On The Fekete-Szegö Problem for Certain Subclasses of Bi-Univalent Functions Involving Fractional $q-$ Calculus Operators

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Abstract: In this paper, we introduce and investigate new subclasses of the function class Σ of bi-univalent functions defined in the open unit disk, which are associated with fractional $q-$ calculus operators satisfying subordinate conditions. Estimates on the coefficients $|a_2|$ and $|a_3|$ are obtained and Fekete-Szegö inequalities for the function class are determined.

Key–Words: Analytic functions, Univalent functions, Bi-univalent functions, Bi-starlike functions, Bi-convex functions, Bi-Mocanu-convex functions, Subordination, q-calculus operator.

1 Introduction

The theory of a special function does not have a specific definition but it is of incredibly important to scientist and engineers who are concerned with Mathematical calculations and have a wide application in physics, Computer, engineering etc .Recently, the theory of special function has been outshining by other fields like real analysis, functional analysis, algebra, topology, differential equations.The generalized hypergeometric functions plays a major role in geometric function theory after the proof of Bieberbach conjecture by de-Branges .Usually, the special functions of mathematical physics are defined by means of power series representations. However, some alternative representations can be used as their definitions. Let us mention the well known Poisson integrals for the Bessel functions and the analytical continuation of the Gauss hypergeometric function via the Euler integral formula. The Rodrigues differential formulae, involving repeated or fractional differentiation are also used as definitions of the classical orthogonal polynomials and their generalizations.

Let $\mathcal{A}$ be the class of analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

normalized by the conditions $f(0) = 0 = f'(0) - 1$ defined in the open unit disk

$$\triangle = \{z \in \mathbb{C} : |z| < 1\}.$$

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on $\triangle$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [16] unified various subclasses of starlike and convex functions for which either of the quantity

$$\frac{z\,f'(z)}{f(z)} \text{ or } 1 + \frac{z\,f''(z)}{f'(z)}$$

is subordinate to a more general superordinate function. For this purpose, they considered an analytic function φ with positive real part in the unit disk $\triangle, \varphi(0) = 1, \varphi'(0) > 0$ and φ maps $\triangle$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination

$$\frac{z\,f'(z)}{f(z)} \prec \varphi(z).$$

Similarly, the class of Ma-Minda convex functions consists of functions $f \in \mathcal{A}$ satisfying the subordination

$$1 + \frac{z\,f''(z)}{f'(z)} \prec \varphi(z).$$

Denote by $\mathcal{M}(\lambda, \phi)$ Ma-Minda Mocanu-convex functions consists of functions $f \in \mathcal{A}$ satisfying the subordination

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \ \ (\lambda \geq 0).$$

In the sequel, it is assumed that φ is an analytic function with positive real part in the unit disk $\triangle$, satisfying $\varphi(0) = 1, \varphi'(0) > 0$ and $\varphi(\triangle)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0). \tag{1.2}$$

For functions $f \in \mathcal{A}$ the Koebe one quarter theorem [7] ensures that the image of $\triangle$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \ (z \in \triangle)$$

and

$$f(f^{-1}(w)) = w \ (|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\triangle$ if both f and f^{-1} are univalent in $\triangle$. Let Σ denote the class of bi-univalent functions (for more details see [2, 3, 4, 5]) defined in the unit disk $\triangle$. Since $f \in \Sigma$ has the Maclaurian series given by (1.1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots. \tag{1.3}$$

A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}^*_\Sigma(\varphi)$ and $\mathcal{K}_\Sigma(\varphi)$. Recently there has been triggering interest to study bi-univalent functions (see [26, 30, 31]).

The study of operators plays an important role in the geometric function theory and its related fields. Many differential operators, integral operators and Hurwitz-Lerch zeta functions[10](also see references cited therein) and generalized hypergeometic functions [9]can be written in terms of convolution and large number of generalizations of the class of univalent function and meromorphic functions have been explored in the literature. The fractional calculus operator has gained importance and popularity due to numerous applications, in particular in engineering and geometric function theory. The fractional *q-calculus* operator is an extension of the ordinary fractional calculus in the $q-$theory (see[9, 13]. Recently Purohit and Raina [19] (also see [15, 20, 24]) investigated applications of fractional *q-calculus* operator to define new classes of functions which are analytic in the open unit disc. To make this paper self contained, we present below the basic definitions and related details of the $q-$calculus,of complex valued function f which are used in the sequel.

The $q-$ shifted factorial is defined for $\alpha, q \in \mathbb{C}$ as a product of n factors by

$$(\alpha; q)_n = \begin{cases} 1 & n = 0 \\ (1-\alpha)(1-\alpha q)\ldots(1-\alpha q^{n-1}) & n \in \mathbb{N} \end{cases}$$

in terms of the basic analogue of the gamma function

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, n > 0. \tag{1.4}$$

Due to Gasper and Rahman [9] ,the recurrence relation for $q - gamma$ function is given by

$$\Gamma_q(1+\alpha) = \frac{(1-q^\alpha)\Gamma_q(\alpha)}{1-q} \tag{1.5}$$

and the *q-binomial* expansion is given by

$$\begin{aligned}(x-y)_v = x^v(-y/x; q)_v &= x^v \prod_{n=0}^{\infty} \frac{1-(y/x)q^n}{1-(y/x)q^{v+n}} \\ &= x^v_1\Phi_0[q^{-v}; -; q, yq^v/x].\end{aligned} \tag{1.6}$$

Also, the Jackson's *q-derivative* and *q-integral* of functions f, defined on the subset of $\mathbb{C}$ are respectively given by

$$D_{q,z} f(z) = \frac{f(z) - f(zq)}{z(1-q)}, (z \neq 0, q \neq 0)$$

and

$$\int_0^z f(t) d(t; q) = z(1-q) \sum_{k=0}^{\infty} q^k f(zq^k).$$

Recall that

$$\lim_{q \to 1^-} \frac{(q^\alpha; q)_n}{(1-q)^n} = (\alpha)_n = \alpha(\alpha+1)...(\alpha+n-1)$$

the familiar Pochhammer symbol. Due to Purohit and Raina, [19], we recall the following definitions of fractional *q-integral* and fractional *q-derivative* operators.

Definition 1.1 *Let the function $f \in \mathcal{A}$ be analytic in a simply-connected region of the $z-$plane containing the origin. The fractional $q-$integral of f of order μ is defined by*

$$\begin{aligned}\mathcal{I}^\mu_{q,z} f(z) &= \mathcal{D}^{-\mu}_{q,z} f(z) \\ &= \frac{1}{\Gamma_q(\mu)} \int_0^z (z-qt)_{\mu-1} f(t) d(t; q), \quad \mu > 0,\end{aligned} \tag{1.7}$$

where $(z-tq)_{\mu-1}$ can be expressed as the $q-$binomial given by (1.6). The series ${}_1\Phi_0[\mu;-;q,z]$ is a single valued when $|arg(z)|<\pi$ and $|z|<1$, therefore the function $(z-tq)_{\mu-1}$ in (1.7) is single valued when $|arg(-tq^{\mu}/z)|<\pi$, $|tq^{\mu}/z|<1$ and $|arg(z)|<\pi$.

Definition 1.2 *For $f(z)$ as in Definition 1.1 the fractional $q-$derivative operator of order μ, is defined by*

$$\mathcal{D}^{\mu}_{q,z}f(z)=\mathcal{D}_{q,z}\mathcal{I}^{1-\mu}_{q,z}f(z) = \frac{1}{\Gamma_q(1-\mu)}D_{q,z}\int_0^z(z-qt)_{-\mu}f(t)d(t;q) \tag{1.8}$$

$(0\le\mu<1)$, *where the function $f(z)$ is constrained, and the multiplicity of the function $(z-tq)_{-\mu}$ is removed as in Definition 1.1.*

Definition 1.3 *Under the hypothesis of Definition 1.2, the extended fractional $q-$derivative of order μ is defined by*

$$D^{\mu}_{q,z}f(z)=D^{m}_{q,z}\mathcal{I}^{m-\mu}_{q,z}f(z), \tag{1.9}$$

$(m-1\le\mu<m,\ m\in\mathbb{N}_0)$.

