The Boundary Value Condition of a Degenerate Parabolic Equation

Huashui Zhan
School of Applied Mathematics
Xiamen University of Technology
CHINA
2012111007@xmut.edu.cn

Abstract: By Fichera-Oleinik Rule, how to give a homogeneous boundary condition to assure the posedness of the equation

$$\partial_{xx}u + u\partial_y u - \partial_t u = f(x, y, t, u), (x, y, t) \in Q_T = \Omega \times (0, T),$$

is researched. By introducing a new kind of entropy solution, in which the trace $\gamma(\widehat{\frac{\partial u}{\partial x_i}})$, $x_i = x$ or y, on the boundary of Ω is avoided. By the parabolic regularization method, the uniformly estimate of the gradient is obtained, and using Kolmogoroff's theorem, the solvability of the equation in $BV(Q_T)$ is obtained.

Key–Words: Degenerate parabolic equation, Fichera-Oleinik Rule, boundary value condition, entropy solution, trace.

1 Introduction

In this paper, we consider the initial boundary value problem of the following partial differential equation

$$\partial_{xx}u + u\partial_y u - \partial_t u = f(\cdot, u), \tag{1}$$

where $(x, y, t) \in Q_T = \Omega \times (0, T)$, $\Omega \subset R^2$ is a bounded domain with the suitably smooth boundary $\partial\Omega$. The equation (1) arises in mathematics finance [1], arises when studying nonlinear physical phenomena such as the combined effects of diffusion and convection of matter [2]. Antonelli, Barucci and Mancino [1] introduced a new model for agent's decision under risk, in which the utility function is the solution of equation (1). Under the assumption of that f is a uniformly Lipschitz continuous function, Crandall, Ishii and Lions [3], Citti, Pascucci and Polidoro [4], Antonelli and Pascucci [5], step by step, they had proved that there is a local classical solution of Cauchy problem of equation (1).

Clearly, equation (1) is a degenerate parabolic equation on account of that it lacks the second order partial derivative term $\partial_{yy}u$. As for the existence and uniqueness of the global weak solution for Cauchy problem of equation (1), there are some different ways to deal with them, for examples, equation (1) is the special case of the degenerate parabolic equations discussed in [6-7] etc. The other related results in the posedness of the degenerate parabolic equations, one can refer to [8-12, 27-28]. However, the author [13] had shown that the global weak solution of equation (1) can not be a classical solution generally. In other words, some blow-up phenomena happen in finite time.

As for the boundary value problem of equation (1). If $\Omega = (0, R) \times (0, N) \subset R^2$ and the nonnegative solutions is considered, according to Fichera-Oleinik Rule, using Oleinik's line method (see [14]], the author had got the local classical solution of (1) in [15],[16] . If $\Omega \subset R^2$ is a general bounded domain with suitably smooth boundary $\partial\Omega$, very recently, the author [17] has researched the posedness of the solutions of equation (1) with the initial value condition

$$u(x, y, 0) = u_0(x, y),\ (x, y) \in \Omega, \tag{2}$$

and the part boundary value condition. In details, only on the portion of the boundary

$$\Sigma_3 = \{(x, y, t) \in \partial\Omega \times [0, T) : n_1(x, y, t) \neq 0\}, \tag{3}$$

is endowed the homogeneous boundary value condition

$$u(x, y, t) = 0,\ (x, y, t) \in \Sigma_3, \tag{4}$$

and $\vec{n} = \{n_1, n_2, 0\}$ is the inner unit normal vector of $\partial\Omega$.

However, [17] did not explain that why the homogeneous boundary value condition (4) is endowed on the portion of the boundary (3). In this paper, we shall give an explanation to the conditions (3)-(4) by Fichera-Oleinik Rule, then we shall give a new kind of entropy solution to the problem (1)-(2)-(4), the advantage of the approach was that the boundary condition was implicitly contained in the entropy inequalities.

