-1}}^{x_{j+1}} \omega_{j-1,1}(x)dx + u'_{j+1} \int\limits_{x_{j-1}}^{x_{j+1}} \omega_{j+1,1}(x)dx,$$

where

$$\int\limits_{x_{j-1}}^{x_{j+1}} \omega_{j,0}(x)\, dx = \frac{16h}{15}, \ \int\limits_{x_{j-1}}^{x_{j+1}} \omega_{j+1,0}(x)\, dx = \frac{7h}{15},$$

$$\int\limits_{x_{j-1}}^{x_{j+1}} \omega_{j-1,0}(x)\, dx = \frac{7h}{15}, \ \int\limits_{x_{j-1}}^{x_{j+1}} \omega_{j-1,1}(x)\, dx = \frac{h^2}{15},$$

$$\int\limits_{x_{j-1}}^{x_{j+1}} \omega_{j+1,1}(x)dx = -\frac{h^2}{15}.$$

Lemma 1. Let function $u(x)$ be such that $u \in C^6([x_{j-1}, x_{j+1}])$. The following quadrature is valid:

$$\int_{x_{j-1}}^{x_{j+1}} u(x)dx = V_j(u) + \frac{h^7}{4725}u^{(6)}(\xi), \quad (3)$$

where $\xi \in [x_{j-1}, x_{j+1}]$,

$$V_j(u) = \frac{h}{15}(7u(x_{j-1}) + 7u(x_{j+1}) + 16u(x_j)) -$$

$$-\frac{h^2}{15}(u'(x_{j+1}) - u'(x_{j-1})).$$

Proof. The construction of the quadrature is evident. The remainder of the quadrature can be found in book [17].

In trigonometric cases we receive quadrature formulae in a similar way. We put

$$\tilde{u}^t(x) = u_{j-1}\omega^t_{j-1,0}(x) + u_j\omega^t_{j,0}(x) + u_{j+1}\omega^t_{j+1,0}(x) +$$

$$+u'_{j-1}\omega^t_{j-1,1}(x) + u'_{j+1}\omega^t_{j+1,1}(x), \ x \in [x_{j-1}, x_{j+1}],$$

where $\omega^t_{k,i}(x)$, $k = j-1, j, j+1$, $i = 0, 1$, have been determined from the system $\tilde{u}^t(x) - u(x) = 0$, when $u(x) = 1, \sin(x), \cos(x), \sin(2x), \cos(2x)$.

For $x_{j+1} - x_j = h$ and $x_j - x_{j-1} = h$ we obtain the following formula:

$$V^T_j(u) = \int\limits_{x_{j-1}}^{x_{j+1}} \tilde{u}^t(x)dx = u_{j-1}I_{j-1,0} + u_jI_{j,0} +$$

$$+u_{j+1}I_{j+1,0} + u'_{j-1}I_{j-1,1} + u'_{j+1}I_{j+1,1}, \quad (4)$$

where

$$I_{j,0} = \frac{2h\cos(h)^2 - 3\sin(2h)/2 + h}{\cos(h)^2 - 2\cos(h) + 1} =$$

$$= 16h/15 + O(h^3),$$

$$I_{j+1,0} = \frac{3\sin(2h)/2 - 4\cos(h)h + h}{2(\cos(h)^2 - 2\cos(h) + 1)} =$$

$$= 7h/15 + O(h^3),$$

$$I_{j-1,0} = \frac{3\sin(2h)/2 - 4\cos(h)h + h}{2(\cos(h)^2 - 2\cos(h) + 1)} =$$

$$= 7h/15 + O(h^3),$$

$$I_{j-1,1} = \frac{h - \frac{\sin(2h)}{2} + 2\cos(h)h - 2\sin(h)}{2(\cos(h) - 1)\sin(h)} =$$

$$= h^2/15 + O(h^4),$$

$$I_{j+1,1} = \frac{\frac{\sin(2h)}{2} - 2\cos(h)h + 2\sin(h) - h}{2(\cos(h) - 1)\sin(h)} =$$

$$= -h^2/15 + O(h^4).$$

3 About approximation of derivative

The following formula is well known:

$$u'(x_j)=\frac{u(x_{j-2})-8u(x_{j-1})+8u(x_{j+1})-u(x_{j+2})}{12h}$$

$$+\frac{h^4}{30}u^{(5)}(\xi_1),\ \xi_1 \in (x_{j-2}, x_{j+2}).$$

For obtaining $u'(x_j)$ for a set of trigonometrical functions we can use the following approximation:

$$\tilde{u}^T(x) = u(x_{j-2})W^T_{j-2} + u(x_{j-1})W^T_{j-1} + u(x_j)W^T_j$$

$$+u(x_{j+1})W^T_{j+1} + u(x_{j+2})W^T_{j+2},$$

where W^T_i we obtain from the system $\tilde{u}^T(x)-u(x) = 0$, when $u(x) = 1, \sin(x), \cos(x), \sin(2x), \cos(2x)$. We can get basic splines as follows:

$W^T_{j+2}(x) = \sin(x/2 - x_{j-2}/2)\sin(x/2 - x_{j-1}/2)\sin(x/2 - x_j/2)\sin(x/2 - x_{j+1}/2)/F^T_1$,