With the aid of the above definitions, and their known extensions we define the Ω^{μ}_q operator involving *q-differintegral* operator.

Let $\Omega^{\mu}_q f(z):\mathcal{A}\to\mathcal{A}$ defined by

$$\Omega^{\mu}_q f(z)=\frac{\Gamma_q(2-\mu)}{\Gamma_q(2)}z^{\mu-1}D^{\mu}_{q,z}f(z) = z+\sum_{n=2}^{\infty}\Gamma^q_n(\mu)a_nz^n, z\in\Delta \tag{1.10}$$

where

$$\Gamma^q_n(\mu)=\frac{\Gamma_q(2-\mu)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\mu)}, \quad (-\infty<\mu<2, 0<q<1). \tag{1.11}$$

Here $D^{\mu}_{q,z}f(z)$ in (1.9) represents, respectively, a fractional $q-$ integral of $f(z)$ of order μ when $-\infty<\mu<0$ and fractional *q-derivative* of $f(z)$ of order μ when $0\le\mu<2$. The function $\Gamma^q_n(\mu)$ is a decreasing function of n if $\frac{\Gamma^q_3(\mu)}{\Gamma^q_2(\mu)}\le 1$.

Throughout the paper for the sake of brevity we let

$$\Gamma_n=\Gamma^q_n(\mu)=\frac{\Gamma_q(2-\mu)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\mu)}; \quad (\mu<2, 0<q<1,) \tag{1.12}$$

unless otherwise stated.

The object of the paper is to estimate the Taylor-Maclaurians coefficients coefficients $|a_2|$ and $|a_3|$ for the functions $f\in\Sigma$. Further, using the techniques of Zaprawa [32, 33] (also see[1, 17, 12]) we obtain the Fekete-Szegö result for the function class $f\in\Sigma$.

2 Coefficients estimates for the function class $\mathcal{M}^q_{\Sigma}(\gamma,\lambda,h)$

Motivated by the works of Srivastava et al. [27] and Goyal and Goswami [11], now we define a new subclass $\mathcal{M}^q_{\Sigma}(\gamma,\lambda,h)$ of bi-univalent functions involving $q-$ calculus operator to estimate the coefficients $|a_2|$ and $|a_3|$ for the functions in the class $\mathcal{M}^q_{\Sigma}(\gamma,\lambda,h)$.

Definition 2.1 *Let $h:\triangle\to\mathbb{C}$ be a convex univalent function such that*

$$h(0)=1 \qquad \textit{and} \qquad \Re\big(h(z)\big)>0 \qquad (z\in\Delta).$$

Suppose also that the function $h(z)$ is given by

$$h(z)=1+\sum_{n=1}^{\infty}B_nz^n \qquad (z\in\Delta). \tag{2.1}$$

A function $f\in\Sigma$ is said to be in the class $\mathcal{M}^q_{\Sigma}(\gamma,\lambda,h)$ if the following conditions are satisfied:

$$e^{i\gamma}\left[(1-\lambda)\frac{z(\Omega^{\mu}_q f(z))'}{\Omega^{\mu}_q f(z)}+\lambda\left(1+\frac{z(\Omega^{\mu}_q f(z))''}{(\Omega^{\mu}_q f(z))'}\right)\right] \prec h(z)\cos\gamma+i\sin\gamma, \tag{2.2}$$

and

$$e^{i\gamma}\left[(1-\lambda)\frac{w(\Omega^{\mu}_q g(w))'}{\Omega^{\mu}_q g(w)}+\lambda\left(1+\frac{w(\Omega^{\mu}_q g(w))''}{(\Omega^{\mu}_q g(w))'}\right)\right] \prec h(w)\cos\gamma+i\sin\gamma \tag{2.3}$$

where $g=f^{-1},\gamma\in(-\frac{\pi}{2},\frac{\pi}{2}), 0\le\lambda\le 1$ and $z,w\in\triangle$.

Remark 2.2 *Taking $\lambda=0$ we get $\mathcal{M}^q_{\Sigma}(\gamma,\lambda,h)\equiv\mathcal{S}^q_{\Sigma}(\gamma,h)$ and if $f\in\mathcal{S}^q_{\Sigma}(\gamma,h)$, then*

$$e^{i\gamma}\frac{z(\Omega^{\mu}_q f(z))'}{\Omega^{\mu}_q f(z)}\prec h(z)\cos\gamma+i\sin\gamma$$

and

$$e^{i\gamma}\frac{w(\Omega^{\mu}_q g(w))'}{\Omega^{\mu}_q g(w)}\prec h(w)\cos\gamma+i\sin\gamma$$

Further by taking $\lambda = 1$ *we get* $\mathcal{M}^q_\Sigma(\gamma,\lambda,h),\equiv \mathcal{K}^q_\Sigma(\gamma,h)$ *and if* $f \in \mathcal{K}^q_\Sigma(\gamma,h)$, *then*

$$e^{i\gamma}\left(1+\frac{z(\Omega^\mu_q f(z))''}{(\Omega^\mu_q f(z))'}\right) \prec h(z)\cos\gamma + i\sin\gamma$$

and

$$e^{i\gamma}\left(1+\frac{w(\Omega^\mu_q g(w))''}{(\Omega^\mu_q g(w))'}\right) \prec h(w)\cos\gamma + i\sin\gamma,$$

where $g = f^{-1}, \gamma \in (-\frac{\pi}{2},\frac{\pi}{2})$ *and* $z,w \in \triangle$.

Remark 2.3 *If we set* $h(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, *then the class*

$$\mathcal{M}^q_\Sigma(\gamma,\lambda,h) \equiv \mathcal{M}^q_\Sigma(\gamma,\lambda,A,B)$$

which is defined as

$$e^{i\gamma}\left[(1-\lambda)\frac{z(\Omega^\mu_q f(z))'}{\Omega^\mu_q f(z)} + \lambda\left(1+\frac{z(\Omega^\mu_q f(z))''}{(\Omega^\mu_q f(z))'}\right)\right] \prec \frac{1+Az}{1+Bz}\cos\gamma + i\sin\gamma, \tag{2.4}$$

and

$$e^{i\gamma}\left[(1-\lambda)\frac{w(\Omega^\mu_q g(w))'}{\Omega^\mu_q g(w)} + \lambda\left(1+\frac{w(\Omega^\mu_q g(w))''}{(\Omega^\mu_q g(w))'}\right)\right] \prec \frac{1+Aw}{1+Bw}\cos\gamma + i\sin\gamma \tag{2.5}$$

where $g = f^{-1}, \gamma \in (-\frac{\pi}{2},\frac{\pi}{2}), 0 \le \lambda \le 1$ *and* $z,w \in \triangle$.