We shall use the general parabolic regularization method to prove the existence of the solution, in other words, the initial boundary value problem of the following equation is considered

$$\varepsilon\Delta u_\varepsilon + \partial_{xx}u_\varepsilon + u_\varepsilon\partial_y u_\varepsilon - \partial_t u_\varepsilon$$

$$= f(x,y,t,u_\varepsilon),(x,y,t)\in\Omega\times(0,T). \quad (5)$$

In order to prove the compactness of $\{u_\varepsilon\}$, we need some estimates on $\{u_\varepsilon\}$. However, since for the limit function u of a certain subsequence of $\{u_\varepsilon\}$, $\widehat{\frac{\partial u}{\partial x}}$ and $\widehat{\frac{\partial u}{\partial y}}$ need not have the trace $\gamma(\widehat{\frac{\partial u}{\partial x_i}})$, $x_i = x$ or y, on the boundary of Ω, we have to make a detour to avoid $\gamma(\widehat{\frac{\partial u}{\partial x_i}})$ in defining the solution, this method is inspired by [18]. At the same time, we use some ideas of [6-7] to prove the existence of the solution. As for the stability of the solutions, it is able to be proved by a similar way as [17].

2 Fichera-Oleinik Rules and the applications

Early as in 50-60s of the last century, Fichera [19-20] and Oleǐnik [21-22] developed and perfected the general theory of second order equation with nonnegative characteristic form, which, in particular contains those degenerating on the boundary. We can call it as Fichera-Oleǐnik Rule. By the rule, for a linear degenerate elliptic equation,

$$\sum_{r,s=1}^{N+1} a^{rs}(x)\frac{\partial^2 u}{\partial x_r\partial x_s} + \sum_{r=1}^{N+1} b_r(x)\frac{\partial u}{\partial x_r} + c(x)u$$

$$= f(x), x\in\widetilde{\Omega}\subset R^{N+1} \quad (6)$$

if one wants to consider the boundary value problem of (6), it needs and only needs to give part boundary condition. In details, let $\{n_s\}$ be the inner unit normal vector of $\partial\widetilde{\Omega}$ and denote that

$$\Sigma_2 = \{x\in\partial\widetilde{\Omega} : a^{rs}n_r n_s = 0, (b_r - a^{rs}_{x_s})n_r < 0\},$$

$$\Sigma_3 = \{x\in\partial\widetilde{\Omega} : a^{rs}n_s n_r > 0\}.$$

Then, to ensure the posedness of equation (6), Fichera-Oleinik Rule tells us that the suitable boundary condition is

$$u|_{\sum_2\bigcup\sum_3} = g(x). \quad (7)$$

In particular, if the matrix (a^{rs}) is positive definite, (7) is just the usual Dirichlet boundary condition.

Now, for the nonlinear heat equation

$$u_t = \triangle A(u), \quad (8)$$

with the existence of A^{-1}, in other words, equation (8) is weakly degenerate, then let $v = A(u), u = A^{-1}(v)$,

$$\triangle v - (A^{-1}(v))_t = 0. \quad (9)$$

According to Fichera-Oleinik Rule, we know that we can require the whole Dirichlet homogeneous boundary condition.

But, when considering the following anisotropic degenerate parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i}(a^{ij}(u)\frac{\partial u}{\partial x_j})$$

$$+\frac{\partial b_i(u)}{\partial x_i} + f(u,x,t),(x,t)\in\Omega\times(0,T), \quad (10)$$

if the inverse matrix $A^{-1} = (a_{ij})^{-1}$ is not existential, we can not deal with it as (9). Rewrite equation (10) as

$$\frac{\partial u}{\partial t} = a^{ij}(u)\frac{\partial^2 u}{\partial x_i\partial x_j} + a^{ij'}(u)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}$$

$$+\sum_{i=1}^{N} b_i'(u)\frac{\partial u}{\partial x_i} + f(u,x,t);\ in\ Q_T = \Omega\times(0,T), \quad (11)$$

the domain is a cylinder $\Omega\times(0,T)$. If we let $t = x_{N+1}$ and regard the degenerate parabolic equation (11) as the form of a "linear" degenerate elliptic equation as (6),

$$(\widetilde{a}^{rs})_{(N+1)\times(N+1)} = \begin{pmatrix} a^{ij} & 0 \\ 0 & 0 \end{pmatrix}.$$