$F^T_1 = \sin(-x_{j+2}/2 + x_{j-2}/2)\sin(-x_{j+2}/2 + x_{j-1}/2)\sin(-x_{j+2}/2 + x_j/2)\sin(-x_{j+2}/2 + x_{j+1}/2)$,

$W^T_{j+1}(x) = -\sin(x/2 - x_{j-2}/2)\sin(x/2 - x_{j-1}/2)\sin(x/2 - x_j/2)\sin(x/2 - x_{j+2}/2)/F^T_2$,

$F^T_2 = \sin(-x_{j+1}/2 + x_{j-2}/2)\sin(-x_{j+1}/2 + x_{j-1}/2)\sin(-x_{j+1}/2 + x_j/2)\sin(-x_{j+2}/2 + x_{j+1}/2)$,

$W^T_j(x) = \sin(x/2 - x_{j-2}/2)\sin(x/2 - x_{j-1}/2)\sin(x/2 - x_{j+1}/2)\sin(x/2 - x_{j+2}/2)/F^T_3$,

$F^T_3 = \sin(x_j/2 - x_{j-2}/2)\sin(x_j/2 - x_{j-1}/2)\sin(-x_{j+1}/2 + x_j/2)\sin(-x_{j+2}/2 + x_j/2)$,

$W^T_{j-1}(x) = \sin(x/2 - x_{j-2}/2)\sin(x/2 - x_j/2)\sin(x/2 - x_{j+1}/2)\sin(x/2 - x_{j+2}/2)/F^T_4$,

$F^T_4 = \sin(-x_{j-1}/2 + x_{j-2}/2)\sin(x_j/2 - x_{j-1}/2)\sin(-x_{j+1}/2 + x_{j-1}/2)\sin(-x_{j+2}/2 + x_{j-1}/2)$,

$W^T_{j-2}(x) = -\sin(x/2 - x_{j-1}/2)\sin(x/2 - x_j/2)\sin(x/2 - x_{j+1}/2)\sin(x/2 - x_{j+2}/2)/F^T_5$,

$F^T_5 = \sin(-x_{j-1}/2 + x_{j-2}/2)\sin(x_j/2 - x_{j-2}/2)\sin(-x_{j+1}/2 + x_{j-2}/2)\sin(-x_{j+2}/2 + x_{j-2}/2)$.

Finally, using the approximation $\tilde{u}^T(x)$ we obtain the following formula:

$$u'(x_j) = F_j(u) + O(h^4), \tag{5}$$

where

$$\frac{C_j}{4\sin(h)(8\cos^4(h/2)-6\cos^2(h/2)+1)},$$

$C_j = \left(\left(8\cos^2(h/2) - 16\cos^4(h/2)\right)u(x_{j-1}) + \left(16\cos^4(h/2) - 8\cos^2(h/2)\right)u(x_{j+1}) + u(x_{j-2}) - u(x_{j+2})\right)$.

It can be shown that the formulae obtained above for the polynomial and trigonometrical functions are connected by the following expression:

$$\frac{C_j}{16\sin(h)(8\cos^4(h/2)-6\cos^2(h/2)+1)} =$$

$$= \frac{u(x_{j-2})}{12h} - \frac{2u(x_{j-1})}{3h} + \frac{2u(x_{j+1})}{3h} - \frac{u(x_{j+2})}{12h} + O(h).$$

4 Left polynomial splines of one variable

Suppose that the values of the function u and its first derivative are known in every grid node x_j. We denote by $\widetilde{u}(x)$ an approximation of the function $u(x)$ on the interval $[x_j, x_{j+1}] \subset [a, b]$:

$$\widetilde{u}(x) = u(x_j)\omega_{j,0}(x) + u(x_{j+1})\omega_{j+1,0}(x)+$$
$$+u'(x_j)\omega_{j,1}(x) + u'(x_{j+1})\omega_{j+1,1}(x)+$$
$$+V_j(u)\,\omega^{<0>}_j(x). \tag{6}$$

The basic splines $\omega_{j,0}(x)$, $\omega_{j+1,0}(x)$, $\omega_{j,1}(x)$, $\omega_{j+1,1}(x)$, $\omega^{<0>}_j(x)$, we obtain from the system:

$$\widetilde{u}(x) \equiv u(x),\ \ u(x) = x^{i-1}, i = 1, 2, 3, 4, 5. \tag{7}$$

Suppose that $supp\,\omega_{k,\alpha} = [x_{k-1}, x_{k+1}]$, $\alpha = 0, 1$, $supp\,\omega^{<0>}_k = [x_k, x_{k+1}]$. It is easy to see that $\omega_{k,0}, \omega_{k,1}, \omega^{<0>}_k \in C^1(R^1)$. We have for $x = x_j + th$, $t \in [0, 1]$ the next formulae:

$$\omega_{j,0}(x_j + th)=(2t + 1)(t - 1)^2, \tag{8}$$

$$\omega_{j+1,0}(x_j + th)=-(1/8)t^2(15t^2 - 14t - 9), \tag{9}$$

$$\omega_{j,1}(x_j + th)=(1/4)th(5t + 4)(t - 1)^2, \tag{10}$$

$$\omega_{j+1,1}(x_j + th)=(1/8)ht^2(5t + 3)(t - 1), \tag{11}$$

$$\omega^{<0>}_j(x_j + th)=(15/16)t^2(t - 1)^2/h. \tag{12}$$

Figures 1, 2, 3 show the graphics of the basic functions $\omega_{j,0}(x)$, $\omega_{j+1,0}(x)$, $\omega_{j,1}(x)$, $\omega_{j+1,1}(x)$, $\omega^{<0>}_j(x)$, when $h = 1$. Figure 3 (right) shows the error of approximation of the Runge function $u(x) = 1/(1 + 25x^2)$ with the polynomial splines, $h = 0.1$, $x \in [-1, 1]$.