Remark 2.4 *If in Remark 2.3, we set ,*

$$A = 1-2\beta, \quad B = -1, \quad (0 \le \beta < 1),$$

that is if we put

$$h(z) = h_\beta(z) = \frac{1+(1-2\beta)z}{1-z},\ 0 \le \beta < 1,$$

then we get

$$\mathcal{M}^q_\Sigma(\gamma,\lambda,\frac{1+(1-2\beta)z}{1-z}) \equiv \mathcal{M}^q_\Sigma(\gamma,\lambda,h_\beta(z))$$

in which $\mathcal{M}^q_\Sigma(\gamma,\lambda,h_\beta(z))$ *denotes the class of functions* $f \in \Sigma$ *such that*

$$\Re\left(e^{i\gamma}\left[(1-\lambda)\frac{z(\Omega^\mu_q f(z))'}{\Omega^\mu_q f(z)} + \lambda\left(1+\frac{z(\Omega^\mu_q f(z))''}{(\Omega^\mu_q f(z))'}\right)\right]\right) > \beta\cos\gamma$$

and

$$\Re\left(e^{i\gamma}\left[(1-\lambda)\frac{w(\Omega^\mu_q g(w))'}{\Omega^\mu_q g(w)} + \lambda\left(1+\frac{w(\Omega^\mu_q g(w))''}{(\Omega^\mu_q g(w))'}\right)\right]\right) > \beta\cos\gamma$$

where $g = f^{-1}, \gamma \in (-\frac{\pi}{2},\frac{\pi}{2}), 0 \le \lambda \le 1$ *and* $z,w \in \triangle$.

Remark 2.5 *Taking* $\lambda = 0$ *we get* $\mathcal{M}^q_\Sigma(\gamma,\lambda,A,B) \equiv \mathcal{S}^q_\Sigma(\gamma,A,B)$ *and if* $f \in \mathcal{S}^q_\Sigma(\gamma,A,B)$, *then*

$$e^{i\gamma}\frac{z(\Omega^\mu_q f(z))'}{\Omega^\mu_q f(z)} \prec \frac{1+Az}{1+Bz}\cos\gamma + i\sin\gamma \tag{2.6}$$

and

$$e^{i\gamma}\frac{w(\Omega^\mu_q g(w))'}{\Omega^\mu_q g(w)} \prec \frac{1+Aw}{1+Bw}\cos\gamma + i\sin\gamma \tag{2.7}$$

where $g = f^{-1}, \gamma \in (-\frac{\pi}{2},\frac{\pi}{2})$ *and* $z,w \in \triangle$.

Remark 2.6 *Taking* $\lambda = 1$ *we get* $\mathcal{M}^q_\Sigma(\gamma,\lambda,A,B),\equiv \mathcal{K}^q_\Sigma(\gamma,A,B)$ *and if* $f \in \mathcal{K}^q_\Sigma(\gamma,A,B)$, *then*

$$e^{i\gamma}\left(1+\frac{z(\Omega^\mu_q f(z))''}{(\Omega^\mu_q f(z))'}\right) \prec \frac{1+Az}{1+Bz}\cos\gamma + i\sin\gamma, \tag{2.8}$$

and

$$e^{i\gamma}\left(1+\frac{w(\Omega^\mu_q g(w))''}{(\Omega^\mu_q g(w))'}\right) \prec \frac{1+Aw}{1+Bw}\cos\gamma + i\sin\gamma, \tag{2.9}$$

where $g = f^{-1}, \gamma \in (-\frac{\pi}{2},\frac{\pi}{2})$ *and* $z,w \in \triangle$.

Remark 2.7 *By taking* $\lambda = 0$ *as in Remark 2.4,we state analogous subclasses as in Remarks 2.5 denoted by*

$$\mathcal{M}^q_\Sigma(\gamma,\lambda,h_\beta(z)) \equiv \mathcal{S}^q_\Sigma(\gamma,\beta)$$

which satisfies the following criteria

$$\Re\left(e^{i\gamma}\frac{z(\Omega^\mu_q f(z))'}{\Omega^\mu_q f(z)}\right) > \beta\cos\gamma$$

and

$$\Re\left(e^{i\gamma}\frac{w(\Omega^\mu_q g(w))'}{\Omega^\mu_q g(w)}\right) > \beta\cos\gamma$$

where $g = f^{-1}, \gamma \in (-\frac{\pi}{2},\frac{\pi}{2}$ *and* $z,w \in \triangle$.

Remark 2.8 *By taking* $\lambda = 1$ *in Remark 2.4,we state analogous subclasses as in Remarks 2.6 denoted by*

$$\mathcal{M}^q_\Sigma(\gamma,\lambda,h_\beta(z)),\equiv \mathcal{K}^q_\Sigma(\gamma,\beta)$$

$$\Re\left(e^{i\gamma}\left[1+\frac{z(\Omega_q^\mu f(z))''}{(\Omega_q^\mu f(z))'}\right]\right)>\beta\cos\gamma$$

and

$$\Re\left(e^{i\gamma}\left[1+\frac{w(\Omega_q^\mu g(w))''}{(\Omega_q^\mu g(w))'}\right]\right)>\beta\cos\gamma$$

where $g=f^{-1},\gamma\in(-\frac{\pi}{2},\frac{\pi}{2})$ *and* $z,w\in\triangle$.

In order to prove our main result for the functions class $\mathcal{M}_\Sigma^q(\gamma,\lambda,h)$ we recall the following lemmas.

Lemma 2.9 *(see [18]). If a function* $p\in\mathcal{P}$ *is given by*

$$p(z)=1+p_1z+p_2z^2+\cdots\qquad(z\in\Delta),$$

then

$$|p_k|\leqq 2\qquad(k\in\mathbb{N}),$$

where $\mathcal{P}$ *is the family of all functions* p, *analytic in* Δ, *for which*

$$p(0)=1\qquad\text{and}\qquad\Re\big(p(z)\big)>0\qquad(z\in\Delta).$$

Lemma 2.10 *(see [21]; see also [7]). Let the function* $\psi(z)$ *given by*

$$\psi(z)=\sum_{n=1}^{\infty}C_nz^n\qquad(z\in\Delta)$$

be convex in Δ. *Suppose also that the function* $\mathfrak{h}(z)$ *given by*

$$\mathfrak{h}(z)=\sum_{n=1}^{\infty}\mathfrak{h}_nz^n$$

is holomorphic in Δ. *If*

$$\mathfrak{h}(z)\prec\psi(z)\qquad(z\in\Delta),$$

then

$$|\mathfrak{h}_n|\leq|C_1|\qquad(n\in\mathbb{N}).$$

Theorem 2.11 *Let* f *given by* (1.1) *be in the class* $\mathcal{M}_\Sigma^q(\gamma,\lambda,h)$. *Then*

$$|a_2|\leq\sqrt{\frac{|B_1|\ \cos\gamma}{2(1+2\lambda)\Gamma_3-(1+3\lambda)\Gamma_2^2}}\tag{2.10}$$

and

$$|a_3|\leq|B_1|\cos\gamma\left(\frac{1}{2(1+2\lambda)\Gamma_3}+\frac{|B_1|\ \cos\gamma}{(1+\lambda)^2\Gamma_2^2}\right),\tag{2.11}$$

where $0\leq\lambda\leq1,\gamma\in(-\frac{\pi}{2},\frac{\pi}{2})$.

proof. Let $f\in\mathcal{M}_\Sigma^q(\gamma,\lambda,h)$ and $g=f^{-1}$. Then from (2.2) and (2.3) we have

$$e^{i\gamma}\left[(1-\lambda)\frac{z(\Omega_q^\mu f(z))'}{\Omega_q^\mu f(z)}+\lambda\left(1+\frac{z(\Omega_q^\mu f(z))''}{(\Omega_q^\mu f(z))'}\right)\right]=p(z)\cos\gamma+i\sin\gamma,\ (z\in\triangle)\tag{2.12}$$

and

$$e^{i\gamma}\left[(1-\lambda)\frac{w(\Omega_q^\mu g(w))'}{\Omega_q^\mu g(w)}+\lambda\left(1+\frac{w(\Omega_q^\mu g(w))''}{(\Omega_q^\mu g(w))'}\right)\right]=q(w)\cos\gamma+i\sin\gamma,\ (w\in\triangle)\tag{2.13}$$

where $p(z)\prec h(z)$ and $q(w)\prec h(w)$ and have the following forms:

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots,\ z\in\triangle\tag{2.14}$$

and

$$q(w)=1+q_1w+q_2w^2+q_3w^3+\cdots,\ w\in\triangle.\tag{2.15}$$

Now, equating the coefficients in (2.12) and (2.13) we get

$$e^{i\gamma}(1+\lambda)\Gamma_2a_2=\ p_1\cos\gamma\tag{2.16}$$

$$e^{i\gamma}[-(1+3\lambda)\Gamma_2^2a_2^2+2(1+2\lambda)\Gamma_3a_3]=\ p_2\cos\gamma\tag{2.17}$$

$$-e^{i\gamma}(1+\lambda)\Gamma_2a_2=\ q_1\cos\gamma\tag{2.18}$$

and

$$e^{i\gamma}[\{4(1+2\lambda)\Gamma_3-(1+3\lambda)\Gamma_2^2\}a_2^2-2(1+2\lambda)\Gamma_3a_3]=q_2\cos\gamma.\tag{2.19}$$

From (2.16) and (2.18) it follows that

$$p_1=-q_1\tag{2.20}$$

and

$$2e^{2i\gamma}(1+\lambda)^2\Gamma_2^2\,a_2^2=(p_1^2+q_1^2)\cos^2\gamma$$

$$a_2^2=\frac{(p_1^2+q_1^2)\cos^2\gamma}{2(1+\lambda)^2\Gamma_2^2}e^{-2i\gamma}.\tag{2.21}$$

Adding (2.17) and (2.19) it follows that

$$a_2^2=\frac{(p_2+q_2)}{4(1+2\lambda)\Gamma_3-2(1+3\lambda)\Gamma_2^2}e^{-i\gamma}\cos\gamma.\tag{2.22}$$

Since by definition, $p(z), q(w) \in h(\triangle)$, by applying Lemma 2.10 in conjunction with the Taylor-Maclaurin expansions (2.1), (2.14) and (2.15), we find that

$$|p_n| := \left|\frac{p^{(n)}(0)}{n!}\right| \leqq |B_1| \qquad (n \in \mathbb{N}) \tag{2.23}$$

and

$$|q_n| := \left|\frac{q^{(n)}(0)}{n!}\right| \leqq |B_1| \qquad (n \in \mathbb{N}). \tag{2.24}$$

we get

$$|a_2|^2 = \frac{|B_1|\cos\gamma}{2(1+2\lambda)\Gamma_3 - (1+3\lambda)\Gamma_2^2} \tag{2.25}$$

which gives the estimate on $|a_2|$ as asserted in (2.10). Subtracting (2.19) from (2.17), we get

$$a_3 - a_2^2 = \frac{(p_2 - q_2)e^{-i\gamma}\cos\gamma}{4(1+2\lambda)\Gamma_3} \tag{2.26}$$

Substituting the value of a_2^2 from (2.21) in (2.26), we get

$$a_3 = \frac{(p_2 - q_2)e^{-i\gamma}\cos\gamma}{4(1+2\lambda)\Gamma_3} + \frac{(p_1^2 + q_1^2)e^{-2i\gamma}\cos^2\gamma}{2(1+\lambda)^2\Gamma_2^2}.$$

Applying Lemma 2.10 once again for the coefficients p_1, p_2, q_1 and q_2, we get

$$|a_3| \leq \frac{|B_1|\cos\gamma}{2(1+2\lambda)\Gamma_3} + \frac{|B_1|^2\cos^2\gamma}{(1+\lambda)^2\Gamma_2^2}$$

which gives the estimate on $|a_3|$ as asserted in (2.11).

By setting $h(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$ from Theorem 2.11, we get the following corollary:

Corollary 2.12 *Let f given by* (1.1) *be in the class $\mathcal{M}_\Sigma^\gamma(\lambda, A, B)$. Then*

$$|a_2| \leq \sqrt{\frac{(A-B)\cos\gamma}{2(1+2\lambda)\Gamma_3 - (1+3\lambda)\Gamma_2^2}} \tag{2.27}$$

and

$$|a_3| \leq \left(\frac{1}{2(1+2\lambda)\Gamma_3} + \frac{(A-B)\cos\gamma}{(1+\lambda)^2\Gamma_2^2}\right)(A-B)\cos\gamma, \tag{2.28}$$

where $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $0 \leq \lambda \leq 1$.

Further, by setting $h(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$ from Theorem 2.11 we get the following corollary:

Corollary 2.13 *Let f be given by* (1.1) *be in the class $\mathcal{M}_\Sigma^\gamma(\lambda, h_\beta)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)\cos\gamma}{2(1+2\lambda)\Gamma_3 - (1+3\lambda)\Gamma_2^2}}$$

and

$$|a_3| \leq \left(\frac{1}{(1+2\lambda)\Gamma_3} + \frac{4(1-\beta)\cos\gamma}{(1+\lambda)^2\Gamma_2}\right)(1-\beta)\cos\gamma,$$

where $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $0 \leq \lambda \leq 1$.

By taking $\lambda = 0$ and $\lambda = 1$ we state following Corollaries:

Corollary 2.14 *Let f given by* (1.1) *be in the class $\mathcal{M}_\Sigma^q(\gamma, 0, h) \equiv \mathcal{S}_\Sigma^q(\gamma, h)$. Then*

$$|a_2| \leq \sqrt{\frac{|B_1|\ \cos\gamma}{2\Gamma_3 - \Gamma_2^2}} \tag{2.29}$$

and

$$|a_3| \leq |B_1|\cos\gamma\left(\frac{1}{2\Gamma_3} + \frac{|B_1|\ \cos\gamma}{\Gamma_2^2}\right), \tag{2.30}$$

where $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Corollary 2.15 *Let f given by* (1.1) *be in the class $\mathcal{M}_\Sigma^q(\gamma, 1, h) \equiv \mathcal{K}_\Sigma^q(\gamma, h)$. Then*

$$|a_2| \leq \sqrt{\frac{|B_1|\ \cos\gamma}{6\Gamma_3 - 4\Gamma_2^2}} \tag{2.31}$$

and

$$|a_3| \leq |B_1|\cos\gamma\left(\frac{1}{6\Gamma_3} + \frac{|B_1|\ \cos\gamma}{4\Gamma_2^2}\right), \tag{2.32}$$

where $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Remark 2.16 *By setting $h(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$ or by setting $h(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$ from Corollaries 2.12, 2.13 and from above Corollary we can state the estimates for $f \in \mathcal{S}_\Sigma^q(\gamma, A, B)$*(or *$f \in \mathcal{S}_\Sigma^q(\gamma, \beta)$) and $f \in \mathcal{K}_\Sigma^q(\gamma, A, B)$*(or *$f \in \mathcal{K}_\Sigma^q(\gamma, \beta)$) respectively.*

3 The Function Class $\mathcal{G}^q_\Sigma(\gamma,\alpha,h)$

Definition 3.1 *Let $h : \triangle \to \mathbb{C}$ be a convex univalent function such that*

$$h(0) = 1 \qquad and \qquad \Re\big(h(z)\big) > 0 \qquad (z \in \Delta).$$

Suppose also that the function $h(z)$ is given by (2.1). For $\alpha \geq 0$, a function $f \in \Sigma$ is said to be in the class $\mathcal{G}^q_\Sigma(\gamma,\alpha,h)$ if the following conditions are satisfied:

$$e^{i\gamma}\left[\left(\frac{z(\Omega^\mu_q f(z))'}{\Omega^\mu_q f(z)}\right)^\alpha \left(1+\frac{z(\Omega^\mu_q f(z))''}{(\Omega^\mu_q f(z))'}\right)^{1-\alpha}\right] \prec h(z)\cos\gamma + i\sin\gamma \tag{3.1}$$

and

$$e^{i\gamma}\left[\left(\frac{w(\Omega^\mu_q g(w))'}{\Omega^\mu_q g(w)}\right)^\alpha \left(1+\frac{w(\Omega^\mu_q g(w))''}{(\Omega^\mu_q g(w))'}\right)^{1-\alpha}\right] \prec h(w)\cos\gamma + i\sin\gamma \tag{3.2}$$

where $g = f^{-1}, \gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $z, w \in \triangle$.