If $a^{ij}(0) = 0$, which means that equation (11) is not only strongly degenerate in the interior of Ω, but also on the boundary $\partial\Omega$. Then $\sum_3$ is an empty set. While

$$\widetilde{b}_s(x,t) = \begin{cases} b_i'(u) + a^{ij'}(u)\frac{\partial u}{\partial x_j}, & 1\le s = i\le N, \\ -1, & s = N+1, \end{cases} \quad (12)$$

Under this observation, according to Fichera-Oleǐnik Rule, the initial value condition (2) is always needed, but on the lateral boundary $\partial\Omega\times(0,T)$, by $a^{ij}(0) = 0$, the portion of the boundary on which we can give the boundary value is

$$\Sigma_p = \{x\in\partial\Omega : (b_i'(0)$$

$$+a^{ij'}(0)\frac{\partial u}{\partial x_j}|_{x\in\partial\Omega} - a^{ij'}(0)\frac{\partial u}{\partial x_j}|_{x\in\partial\Omega})n_i < 0\}$$

$$= \{x \in \partial\Omega : b_i'(0)n_i < 0\}. \tag{13}$$

where $\{n_i\}$ be the unit inner normal vector of $\partial\Omega$.

Though (13) seems reasonable and beautiful, whether the term $\frac{\partial u}{\partial x_i}|_{x\in\partial\Omega}$ has a explicit definition is unclearly, unless that equation (10) has a classical solution. In fact, due to the strongly degenerate property of (a^{ij}), equation (10) generally only has weak solution. For example, if we consider the solution of equation (10) in BV sense, then we can not define the trace of $\frac{\partial u}{\partial x_i}$ on $\partial\Omega$, which means that we also can not define

$$\Sigma_p = \{x \in \partial\Omega : (b_i'(0)$$

$$+a^{ij'}(0)\frac{\partial u}{\partial x_j}|_{x\in\partial\Omega} - a^{ij'}(0)\frac{\partial u}{\partial x_j}|_{x\in\partial\Omega})n_i < 0\}.$$

Fortunately, only if $b_i(s)$ is derivable, then

$$\Sigma_p = \{x \in \partial\Omega : b_i'(0)n_i < 0\}. \tag{14}$$

has a definite sense, and we can conjecture that we can require the homogeneous boundary value condition on it, one can refer to [27].

If without the assumption of that $a^{ij}(0) = 0$, according to Fichera-Oleinik Rule, except the initial value (2), the suitable homogeneous boundary value for equation (11) is

$$u\mid_{\sum_p} = 0, \tag{15}$$

where

$$\sum_p = \{x \in \partial\Omega : b_i'(0)n_i(x) < 0\}$$

$$\bigcup \{x \in \partial\Omega : a^{ij}(0)n_i(x)n_j(x) > 0\}.$$

Let us come back to the main equation (1) in our paper. By comparing (1) to (6), we has the special form

$$(a^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$b_2(u) = \frac{1}{2}u^2,\ b_1(u) = 0.$$

By the above discussion, according to Fichera-Oleǐnik Rule, the initial value condition (2) is always needed, but on the lateral boundary $\partial\Omega \times (0,T)$, it is not difficult to observe that

$$\Sigma_p = \{x \in \partial\widetilde{\Omega} : a^{ij}n_in_j > 0\} = \{x \in \partial\widetilde{\Omega} : n_1(x) \neq 0\}.$$

Thus we give a reasonable explanation of the boundary value condition (4).