Let us take $\tilde{U}(x)$, $x \in [a, b]$, such that $\tilde{U}(x) = \tilde{u}(x)$, $x \in [x_j, x_{j+1}]$. Let $\|u\|_{[a,b]} = \max_{[a,b]} |u(x)|$.

Theorem 1. Let function $u(x)$ be such that $u \in C^5([a, b])$. For approximation $u(x)$, $x \in [x_j, x_{j+1}]$ by (6), (8) – (12) we have:

$$|\tilde{u}(x) - u(x)|_{[x_j,x_{j+1}]} \le K_1 h^5 \|u^{(5)}\|_{[x_{j-1},x_{j+1}]}, \tag{13}$$

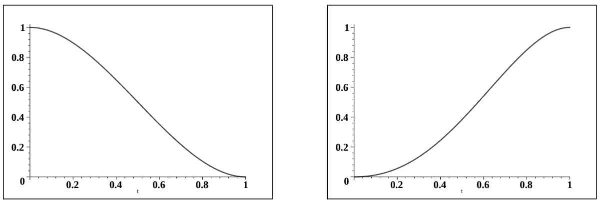

Figure 1: Plots of the basic functions: $\omega_{j,0}(x)$ (left), $\omega_{j+1,0}(x)$ (right)

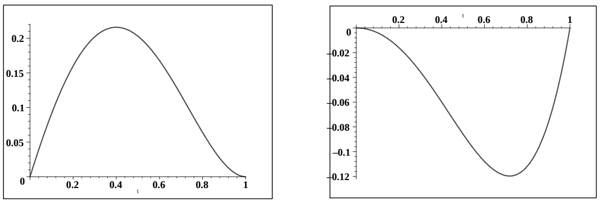

Figure 2: Plots of the basic functions: $\omega_{j,1}(x)$, when $h = 1$(left), $\omega_{j+1,1}(x)$, when $h = 1$ (right)

$$|\tilde{u}'(x) - u'(x)|_{[x_j,x_{j+1}]} \leq K_2 h^4 \|u^{(5)}\|_{[x_{j-1},x_{j+1}]}, \quad (14)$$

where $K_1 = 0.0225,\ K_2 = 0.0994$,

$$|\tilde{U}(x) - u(x)|_{[a+h,b]} \leq K_1 h^5 \|u^{(5)}\|_{[a,b]}. \quad (15)$$

Proof. Inequality (13) follows from Taylor's theorem and the inequalities:
$|\omega_{j,0}(x)| \leq 1$, $|\omega_{j+1,0}(x)| \leq 1$,
$|\omega_{j,1}(x)| \leq 0.216h$, $|\omega_{j+1,1}(x)| \leq 0.1198h$,
$|\omega_j^{<0>}(x)| \leq 0.0586/h$.
We have the next expressions for derivatives of basic functions:
$\omega'_{j,0}(x_j + th) = 6t(t-1)/h$,
$\omega'_{j+1,0}(x_j + th) = -(3/4)t(-7t - 3 + 10t^2)/h$,
${\omega'}_j^{<0>}(x_j + th) = (15/8)t(1 + 2t^2 - 3t)/h^2$,
$\omega'_{j,1}(x_j + th) = -(3/2)t - (9/2)t^2 + 5t^3 + 1$,
$\omega'_{j+1,1}(x_j + th) = -(3/4)t - (3/4)t^2 + (5/2)t^3$.
Inequality (14) follows from Taylor's theorem and the inequalities:
$|\omega'_{j,0}(x)| \leq 1.5/h$, $|\omega'_{j+1,0}(x)| \leq 1.626/h$,
$|\omega'_{j,1}(x)| \leq 1$, $|\omega'_{j+1,1}(x)| \leq 1$,
$|{\omega'}_j^{<0>}(x)| \leq 0.181/h^2$.
Inequality (15) follows from (13).

Theorem 2. Let function $u(x)$ be such that $u \in C^5([a,b])$, $\tilde{u}(x)$, $x \in [x_j, x_{j+1}]$ has form (6), (8) – (12) We have:

$$\int_{x_j}^{x_{j+1}} (\tilde{u}(x) - u(x))dx \leq 0.0081 h^6 \|u^{(5)}\|_{[x_{j-1},x_{j+1}]}.$$

The proof is similar to Theorem 1.