By setting $h(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$ or by setting $h(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$ we define new subclasses $\mathcal{G}^q_\Sigma(\gamma,\alpha,A,B)$ and $\mathcal{G}^q_\Sigma(\gamma,\alpha,\beta)$ from the above definition as defined in Remarks 2.3 and 2.4 of previous section.

Remark 3.2 *By taking $\alpha = 0$ and $\alpha = 1$, the class $\mathcal{G}^q_\Sigma(\gamma,\alpha,h)$ leads to the classes defined in Remark 2.2 and for various choices of $h(z)$ we can state the subclasses mentioned in Remarks 2.5, 2.6 , 2.7 and 4.9 .*

In the following theorem we obtain initial Taylor coefficients for $f \in \mathcal{G}^q_\Sigma(\gamma,\alpha,h)$.

Theorem 3.3 *Let f given by (1.1) be in the class $\mathcal{G}^q_\Sigma(\gamma,\alpha,h)$. Then*

$$|a_2| \leq \sqrt{\frac{2|B_1|\ \cos\gamma}{4(3-2\alpha)\Gamma_3 + [(\alpha-2)^2 - 3(4-3\alpha)]\Gamma_2^2}} \tag{3.3}$$

and

$$|a_3| \leq |B_1|\cos\gamma\left(\frac{1}{2(3-2\alpha)\Gamma_3} + \frac{|B_1|\ \cos\gamma}{(2-\alpha)^2\Gamma_2^2}\right), \tag{3.4}$$

where $\alpha \geq 0, \gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

proof. Let $f \in \mathcal{G}^q_\Sigma(\gamma,\alpha,h)$ and $g = f^{-1}$. Then from (3.1) and (3.2), we have

$$e^{i\gamma}\left[\left(\frac{z(\Omega^\mu_q f(z))'}{\Omega^\mu_q f(z)}\right)^\alpha \left(1+\frac{z(\Omega^\mu_q f(z))''}{(\Omega^\mu_q f(z))'}\right)^{1-\alpha}\right] = p(z)\cos\gamma + i\sin\gamma,\ (z \in \triangle) \tag{3.5}$$

and

$$e^{i\gamma}\left[\left(\frac{w(\Omega^\mu_q g(w))'}{\Omega^\mu_q g(w)}\right)^\alpha \left(1+\frac{w(\Omega^\mu_q g(w))''}{(\Omega^\mu_q g(w))'}\right)^{1-\alpha}\right] = q(w)\cos\gamma + i\sin\gamma,\ (w \in \triangle) \tag{3.6}$$

where $p(z) \prec h(z)$ and $q(w) \prec h(w)$ and of the forms as given in (2.14) and (2.15) respectively. Now, equating the coefficients in (3.5) and (3.6), we get

$$e^{i\gamma}(2-\alpha)\Gamma_2 a_2 = p_1\cos\gamma \tag{3.7}$$

$$e^{i\gamma}[\{(\alpha-2)^2 - 3(4-3\alpha)\}\frac{\Gamma_2^2 a_2^2}{2} + 2(3-2\alpha)\Gamma_3 a_3] = p_2\cos\gamma \tag{3.8}$$

$$-e^{i\gamma}(2-\alpha)\Gamma_2 a_2 = q_1\cos\gamma \tag{3.9}$$

and

$$e^{i\gamma}[\{4(3-2\alpha)\Gamma_3 + \frac{1}{2}[(\alpha-2)^2 - 3(4-3\alpha)]\Gamma_2^2\}a_2^2 - 2(3-2\alpha)\Gamma_3 a_3] = q_2\cos\gamma. \tag{3.10}$$

From (3.7) and (3.9) it follows that

$$p_1 = -q_1 \tag{3.11}$$

and

$$2e^{2i\gamma}(2-\alpha)^2\Gamma_2^2\, a_2^2 = (p_1^2 + q_1^2)\cos^2\gamma$$

$$a_2^2 = \frac{(p_1^2+q_1^2)\cos^2\gamma}{2(2-\alpha)^2\Gamma_2^2}e^{-2i\gamma}. \tag{3.12}$$

Adding (3.8) and (3.10) it follows that

$$a_2^2 = \frac{(p_2+q_2)e^{-i\gamma}\cos\gamma}{4(3-2\alpha)\Gamma_3 + \{(\alpha-2)^2 - 3(4-3\alpha)\}\Gamma_2^2}. \tag{3.13}$$

Since by definition, $p(z), q(w) \in h(\triangle)$, by applying Lemma 2.10 in conjunction with the Taylor-Maclaurin expansions (2.1), (2.14) and (2.15), we find from (2.23) and (2.24) we get

$$|a_2|^2 = \frac{2|B_1|\cos\gamma}{4(3-2\alpha)\Gamma_3 + \{(\alpha-2)^2 - 3(4-3\alpha)\}\Gamma_2^2} \tag{3.14}$$

which gives the estimate on $|a_2|$ as asserted in (3.3).
Subtracting (3.10) from (3.8), we get

$$a_3 - a_2^2 = \frac{(p_2-q_2)e^{-i\gamma}\cos\gamma}{4(3-2\alpha)\Gamma_3} \tag{3.15}$$

Substituting the value of a_2^2 from (3.12) in (3.15) we get

$$a_3 = \frac{(p_2-q_2)e^{-i\gamma}\cos\gamma}{4(3-2\alpha)\Gamma_3} + \frac{(p_1^2+q_1^2)e^{-2i\gamma}\cos^2\gamma}{2(2-\alpha)^2\Gamma_2^2}.$$

Applying Lemma 2.10 once again for the coefficients p_1, p_2, q_1 and q_2, we get

$$|a_3| \leq \frac{|B_1|\cos\gamma}{2(3-2\alpha)\Gamma_3} + \frac{|B_1|^2\cos^2\gamma}{(2-\alpha)^2\Gamma_2^2}$$

which gives the estimate on $|a_3|$ as asserted in (3.4).

By setting $h(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$ from Theorem 3.3, we get the following corollary:

Corollary 3.4 *Let f given by* (1.1) *be in the class $\mathcal{G}_\Sigma^q(\gamma, \alpha A, B)$. Then*

$$|a_2| \leq \sqrt{\frac{2(A-B)\cos\gamma}{4(3-2\alpha)\Gamma_3 + \{(\alpha-2)^2 - 3(4-3\alpha)\}\Gamma_2^2}}$$

and

$$|a_3| \leq \left(\frac{1}{2(3-2\alpha)\Gamma_3} + \frac{(A-B)\cos\gamma}{(2-\alpha)^2\Gamma_2^2}\right)(A-B)\cos\gamma,$$

where $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\alpha \geq 0$.

Further, by setting $h_\beta(z) = h(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$ from Theorem 3.3 we get the following corollary:

Corollary 3.5 *Let f be given by* (1.1) *be in the class $\mathcal{G}_\Sigma^q(\gamma, \alpha, \beta)$. Then*

$$|a_2| \leq \sqrt{\frac{4(1-\beta)\cos\gamma}{4(3-2\alpha)\Gamma_3 + \{(\alpha-2)^2 - 3(4-3\alpha)\}\Gamma_2^2}}$$

and

$$|a_3| \leq \left(\frac{1}{(3-2\alpha)\Gamma_3} + \frac{4(1-\beta)\cos\gamma}{(2-\alpha)^2\Gamma_2}\right)(1-\beta)\cos\gamma,$$

where $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\alpha \geq 0$.