3 The definition of the solution

Following references [23-24], $u \in BV(Q_T), Q_T = R^N \times (0,T)$, if and only if $u \in L^1_{loc}(Q_T)$ and

$$\int_0^T \int_{B_\rho} \mid u(x_1+h_1,\cdots,x_N+h_N,t+h_{N+1})$$

$$-u(x,t)\mid dxdt \leq K \mid h \mid,$$

where

$$B_\rho = \{x \in R^N; \mid X \mid < \rho\},\ h = (h_1,h_2,\cdots h_N,h_{N+1})$$

and K is a positive constant. This is equivalent to that the generalized derivatives of every function in $BV(Q_T)$ are regular Radon measures on Q_T.

Let Γ_u be the set of all jump points of $u \in BV(Q_T)$, v the normal of Γ_u at $X = (x,t)$, $u^+(X)$ and $u^-(X)$ the approximate limits of u at $X \in \Gamma_u$ with respect to $(v, Y - X) > 0$ and $(v, Y - X) < 0$ respectively. For continuous function $p(u,x,t)$ and $u \in BV(Q_T)$, define

$$\widehat{p}(x,t,u) = \int_0^1 p(x,t,\tau u^+ + (1-\tau)u^-)d\tau,$$

which is called the composite mean value of p. For a given t, we denote $\Gamma_u^t,\ H^t, (v_1^t,\ \cdots, v_N^t)$ and $u_\pm^t$ as all jump points of $u(\cdot,t)$, Housdorff measure of Γ_u^t, the unit normal vector of Γ_u^t, and the asymptotic limit of $u(\cdot,t)$ respectively. Moreover, if $f(s) \in C^1(R),\ u \in BV(Q_T)$, then $f(u) \in BV(Q_T)$ and

$$\frac{\partial f(u)}{\partial x_i} = \widehat{f'}(u)\frac{\partial u}{\partial x_i},\ i = 1,2,\cdots,N. \tag{16}$$

Let $S_\eta(s) = \int_0^s h_\eta(\tau)d\tau$ for small $\eta > 0$. Here $h_\eta(s) = \frac{2}{\eta}(1 - \frac{|s|}{\eta})_+$. Obviously $h_\eta(s) \in C(R)$ and

$$h_\eta(s) \geq 0,\ \mid sh_\eta(s) \mid \leq 1,\ \mid S_\eta(s) \mid \leq 1;$$

$$\lim_{\eta\to 0} S_\eta(s) = sgn(s),\ \lim_{\eta\to 0} sS_\eta'(s) = 0, \tag{17}$$

where sgn represents the sign function. In what follows, the dimension of the space variables is $N = 2$.

Definition 1 *A function* u *is said to be the entropy solution of (1)-(2)-(4), if*

1. $u \in BV(Q_T) \cap L^\infty(Q_T)$*, and there exist the function* $g^1 \in L^2(Q_T)$*, such that*

$$\int\int_{Q_T} g^1(x,y,t)\varphi(x,y,t)dxdydt$$

$$= \int\int_{Q_T} \frac{\partial u}{\partial x}\varphi(x,y,t)dxdydt, \tag{18}$$

for any $\varphi(x,y,t) \in L^2(Q_T)$.

2. For any $\varphi_1, \varphi_2 \in C^2(\bar{Q}_T)$, $\varphi_1 \geq 0$, $\varphi_{1x} \mid_{\partial\Omega}= 0$, $\varphi_1 \mid_{\Sigma\setminus\Sigma_3}= 0$, $\varphi_1 \mid_{\partial\Omega\times[0,T]}= \varphi_2 \mid_{\partial\Omega\times[0,T]}$, *and* $supp\varphi_2$, $supp\varphi_1 \subset \bar{\Omega}\times(0,T)$, *for any* $k \in R$, *for any small* $\eta > 0$, u *satisfies*