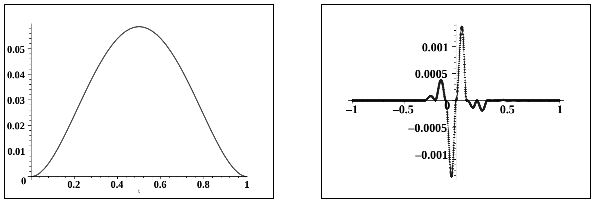

Figure 3: Plots of the basic functions: $\omega_j^{<0>}(x)$, when $h = 1$ (left), and the error of approximation of the Runge function with the polynomial splines, $h = 0.1$, $x \in [-1, 1]$ (right)

5 New left polynomial approximations of one variable

We denote by $\widetilde{u}^n(x)$ an approximation of the function $u(x)$:

$$\widetilde{u}^n(x) = u(x_j)\omega^n_{j,0}(x) + u(x_{j+1})\omega^n_{j+1,0}(x) +$$
$$+ u^p_j \omega^n_{j,1}(x) + u^p_{j+1}\omega^n_{j+1,1}(x) +$$
$$+ V_j(u)\, \omega_j^{n<0>}(x), \quad (16)$$

$x \in [x_j, x_{j+1}]$, where $V_j(u)$ has form (2),

$$u^p_j = \frac{(u(x_{j-2}) - 8u(x_{j-1}) + 8u(x_{j+1}) - u(x_{j+2}))}{12h},$$

$$u^p_{j+1} = \frac{(u(x_{j-1}) - 8u(x_j) + 8u(x_{j+2}) - u(x_{j+3}))}{12h},$$

$\omega^n_{j,0}(x_j + th) = (2t+1)(t-1)^2$,
$\omega^n_{j+1,0}(x_j + th) = -(1/8)t^2(15t^2 - 14t - 9)$,
$\omega^n_{j,1}(x_j + th) = (1/4)th(5t+4)(t-1)^2$,
$\omega_j^{n<0>}(x_j + th) = \frac{15t^2(t-1)^2}{16h}$,
$\omega^n_{j+1,1}(x_j + th) = (1/8)ht^2(5t+3)(t-1)$.
Another form of (16) is next:

$$\widetilde{u}^n(x) = u(x_{j-2})\omega^n_{j,1}(x)/(12h) +$$
$$u(x_{j-1})(-8\omega^n_{j,1}(x)/(12h) + \omega^n_{j+1,1}(x)) +$$
$$u(x_j)(8\omega^n_{j+1,1}(x)) + \omega^n_{j,0}(x)) +$$
$$u(x_{j+1})(8\omega^n_{j+1,0}(x) + \omega^n_{j,1}(x)/(12h)) +$$
$$u(x_{j+2})(-\omega^n_{j,1}(x)/(12h) + 8\omega^n_{j+1,1}(x)/(12h)) +$$
$$u(x_{j+3})(-\omega^n_{j+1,1}(x)/(12h)) + V_j(u)\, \omega_j^{n<0>}(x),$$

$x \in [x_j, x_{j+1}]$.

Theorem 3. Let function $u(x)$ be such that $u \in C^6([a,b])$. For approximation $u(x)$, $x \in [x_j, x_{j+1}]$ by (16) we have:

$$|\tilde{u}(x) - u(x)|_{[x_j,x_{j+1}]} \leq K_3 h^5 \|u^{(5)}\|_{[x_{j-2},x_{j+3}]} +$$
$$+ K_4 h^6 \|u^{(6)}\|_{[x_{j-1},x_{j+1}]}, \quad (17)$$

where $K_3 = 0.0624$, $K_4 = 0.00007$.

Proof. Inequality (17) follows from Taylor's theorem, Lemma 1 and the inequalities:
$|\omega^n_{j+1,0}(x)| \le 1$,
$|\omega^n_{j,1}(x)| \le 0.216h$, $|\omega^n_{j+1,1}(x)| \le 0.1198h$,
$|\omega^{n<0>}_j(x)| \le 0.0586/h$.

6 Left trigonometrical splines of one variable

We denote by $\widetilde{u}^T(x)$ an approximation of the function $u(x)$ on the interval $[x_j, x_{j+1}] \subset [a,b]$:

$$\widetilde{u}^T(x) = u(x_j)\omega^T_{j,0}(x) + u(x_{j+1})\omega^T_{j+1,0}(x)+$$
$$+u'(x_j)\omega^T_{j,1}(x) + u'(x_{j+1})\omega^T_{j+1,1}(x)+$$
$$+V^T_j(u)\,\omega^{<0>T}_j(x), \tag{18}$$

where $V^T_j(u)$ has form (4). The basic splines $\omega^T_{j,0}(x)$, $\omega^T_{j+1,0}(x)$, $\omega^T_{j,1}(x)$, $\omega^T_{j+1,1}(x)$, $\omega^{<0>T}_j(x)$, we obtain from the system:

$$\widetilde{u}^T(x) \equiv u(x),\ u(x) = 1, \sin(kx), \cos(kx),\ k = 1, 2. \tag{19}$$

Suppose that $supp\,\omega^T_{k,\alpha} = [x_{k-1}, x_{k+1}]$, $\alpha = 0, 1$, $supp\,\omega^{<0>T}_k = [x_k, x_{k+1}]$. It is easy to see that $\omega^T_{k,0}, \omega^T_{k,1}, \omega^{<0>T}_k \in C^1(R^1)$. We have for $x = x_j + th$, $t \in [0,1]$ the next formulae:

$\omega^T_{j,0}(x_j + th) = (7\sin(h(-1+2t)) + 16h\cos(th) - 12h\cos(h(-1+2t)) + 8h\cos(h(t-3)) - 24h\cos(h(t+1)) - 2\sin(h(4+t)) + 2\sin(h(t-4)) + \sin(h(2t+3)) - 7\sin(h(2t-3)) - 4h\cos(h(2t-3)) - 4\sin(2h(t+1)) + 4\sin(2h(t-1)) - \sin(h(1+2t)) + 8\sin(h(3+t)) + 16h\cos(2th) - 4\sin(h(t+1)) + 3\sin(3h) - 14\sin(h) + \sin(5h) - 8\sin(h(t-1)) + 4\sin(h(t-3)) - 4\sin(4h) + 8\sin(2h))/(\sin(5h) + 15\sin(3h) - 18\sin(h) + 32h - 8\sin(4h) - 36\cos(h)h + 4h\cos(3h))$,

$\omega^T_{j+1,0}(x_j + th) = -(7\sin(h(-1+2t)) + 12h\cos(h(-1+2t)) + 4h\cos(h(1+2t)) - 16h\cos(h(t-1)) - 2\sin(h(4+t)) + 2\sin(h(t-4)) + \sin(h(2t+3)) - 7\sin(h(2t-3)) - 4\sin(2h(t+1)) + 4\sin(2h(t-1)) + 24h\cos(h(-2+t)) - 8h\cos(h(2+t)) - 16h\cos(2h(t-1)) - \sin(h(1+2t)) + 8\sin(h(3+t)) - 4\sin(h(t+1)) - 12\sin(3h) + 4\sin(h) - 8\sin(h(t-1)) + 4\sin(h(t-3)) + 4\sin(4h) + 8\sin(2h))/(\sin(5h) + 15\sin(3h) - 18\sin(h) + 32h - 8\sin(4h) - 36\cos(h)h + 4h\cos(3h))$,

$\omega^T_{j,1}(x_j + th) = (4\cos(2th) - \cos(th) + 2\cos(h(1+2t)) + 6\cos(h(t-1)) - 8\cos(h(t+1)) + 2\cos(h(3+t)) - 2\cos(h(2t-3)) - 12h\sin(th) + 3\cos(h(-2+t)) - 3\cos(h(2+t)) - 3\cos(2h(t-1)) - \cos(2h(t+1)) - 4h\sin(h(-2+t)) - 8h\sin(h(t-1)) + 8h\sin(h(-1+2t)) + 4h\sin(2h(t-1)) - 3 + \cos(h(t-4)) + 4\cos(2h) - \cos(4h))/(\sin(4h) + 2\sin(2h) - 6\sin(3h) + 10\sin(h) + 4h\cos(2h) - 12h + 8\cos(h)h)$,

$\omega^T_{j+1,1}(x_j + th) = (-1 + \cos(th))(\cos(th)\cos(h) + \sin(th)\sin(h) - 1)/(\cos^2(h)\sin(h) - \sin(2h) - 2\sin(h) + \cos(h)h + 2h)$,

$\omega^{<0>T}_j(x_j + th) = (-4 + 4\cos(2th) - 4\cos(h(-1+2t)) + 4\cos(h(1+2t)) + 7\cos(h(t-1)) - 9\cos(h(t+1)) + \cos(h(t-3)) + \cos(h(3+t)) - 8h\sin(th) + 2\cos(h(-2+t)) - 2\cos(h(2+t)) - 3\cos(2h(t-1)) - \cos(2h(t+1)) - 4h\sin(h(t+1)) - 12h\sin(h(t-1)) + 4h\sin(2th) + 8h\sin(h(-1+2t)) + 4\cos(2h) - 2\cos(3h) + 2\cos(h))/(\sin(4h) + 2\sin(2h) - 6\sin(3h) + 10\sin(h) + 4h\cos(2h) - 12h + 8\cos(h)h)$.

It can be shown, that
$\omega^T_{j,0}(x_j + th) = (2t+1)(-1+t)^2 + O(h^2)$,
$\omega^T_{j+1,0}(x_j + th) = -t^2(-14t - 9 + 15t^2)/8 + O(h^2)$,
$\omega^T_{j,1}(x_j + th) = t(5t+4)(-1+t)^2 h/4 + O(h^3)$,
$\omega^T_{j+1,1}(x_j + th) = t^2(5t+3)(-1+t)h/8 + O(h^3)$,
$\omega^{<0>T}_k(x_j + th) = 15/(h16t^2(-1+t)^2) + O(h)$.

Table 1 shows the errors $R^I = \max_{x\in[a,b]} |\tilde{u} - u|$, $R^T = \max_{x\in[a,b]} |\tilde{u}^T - u|$ when $[a,b] = [-1,1]$, $h = 0.1$. Calculations were done in Maple, Digits=25.

Table 1.

$u(x)$	R^I	R^T
x^4	0.0	$0.8695e-6$
$1/(1+25x^2)$	$0.1417e-2$	$0.140e-2$
$\sin(5x) - \cos(5x)$	$0.2913e-4$	$0.2352e-4$

So, approximation with trigonometric splines gives smaller approximation errors in the approximation of trigonometric functions, than approximation with polynomial splines.