Remark 3.6 *Taking $\alpha = 0$ (or $\alpha = 1$) we state the initial coefficient estimates for $f \in \mathcal{S}_\Sigma^q(\gamma, h)$(or $f \in \mathcal{K}_\Sigma^q(\gamma, h)$)respectively. Further by setting $h(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$ or by setting $h(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$ we can state the estimates for $f \in \mathcal{S}_\Sigma^q(\gamma, A, B)$(or $f \in \mathcal{S}_\Sigma^q(\gamma, \beta)$) and $f \in \mathcal{K}_\Sigma^q(\gamma, A, B)$(or $f \in \mathcal{K}_\Sigma^q(\gamma, \beta)$) respectively.*

Motivated by the earlier work on bi-univalent double zeta functions[12] and also by work of Srivastava et al[26], we define the following new subclass.

4 Coefficient Estimates for $f \in \mathcal{T}_\Sigma^q(\gamma, \lambda, h)$

Definition 4.1 *Let $h : \Delta \to \mathbb{C}$ be a convex univalent function in Δ such that $h(0) = 1$ and $\Re(h(z)) > 0$, $(z \in \Delta)$ and $h(z)$ is of the form(2.1). A function $f \in \Sigma$ given by* (1.1) *is said to be in the class $\mathcal{T}_\Sigma^q(\gamma, \lambda, h)$, if it satisfies the following conditions:*

$$e^{i\gamma}\left(\frac{z(\Omega_q^\mu f(z))'}{(1-\lambda)z + \lambda\Omega_q^\mu f(z)}\right) \prec h(z)cos\gamma + isin\gamma \tag{4.1}$$

and

$$e^{i\gamma}\left(\frac{w(\Omega_q^\mu g(w))'}{(1-\lambda)w + \lambda\Omega_q^\mu g(w)}\right) \prec h(w)cos\gamma + isin\gamma \tag{4.2}$$

where $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2}), 0 \leq \lambda \leq 1$, the function $g = f-1$ and $z, w \in \Delta$.

Definition 4.2 *Let $h : \Delta \to \mathbb{C}$ be a convex univalent function in $\mathbb{U}$ such that $h(0) = 1$ and $\Re(h(z)) > 0$, $(z \in \Delta)$. A function $f \in \Sigma$ given by* (1.1) *is said to be in the class $\mathcal{P}_\Sigma^q(\gamma, \lambda, h)$, if it satisfies the following conditions:*

$$e^{i\gamma}\left(\frac{z(\Omega_q^\mu f(z))' + z^2(\Omega_q^\mu f(z))''}{(1-\lambda)z + \lambda z(\Omega_q^\mu f(z))'}\right) \prec h(z)cos\gamma + isin\gamma \tag{4.3}$$

and

$$e^{i\gamma}\left(\frac{w(\Omega_q^\mu g(w))' + w^2(\Omega_q^\mu g(w))''}{(1-\lambda)w + \lambda w(\Omega_q^\mu g(w))'}\right) \prec h(w)cos\gamma + isin\gamma \tag{4.4}$$

where $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2}), 0 \leq \lambda \leq 1$, the function $g = f^{-1}$ and $z, w \in \Delta$.

Definition 4.3 *Let $h : \Delta \to \mathbb{C}$ be a convex univalent function in $\mathbb{U}$ such that $h(0) = 1$ and $\Re(h(z)) > 0$, $(z \in \Delta)$. A function $f \in \Sigma$ given by* (1.1) *is said to be in the class $\mathcal{T}_\Sigma^q(\gamma, 0, h) \equiv \mathcal{H}_\Sigma^q(\gamma, h)$, if it satisfies the following conditions:*

$$e^{i\gamma}\left((\Omega_q^\mu f(z))'\right) \prec h(z)cos\gamma + isin\gamma \tag{4.5}$$

and

$$e^{i\gamma}\left((\Omega_q^\mu g(w))'\right) \prec h(w)cos\gamma + isin\gamma \tag{4.6}$$

A function $f \in \Sigma$ given by (1.1) *is said to be in the class $\mathcal{P}^q_\Sigma(\gamma,0,h) \equiv \mathcal{Q}^q_\Sigma(\gamma,h)$, if it satisfies the following conditions:*

$$e^{i\gamma}\left((\Omega^\mu_q f(z))' + z(\Omega^\mu_q f(z))''\right) \prec h(z)cos\gamma + isin\gamma \tag{4.7}$$

and

$$e^{i\gamma}\left((\Omega^\mu_q g(w))' + w^2(\Omega^\mu_q g(w))''\right) \prec h(w)cos\gamma + isin\gamma \tag{4.8}$$

where $\gamma \in (-\frac{\pi}{2},\frac{\pi}{2}), 0 \le \lambda \le 1,$ *the function* $g = f^{-1}$ *and* $z,w \in \Delta$.

Remark 4.4 *If we set* $h(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, *then the class* $\mathcal{M}^q_\Sigma(\gamma,\lambda,h) \equiv \mathcal{M}^q_\Sigma(\gamma,\lambda,A,B)$ *denotes the class of functions* $f \in \Sigma$, *satisfying the following conditions:*

$$e^{i\gamma}\left(\frac{z(\Omega^\mu_q f(z))'}{(1-\lambda)z+\lambda\Omega^\mu_q f(z)}\right) \prec \left(\frac{1+Az}{1+Bz}\right)cos\gamma + isin\gamma$$

and

$$e^{i\gamma}\left(\frac{w(\Omega^\mu_q g(w))'}{(1-\lambda)w+\lambda\Omega^\mu_q g(w)}\right) \prec \left(\frac{1+Aw}{1+Bw}\right)cos\gamma + isin\gamma$$

where $\gamma \in (-\frac{\pi}{2},\frac{\pi}{2}), 0 \le \lambda \le 1,$ *the function* $g = f^{-1}$ *and* $z,w \in \Delta$.

Remark 4.5 *If we set* $h(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \le \beta < 1$ *then the class* $\mathcal{T}^q_\Sigma(\gamma,\lambda,h) \equiv \mathcal{T}^q_\Sigma(\gamma,\lambda,\beta)$ *denotes the class of functions* $f \in \Sigma$, *such that*

$$\Re\left(e^{i\gamma}\frac{z(\Omega^\mu_q f(z))'}{(1-\lambda)z+\lambda\Omega^\mu_q f(z)}\right) > \beta\ cos\gamma$$

and

$$\Re\left(e^{i\gamma}\frac{w(\Omega^\mu_q g(w))'}{(1-\lambda)w+\lambda\Omega^\mu_q g(w)}\right) > \beta\ cos\gamma$$

where $\gamma \in (-\frac{\pi}{2},\frac{\pi}{2}), 0 \le \lambda \le 1,$ *the function* $g = f^{-1}$ *and* $z,w \in \Delta$.

By taking $h(z) = \frac{1+Az}{1+Bz}, (-1 \le B < A \le 1)$or $h(z) = \frac{1+(1-2\beta)z}{1-z}$, $(\ 0 \le \beta < 1)$ we state analogous subclasses of $\mathcal{P}^q_\Sigma(\gamma,\lambda,h)$ as in above remarks 4.4 and 4.5 respectively.