$$\int\int_{Q_T}[I_\eta(u-k)\varphi_{1t} - B_\eta(u,k)\varphi_{1y} + I_\eta(u-k)\varphi_{1xx}$$

$$-f(\cdot,u)S_\eta(u-k)\varphi_1 - S'_\eta(u-k)(\partial_x u)^2\varphi_1]dxdydt$$

$$+S_\eta(k)\int\int_{Q_T}[u\varphi_{2t} + \frac{u^2}{2}\varphi_{2y} + u\varphi_{2xx}]dxdydt$$

$$-\frac{1}{2}\int_0^T\int_{\Sigma_3}\int_k^0 s^2 S_\eta(s-k)ds\varphi_1 n_2 dxd\sigma \geq 0. \quad (19)$$

for any $k \in R,\ \eta > 0$. *Here*

$$B_\eta(u,k) = -\int_k^u sS_\eta(s-k)ds,$$

$$I_\eta(u-k) = \int_0^{u-k} S_\eta(s)ds.$$

3. The initial value is satisfies in the sense

$$\lim_{t\to 0}\int_\Omega \mid u(x,y,t) - u_0(x,y) \mid dxdy = 0. \quad (20)$$

Clearly, by (19), we have

$$\int\int_{Q_T}[I_\eta(u-k)\varphi_{1t} - B_\eta(u,k)\varphi_{1y} + I_\eta(u-k)\varphi_{1xx}$$

$$-f(\cdot,u)S_\eta(u-k)\varphi_1]dxdydt$$

$$+S_\eta(k)\int\int_{Q_T}[u\varphi_{2t} + \frac{u^2}{2}\varphi_{2y} + u\varphi_{2xx}]dxdydt$$

$$-\frac{1}{2}\int_0^T\int_{\sum_3}\int_k^0 s^2 S_\eta(s-k)ds\varphi_1 n_2 dxd\sigma \geq 0.$$

Let $\eta \to 0$ in this inequality. One has

$$\int\int_{Q_T}[|u-k|\varphi_{1t} + \frac{1}{2}sgn(u-k)(u^2-k^2)\varphi_{1y} + |u-k|\varphi_{1xx}$$

$$-f(\cdot,u)sgn(u-k)\varphi_1]dxdydt$$

$$+sgn(k)\int\int_{Q_T}[u\varphi_{2t} + \frac{u^2}{2}\varphi_{2y} + u\varphi_{2xx}]dxdydt$$

$$-\frac{1}{2}\int_0^T\int_{\sum_3}\int_k^0 s^2 S_\eta(s-k)ds\varphi_1 n_2 dxd\sigma \geq 0.$$

Moreover, let $\varphi_2 = 0$ and so $\varphi_1 \mid_{\Sigma_3}= 0$. We have

$$\int\int_{Q_T}[|u-k|\varphi_{1t} + \frac{1}{2}sgn(u-k)(u^2-k^2)\varphi_{1y}$$

$$+|u-k|\varphi_{1xx} - f(\cdot,u)sgn(u-k)\varphi_1]dxdt \geq 0.$$

This is just the entropy solution defined in [23-24]. Thus if u is the entropy solution in Definition 1, then u is a entropy solution defined in general cases.

By the way, it is clear of that the function $\varphi(x,t) \in C_0^\infty(Q_T)$ satisfies the request of function φ_1 in Definition 2.1. We do not choose the function $\varphi(x,t) \in C_0^\infty(Q_T)$ as the testing function in Definition 1, since we need the boundary value of φ_1, on which the relationship between itself with another testing function φ_2 can be based. Thus, we can succeed to avoid the trace of $\gamma(\widehat{\frac{\partial u}{\partial x_i}})$ in defining the solution.

We shall prove the following theorem.