Theorem 4. The error of the approximation by the splines (18) is as follows:

$$|\widetilde{u}(x) - u(x)| \le Kh^5\|4u' + 5u''' + u^V\|_{[x_{j-1},x_{j+1}]}, \tag{20}$$

where $x \in [x_j, x_{j+1}]$, $K > 0$.

Proof. The function $u(x)$ on $[x_j, x_{j+1}]$ can be written in the form (see [15]): $u(x) = \frac{2}{3}\int_{x_j}^{x}(4u'(\tau) + 5u'''(\tau) + u^V(\tau))\sin^4(x/2 - \tau/2)d\tau + c_1 + c_2\sin(x) + c_3\cos(x) + c_4\sin(2x) + c_5\cos(2x)$, where c_i, $i = 1$, 2, 3, 4, 5 are arbitrary constants. Using the method from [15] we obtain (20).

Remark. Substituting $u'(x_j)$, $u'(x_{j+1})$ from formula (18) with expression 5, we obtain the following error of approximation:

$$|\widetilde{u}(x) - u(x)| \le Kh^5\|4u' + 5u''' + u^V\|_{[x_{j-2},x_{j+3}]},$$

where $x \in [x_j, x_{j+1}],\ K > 0$.

7 Comparing with the Hermit interpolation

Here we shall compare the polynomial approximation that has been constructed above and the Hermite approximation:

$$\widetilde{u}^H(x) = u(x_{j-1})\omega_{j-1,0}(x) + u(x_j)\omega_{j,0}(x)$$
$$+u(x_{j+1})\omega_{j+1,0}(x) + u'(x_j)\omega_{j,1}(x)+$$
$$u'(x_{j+1})\omega_{j+1,1}(x), x \in [x_j, x_{j+1}].$$

The basic splines $\omega_{j-1,0}(x), \omega_{j,0}(x),\ \omega_{j+1,0}(x), \omega_{j,1}(x), \omega_{j+1,1}(x)$ we obtain from the system:

$$\widetilde{u}^H(x) \equiv u(x),\ \ u(x) = x^{i-1}, i = 1, 2, 3, 4, 5. \quad (21)$$

Suppose that $supp\,\omega_{k,0} = [x_{k-1}, x_{k+2}]$, $supp\,\omega_{k,1} = [x_{k-1}, x_{k+1}]$. It is easy to see that $\omega_{k,0}, \omega_{s,1}, \in C^1(R^1)$, $k = j-1, j, j+1$, $s = j, j+1$.

We have for $x = x_j + th$, $t \in [0, 1]$, the next formulae:

$$\omega_{j,0}(x_j + th)=(-1 + t)^2(t + 1)^2, \quad (22)$$

$$\omega_{j+1,0}(x_j + th)= - (1/4)t^2(t + 1)(5t - 7), \quad (23)$$

$$\omega_{j,1}(x_j + th)=th(t + 1)(-1 + t)^2, \quad (24)$$

$$\omega_{j+1,1}(x_j + th)=(1/2)ht^2(-1 + t)(t + 1), \quad (25)$$

$$\omega_{j-1,0}(x_j + th)=(1/4)t^2(-1 + t)^2. \quad (26)$$

Table 2 shows the errors $R^I = \max_{x\in[a,b]} |\tilde{u} - u|$, where $\tilde{u}(x)$ is defined by (16), and $R^{II} = \max_{x\in[a,b]} |\tilde{u}^H - u|$, when $[a, b] = [-1, 1]$, $h = 0.1$. Calculations were done in Maple, Digits=15.

Table 2.

$u(x)$	R^I	R^{II}
x^4	0.0	0.0
$1/(1 + 25x^2)$	$0.1417e-2$	$0.1531e-2$
$\sin(5x) - \cos(5x)$	$0.2913e-4$	$0.3466e-4$

8 Comparing with the Lagrange interpolation

Here we compare the polynomial approximation that has been constructed above and the polynomial Lagrange approximation:

$$\tilde{u}^L(x) = u(x_{j-2})W^L_{j-2} + u(x_{j-1})W^L_{j-1} + u(x_j)W^L_j$$
$$+u(x_{j+1})W^L_{j+1} + u(x_{j+2})W^L_{j+2},\ x \in [x_j, x_{j+1}].$$

Table 3 shows the errors $R^I = \max_{x\in[a,b]} |\tilde{u} - u|$, where $\tilde{u}(x)$ is defined by (16), and $R^{III} = \max_{x\in[a,b]} |\tilde{u}^L - u|$ when $[a, b] = [-1, 1]$, $h = 0.1$. Calculations were done in Maple, Digits=15.

Here we compare trigonometrical approximation (18) that has been constructed above and the trigonometrical approximation of Lagrange type:

$$\tilde{u}^T(x) = u(x_{j-2})W^T_{j-2} + u(x_{j-1})W^T_{j-1} + u(x_j)W^T_j$$
$$+u(x_{j+1})W^T_{j+1} + u(x_{j+2})W^T_{j+2},\ x \in [x_j, x_{j+1}].$$

Table 3 shows the errors $R^T = \max_{x\in[a,b]} |\tilde{u} - u|$, where $\tilde{u}(x)$ is defined by (18), and $R^{IV} = \max_{x\in[a,b]} |\tilde{u}^T - u|$ when $[a, b] = [-1, 1]$, $h = 0.1$. Calculations were done in Maple, Digits=15.