Theorem 4.6 *Let the function f given by* (1.1) *be in the class $\mathcal{T}^q_\Sigma(\gamma,\lambda,h)$. Suppose also that B_1 is given as in the Taylor-Maclaurin expansion (5.9) of the function $h(z)$. Then*

$$|a_2| \le \sqrt{\frac{|B_1|cos\gamma}{(\lambda^2-2\lambda)\Gamma_2^2+(3-\lambda)\Gamma_3}} \tag{4.9}$$

and

$$|a_3| \le \frac{|B_1|cos\gamma}{(3-\lambda)\Gamma_3} + \left(\frac{|B_1|cos\gamma}{(2-\lambda)\Gamma_2}\right)^2 \tag{4.10}$$

where $\gamma \in (-\frac{\pi}{2},\frac{\pi}{2})$,$0 \le \lambda \le 1$

proof. From (4.1) and (4.2), we have

$$e^{i\gamma}\left(\frac{z(\Omega^\mu_q f(z))'}{(1-\lambda)z+\lambda\Omega^\mu_q f(z)}\right) = p(z)cos\gamma+isin\gamma, \quad (z \in \mathbb{U}) \tag{4.11}$$

and

$$e^{i\gamma}\left(\frac{w(\Omega^\mu_q g(w))'}{(1-\lambda)w+\lambda\Omega^\mu_q g(w)}\right) = q(w)cos\gamma+isin\gamma, \quad (w \in \mathbb{U}) \tag{4.12}$$

where $p(z) \prec h(z),\ (z \in \mathbb{U})$ and $q(w) \prec h(w)\ (w \in \mathbb{U})$ are in the above-defined class $\mathcal{P}$ given in (2.14) and (2.15) respectively.

Now, equating the coefficients in (4.11) and (4.12), we get

$$e^{i\gamma}(2-\lambda)\Gamma_2 a_2 = p_1 cos\gamma \tag{4.13}$$

$$e^{i\gamma}\{(\lambda^2-2\lambda)\Gamma_2^2a_2^2 + (3-\lambda)\Gamma_3 a_3\} = p_2 cos\gamma \tag{4.14}$$

$$-e^{i\gamma}(2-\lambda)\Gamma_2 a_2 = q_1 cos\gamma \tag{4.15}$$

and

$$e^{i\gamma}\{(\lambda^2-2\lambda)\Gamma_2^2a_2^2 + (3-\lambda)(2a_2^2-a_3)\Gamma_3\} = q_2 cos\gamma. \tag{4.16}$$

Proceeding on lines similar to Theorem 2.11 and applying the Lemma2.10 we get the desired estimates

$$|a_2|^2 \le \frac{|B_1|cos\gamma}{(\lambda^2-2\lambda)\Gamma_2^2+(3-\lambda)\Gamma_3}$$

which gives the estimate on $|a_2|$ as asserted in (4.9). Further

$$|a_3| \le \frac{|B_1|cos\gamma}{(3-\lambda)\Gamma_3} + \left(\frac{|B_1|cos\gamma}{(2-\lambda)\Gamma_2}\right)^2$$

which gives the estimate on $|a_3|$ as asserted in (4.10).

By taking $h(z) = \frac{1+Az}{1+Bz}, (-1 \le B < A \le 1)$ and $h(z) = \frac{1+(1-2\beta)z}{1-z}$, $(\ 0 \le \beta < 1)$ we state the following corollaries for the function classes defined in Remark 4.4 and Remark 4.5 without proof.

Corollary 4.7 *Let the function f given by* (1.1) *be in the class $\mathcal{T}^q_\Sigma(\gamma,\lambda,A,B)$. Then*

$$|a_2| \le \sqrt{\frac{(A-B)cos\gamma}{(\lambda^2-2\lambda)\Gamma_2^2+(3-\lambda)\Gamma_3}}$$

and

$$|a_3| \le \frac{(A-B)cos\gamma}{(3-\lambda)\Gamma_3} + \left(\frac{(A-B)cos\gamma}{(2-\lambda)\Gamma_2}\right)^2$$

where $\gamma \in (-\frac{\pi}{2},\frac{\pi}{2}), 0 \le \lambda \le 1.$

Corollary 4.8 *Let the function f given by* (1.1) *be in the class $\mathcal{T}^q_\Sigma(\gamma,\lambda,\beta)$. Then*

$$|a_2| \le \sqrt{\frac{2(1-\beta)cos\gamma}{(\lambda^2-2\lambda)\Gamma_2^2+(3-\lambda)\Gamma_3}}$$

and

$$|a_3| \leq \frac{2(1-\beta)cos\gamma}{(3-\lambda)\Gamma_3} + \left(\frac{2(1-\beta)cos\gamma}{(2-\lambda)\Gamma_2}\right)^2$$

where $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2}), 0 \leq \lambda \leq 1.$

Theorem 4.9 *Let the function* f *given by* (1.1) *be in the class* $\mathcal{P}^q_\Sigma(\gamma, \lambda, h)$. *Suppose also that* B_1 *is given as in the Taylor-Maclaurin expansion (5.9) of the function* h. *Then*

$$|a_2| \leq \sqrt{\frac{|B_1|cos\gamma}{4(\lambda^2-2\lambda)\Gamma^2 + 3(3-\lambda)\Gamma_3}}$$

and

$$|a_3| \leq \frac{|B_1|cos\gamma}{3(3-\lambda)\Gamma_3} + \left(\frac{|B_1|cos\gamma}{2(2-\lambda)\Gamma_2^2}\right)^2$$

where $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2}), 0 \leq \lambda \leq 1.$

proof. From (4.3) and (4.4), and proceeding as in Theorem 4.6,we get

$$2e^{i\gamma}(2-\lambda)\Gamma_2 a_2 = p_1 cos\gamma$$

$$e^{i\gamma}\{4(\lambda^2-2\lambda)\Gamma_2^2 a_2^2 + 3(3-\lambda)\Gamma_3 a_3\} = p_2 cos\gamma$$

$$-2e^{i\gamma}(2-\lambda)\Gamma_2 a_2 = q_1 cos\gamma$$

and

$$e^{i\gamma}\{4(\lambda^2-2\lambda)\Gamma_2^2 a_2^2 + 3(3-\lambda)(2a_2^2 - a_3)\Gamma_3\} = q_2 cos\gamma$$

Proceeding on lines similar to Theorem 2.11 and applying the Lemma2.10 we get the desired estimates.

By taking $h(z) = \frac{1+Az}{1+Bz}$,$(-1 \leq B < A \leq 1)$ or $h(z) = \frac{1+(1-2\beta)z}{1-z}$, $(\ 0 \leq \beta < 1)$ we state corollaries analogous to the Corollaries 4.7 and 4.8 for $f \in \mathcal{P}^q_\Sigma(\gamma, \lambda, h)$ respectively.

On specializing the parameters λ, we define various interesting new subclasses , analogous to the the function classes studied in [6, 8, 25] associated with q-calculus operator and estimates $|a_2|$ and $|a_3|$ (which are asserted in Theorem 4.6) can be derived easily and so we omit the details. Further by choosing $\gamma = 0$ we state the results analogous to the results studied in [2, 8, 6, 25].

5 Fekete-Szegö inequality

Making use of the values of a_2^2 and a_3, and motivated by the recent work of Zaprawa [32] we prove the following Fekete-Szegö result for the function class $\mathcal{M}^q_\Sigma(\gamma, \lambda, h)$. We state the following lemmas given by Zaprawa [33] (also see [17, 12]).

Lemma 5.1 *Let* $k \in \mathbb{R}$ *and* $z_1, z_2 \in \mathbb{C}$. *If* $|z_1| < R$ *and* $|z_2| < R$ *then*

$$|(k+1)z_1 + (k-1)z_2| \leq \begin{cases} 2|k|R, & |k| \geq 1 \\ 2R & |k| \leq 1. \end{cases} \tag{5.1}$$

Lemma 5.2 *Let* $k, l \in \mathbb{R}$ *and* $z_1, z_2 \in \mathbb{C}$. *If* $|z_1| < R$ *and* $|z_2| < R$ *then*

$$|(k+l)z_1 + (k-l)z_2| \leq \begin{cases} 2|k|R, & |k| \geq |l| \\ 2|l|R & |k| \leq |l|. \end{cases} \tag{5.2}$$

Theorem 5.3 *Let the function* f *given by* (1.1) *be in the class* $\mathcal{M}^q_\Sigma(\gamma, \lambda, h)$ *and* $\eta \in \mathbb{R}$. *Then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} 2B_1\ cos\gamma|\tau(\eta)|, \\ \quad |\tau(\eta)| > \frac{1}{4(1+2\lambda)\Gamma_3}, \\ \frac{B_1 cos\gamma}{2(1+2\lambda)\Gamma_3}, \\ \quad |\tau(\eta)| \leq \frac{1}{4(1+2\lambda)\Gamma_3}. \end{cases} \tag{5.3}$$

proof. From (2.26), we get

$$a_3 = a_2^2 + \frac{(p_2 - q_2)e^{-i\gamma}\cos\gamma}{4(1+2\lambda)\Gamma_3} \tag{5.4}$$

Substituting for a_2^2 given by (2.22) and by simple calculation we get

$$a_3 - \eta a_2^2 = e^{-i\gamma}\cos\gamma\left[\left(\tau(\eta) + \frac{1}{4(1+2\lambda)\Gamma_3}\right)p_2 + \left(\tau(\eta) - \frac{1}{4(1+2\lambda)\Gamma_3}\right)q_2\right],$$

where

$$\tau(\eta) = \frac{1-\eta}{4(1+2\lambda)\Gamma_3 - 2(1+3\lambda)\Gamma_2^2}.$$

Since all B_j are real and $B_1 > 0$, we have

$$|a_3 - \eta a_2^2| \leq \begin{cases} 2B_1\ cos\gamma|\tau(\eta)|, \\ \quad |\tau(\eta)| > \frac{1}{4(1+2\lambda)\Gamma_3}, \\ \frac{B_1 cos\gamma}{2(1+2\lambda)\Gamma_3}, \\ \quad |\tau(\eta)| \leq \frac{1}{4(1+2\lambda)\Gamma_3}. \end{cases}$$

which completes the proof. By taking $\lambda = 0$ we deduce the following:

Theorem 5.4 *Let the function* f *given by* (1.1) *be in the class* $\mathcal{S}^q_\Sigma(\gamma, h)$ *and* $\eta \in \mathbb{R}$. *Then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} 2B_1\ cos\gamma|\tau(\eta)|, |\tau(\eta)| > \frac{1}{4\Gamma_3}, \\ \frac{B_1 cos\gamma}{2\Gamma_3}, |\tau(\eta)| \leq \frac{1}{4\Gamma_3}. \end{cases} \tag{5.5}$$

where

$$\tau(\eta) = \frac{1-\eta}{4\Gamma_3 - 2\Gamma_2^2}.$$

By taking $\lambda = 1$ in Theorem 5.7we deduce the following:

Theorem 5.5 *Let the function f given by* (1.1) *be in the class $\mathcal{K}^q_\Sigma(\gamma, h)$ and $\eta \in \mathbb{R}$. Then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} 2B_1 \, cos\gamma |\tau(\eta)|, |\tau(\eta)| > \frac{1}{12\Gamma_3}, \\ \frac{B_1 cos\gamma}{6\Gamma_3}, |\tau(\eta)| \leq \frac{1}{12\Gamma_3}. \end{cases} \tag{5.6}$$

where

$$\tau(\eta) = \frac{1-\eta}{12\Gamma_3 - 8\Gamma_2^2}.$$

Proceeding on lines similar to the proof of Theorem 5.7we state the following Fekete-Szegö inequality theorems without proof.

Theorem 5.6 *Let the function f given by* (1.1) *be in the class $\mathcal{G}^q_\Sigma(\gamma, \alpha, h)$ and $\eta \in \mathbb{R}$. Then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} 2B_1 \, cos\gamma |\tau(\eta)|, \\ \quad |\tau(\eta)| > \frac{1}{4(3-2\alpha)\Gamma_3}, \\ \frac{B_1 cos\gamma}{2(3-2\alpha)\Gamma_3}, \\ \quad |\tau(\eta)| \leq \frac{1}{4(3-2\alpha)\Gamma_3}. \end{cases} \tag{5.7}$$

where

$$\tau(\eta) = \frac{1-\eta}{4(3-2\alpha)\Gamma_3 + \{(\alpha-2)^2 - 3(4-3\alpha)\}\Gamma_2^2}.$$

Theorem 5.7 *Let the function f given by* (1.1) *be in the class $\mathcal{T}^q_\Sigma(\beta, \lambda, h)$ and $\eta \in \mathbb{R}$, then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} 2B_1 \, cos\gamma |\tau(\eta)|, \\ |\tau(\eta)| > \frac{1}{2(3-\lambda)\Gamma_3}, \\ \frac{B_1 cos\gamma}{(3-\lambda)\Gamma_3}, \\ |\tau(\eta)| \leq \frac{1}{2(3-\lambda)\Gamma_3}. \end{cases} \tag{5.8}$$

where

$$\tau(\eta) = \frac{1-\eta}{2(\lambda^2 - 2\lambda)\Gamma_2^2 + 2(3-\lambda)\Gamma_3}. \tag{5.9}$$

Theorem 5.8 *Let the function f given by* (1.1) *be in the class $\mathcal{P}^q_\Sigma(\beta, \lambda, h)$ and $\eta \in \mathbb{R}$, then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} 2B_1 \, cos\gamma |\tau(\eta)|, \\ |\tau(\eta)| > \frac{1}{6(3-\lambda)\Gamma_3}, \\ \frac{B_1 cos\gamma}{3(3-\lambda)\Gamma_3}, \\ |\tau(\eta)| \leq \frac{1}{6(3-\lambda)\Gamma_3}. \end{cases} \tag{5.10}$$

where

$$\tau(\eta) = \frac{1-\eta}{8(\lambda^2 - 2\lambda)\Gamma_2^2 + 6(3-\lambda)\Gamma_3}. \tag{5.11}$$

Theorem 5.9 *Let the function f given by* (1.1) *be in the class $\mathcal{H}^q_\Sigma(\beta, h)$ and $\eta \in \mathbb{R}$. Then*

$$|a_3 - \eta a_2^2| \leq \begin{cases} 2B_1 \, cos\gamma |\tau(\eta)|, \\ \quad |\tau(\eta)| > \frac{1}{6\Gamma_3}, \\ \frac{B_1 cos\gamma}{3\Gamma_3}, \\ \quad |\tau(\eta)| \leq \frac{1}{6\Gamma_3}. \end{cases} \tag{5.12}$$

where

$$\tau(\eta) = \frac{1-\eta}{6\Gamma_3}.$$

6 Conclusion

On specializing the parameter λ, as mentioned in Remark 2.5 and Remark 2.6 we can deduce other interesting corollaries and consequences of our main results (which are asserted by Theorems 2.11 and 5.7).From Corollary 2.13,we can state the results for the functions f in the subclasses ($\mathcal{S}^\gamma_\Sigma(\lambda, \beta)$ and $\mathcal{K}^\gamma_\Sigma(\lambda, \beta)$) defined in Remark 2.7. The details involved may be left as an exercise for the interested reader.Further $q \to 1^-$ and taking $\gamma = 0$ our result given in Theorem 2.11improved the results obtained in [29] and also for the classes discussed in this study.

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