Table 3.

$u(x)$	R^{IT}	R^{IV}
x^4	0.0	$0.1601e-4$
$1/(1 + 25x^2)$	$0.1417e-2$	$0.1228e-1$
$\sin(5x) - \cos(5x)$	$0.2913e-4$	$0.4092e-3$

Tables 2,3 show that approximation by splines which uses the values of integrals over subintervals or uses quadrature formulae gives smaller errors of approximation than without this information, for example Lagrange or Hermite splines.

9 About approximations with two variables

Suppose that n, m are natural numbers, while a, b, c, d are real numbers, $h_x = (b-a)/n$, $h_y = (d-c)/m$. Let us build the grid of interpolation nodes $x_j = a + jh_x$, $j = 0, 1, \ldots, n$, $y_k = c + kh_y$, $k = 0, 1, \ldots, m$. On every line parallel to axis y, we can construct the approximation in the form:

$$\widetilde{u}(y) = u(y_k)\omega_{k,0}(y) + u(y_{k+1})\omega_{k+1,0}(y)+$$
$$u'(y_k)\omega_{k,1}(y) + u'(y_{k+1})\omega_{k+1,1}(y)+$$
$$+V_k\,\omega^{<0>}_k(y),\ \ y \in [y_k, y_{k+1}]. \quad (27)$$

Now the formulae for $\omega_{k,0}(y)$, $\omega_{k+1,0}(y)$, $\omega_{k,1}(y)$, $\omega_{k+1,1}(y)$, $\omega^{<0>}_k(y)$ are similar to the previous ones.

If $(x, y) \in \Omega_{j,k}$ then we get the next expression using the tensor product:

$$\tilde{u}(x, y) = \sum_{i=0}^{1}\sum_{p=0}^{1} u(x_{j+i}, y_{k+p})\omega_{j+i,0}(x)\omega_{k+p,0}(y)+$$

$$+\sum_{i=0}^{1}\sum_{p=0}^{1} u'_y(x_{j+i}, y_{k+p})\omega_{j+i,0}(x)\omega_{k+p,1}(y)+$$

$$+\sum_{i=0}^{1} V_{j+i,k}(x)\omega_{j+i,0}(x)\omega_k^{<0>}(y)+$$

$$+\sum_{i=0}^{1} V_{j,k+i}\omega_j^{<0>}(x)\omega_{k+i,0}(y)+$$

$$+\sum_{i=0}^{1} S_{j,k+i}\omega_j^{<0>}(x)\omega_{k+i,1}(y)+$$

$$W_{j,k}\omega_k^{<0>}(y)\omega_j^{<0>}(x)+$$

$$+\sum_{i=0}^{1} u'_x(x_j, y_{k+i})dt\omega_{j,0}(x)\omega_{k+i,0}(y)+$$

$$+\sum_{i=0}^{1} u''_{xy}(x_j, y_{k+i})dt\omega_{j,0}(x)\omega_{k+i,1}(y)+$$

$$+\sum_{i=0}^{1} P_{j+i,k}\omega_{j+i,1}(x)\omega_k^{<0>}(y), \qquad (28)$$

where

$$V_{j+i,k} = \frac{(y_{k+1}-y_{k-1})}{30}(7u(x_{j+i}, y_{k-1})+$$

$$7u(x_{j+i}, y_{k+1}) + 16u(x_{j+i}, y_k))-$$

$$\frac{(y_{k+1}-y_{k-1})^2}{60}(u'_y(x_{j+i}, y_{k+1}) - u'_y(x_{j+i}, y_{k-1})),$$

$$V_{j,k+i} = \frac{(x_{j+1}-x_{j-1})}{30}(7u(x_{j-1}, y_{k+i})+$$

$$7u(x_{j+1}, y_{k+i}) + 16u(x_j, y_{k+i}))-$$

$$\frac{(x_{j+1}-x_{j-1})^2}{60}(u'_x(x_{j+1}, y_{k+i}) - u'_x(x_{j-1}, y_{k+i})),$$

$$S_{j,k+i} = \frac{(x_{j+1}-x_{j-1})}{30}(7u'_y(x_{j-1}, y_{k+i})+$$

$$7u'_y(x_{j+1}, y_{k+i}) + 16u_y(x_j, y_{k+i}))-$$

$$\frac{(x_{j+1}-x_{j-1})^2}{60}(u''_{xy}(x_{j+1}, y_{k+i}) - u''_{xy}(x_{j-1}, y_{k+i})),$$

$$P_{j+i,k} = \frac{(y_{k+1}-y_{k-1})}{30}(7u'_x(x_{j+i}, y_{k-1})+$$

$$7u'_x(x_{j+i}, y_{k+1}) + 16u'_x(x_{j+i}, y_k))-$$

$$\frac{(y_{k+1}-y_{k-1})^2}{60}(u''_{yx}(x_{j+i}, y_{k+1}) - u''_{yx}(x_{j+i}, y_{k-1})),$$

$$W_{jk} = \frac{(y_{k+1}-y_{k-1})}{30}(7G(x_j, y_{k-1})+$$

$$7G(x_j, y_{k+1}) + 16G(x_j, y_k))-$$

$$\frac{(y_{k+1}-y_{k-1})^2}{60}(G'y(x_j, y_{k+1}) - G'y(x_j, y_{k-1})),$$

$$G(x_j, y) = \frac{(x_{j+1}-x_{j-1})}{30}(7u(x_{j-1}, y)+$$

$$7u(x_{j+1}, y) + 16u(x_j, y))-$$

$$\frac{(x_{j+1}-x_{j-1})^2}{60}(u'_x(x_{j+1}, y) - u'_x(x_{j-1}, y)).$$

Figures 4 and 5 show approximations and the errors of approximations $\tilde{u}(x,y) - u(x,y)$ by (28), (8)–(12), (22)–(26) of functions $u_1(x,y) = \sin(3x - 3y)\cos(3x-3y)$, $u_2(x,y) = (x-y)^2(x+y)^2$, when $[a,b] = [-1,1]$, $[c,d] = [-1,1]$, $h_x = h_y = h = 0.2$. Calculations were done in Maple, Digits=15.

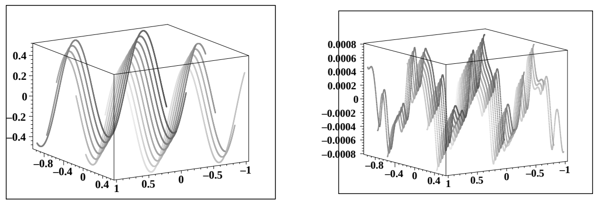

Figure 4: Plots of the functions: $\tilde{u}(x,y) = \sin(3x - 3y)\cos(3x-3y)$ (left) and $\tilde{u}(x,y) - u(x,y)$ (right) when $h = 0.2$, $[-1,1]\times[-1,1]$

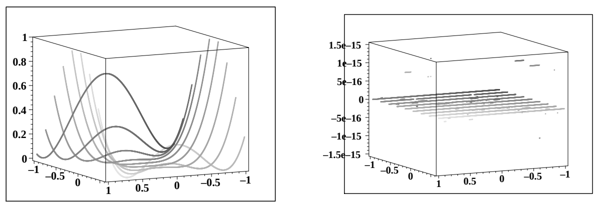

Figure 5: Plots of the functions: $\tilde{u}(x,y) = (x - y)^2(x+y)^2$ (left) and $\tilde{u}(x,y) - u(x,y)$ (right) when $h = 0.2$, $[-1,1]\times[-1,1]$

10 Conclusion

Basic splines can be applied for solving various mathematical problems. We can obtain the formulae of our polynomial basic splines in the following way. In the interval $[x_{j-1}, x_j]$ we obtain basic splines from the system:

$$\widetilde{u}(x) \equiv u(x), \quad u(x) = x^{i-1}, i = 1,2,3,4,5,$$

where $\widetilde{u}(x) = u(x_{j-1})\omega_{j-1,0}(x) + u(x_j)\omega_{j,0}(x) + u'(x_{j-1})\omega_{j-1,1}(x) + u'(x_j)\omega_{j,1}(x) + V_{j-1}\,\omega_{j-1}^{<0>}(x)$.

If we take the basic splines with the same numbers from $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$ then we have:

$$\omega_{j,0}(x_j+th)=\begin{cases}-\frac{15}{8}t^4-\frac{23}{4}t^3-\frac{39}{8}t^2+1, t\in[-1,0],\\ 2t^3-3t^2+1, \qquad t\in[0,1],\\ 0, \qquad t\notin[-1,1],\end{cases}$$

$$\omega_{j,1}(x_j+th)=\begin{cases}\frac{5h}{8}t^4+\frac{9h}{4}t^3+\frac{21h}{8}t^2+th,\\ \qquad t\in[-1,0],\\ \frac{5h}{4}t^4-\frac{3h}{2}t^3-\frac{3h}{4}t^2+th, t\in[0,1],\\ 0, \qquad t\notin[-1,1],\end{cases}$$

$$\omega_j^{<0>}(x_j+th)=\begin{cases}\frac{15}{16h}t^2(t-1)^2,\ t\in[0,1]\\ 0, \qquad t\notin[0,1].\end{cases}$$

Figure 6 shows the plots of the basic splines $\omega_{j,0}, \omega_{j,1}$. The plot of the basic spline $\omega_j^{<0>}$ is shown in Figure 3.

The construction of the nonpolynomial splines with the same properties and their application for the solving of different problems will be regarded in further papers.

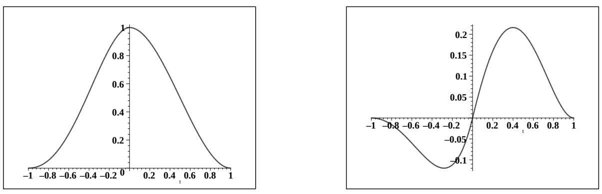

Figure 6: Plots of the basic functions: $\omega_{j,0}(jh+th)$ (left), and $\omega_{j,1}(jh+th)$, when $h=1$ (right)

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