Theorem 2 *Suppose* $u_0(x) \in L^\infty(\Omega)\bigcap C^2(\Omega)$. *If* f_x, f_y, f_t *are bounded functions, and* f_u *is bounded too when* u *is bounded, then the equation (1) with the initial boundary value conditions (2)(3) has a entropy solution in the sense of Definition 1.*

4 Proof of Theorem 2

Lemma 3 [25] *Assume that* $\Omega \subset R^N$ *is an open bounded set and let* $g_k, f \in L^q(\Omega)$, *as* $k \to \infty$, $g_k \rightharpoonup f$ *weakly in* $L^q(\Omega), 1 \leq q < \infty$. *Then*

$$\inf\lim_{k\to\infty} \parallel g_k \parallel^q_{L^q(\Omega)} \geq \parallel g \parallel^q_{L^q(\Omega)} .$$

We now consider the following regularized problem

$$\varepsilon\Delta u + \partial_{xx}u + u\partial_y u - \partial_t u = f(\cdot,u), (x,y,t) \in \Omega\times(0,T), \quad (21)$$

with the compatible initial value (2) and the homogeneous boundary value condition

$$u(x,y,t) = 0, (x,y,t) \in \Sigma = \partial\Omega\times[0,T). \quad (22)$$

Under the assumptions of Theorem 2, it is well known that there is a classical solution u_ε of the initial boundary value problem of (21) with (2)-(22), e.g. one can refer to the chapter 8 of [26].

We need to make some estimates for u_ε of (21). Firstly, since $u_0(x) \in L^\infty(\Omega)$, by the maximum principle, we have

$$\mid u_\varepsilon \mid \leq \parallel u_0 \parallel_{L^\infty} \leq c. \quad (23)$$

Secondly, let's make the BV estimates of u_ε.

Lemma 4 [18] *Let* u_ε *be the solution of (21) with (2)-(22). If the assumptions of Theorem 2 are true, then*

$$\varepsilon\int_{\partial\Omega}|\frac{\partial u_\varepsilon}{\partial n}|d\sigma \leq c_1 + c_2(|gradu_\varepsilon|_{L^1(\Omega)} + |\frac{\partial u_\varepsilon}{\partial t}|_{L^1(\Omega)}).$$

with constants $c_i, i = 1,2$ *independent of* ε, *where* $\vec{n} = \{n_1, n_2\}$ *is the inner normal vector of* Ω.

Theorem 5 *Let u_ε be the solution of (21)-(2)-(22). If the assumptions of Theorem 2 are true, then*

$$|gradu_\varepsilon|_{L^1(\Omega)} \le c. \tag{24}$$

where $|gradu|^2 = \sum_{i=1}^{2} |\frac{\partial u}{\partial x_i}|^2 + |\frac{\partial u}{\partial t}|^2$, c is independent of ε, and $x_1 = x, x_2 = y$.

Proof In what follows, we simply denote the solution of (21)-(2)-(22), u_ε, as u, denote $x_1 = x, x_2 = y, x_3 = t$, $dx = dx_1dx_2$, and the dual index of i represents the sum from 1 to 2, the dual index of s or p represents the sum from 1 to 3. Differentiate (21) with respect to x_s, $s = 1, 2, 3$, and sum up for s after multiplying the resulting relation by $u_{x_s}\frac{S_\eta(|gradu|)}{|gradu|}$, then integrating over Ω yields

$$\int_\Omega \frac{\partial u_{x_s}}{\partial t} u_{x_s} \frac{S_\eta(|gradu|)}{|gradu|} dx$$

$$= \int_\Omega \frac{\partial}{\partial t} \int_0^{|gradu|} S_\eta(\tau) d\tau dx = \frac{d}{dt} \int_\Omega I_\eta(|gradu|) dx. \tag{25}$$

$$\begin{aligned} &\int_\Omega \frac{\partial}{\partial x_s} \frac{\partial}{\partial x} (\frac{\partial u}{\partial x}) u_{x_s} \frac{S_\eta(|gradu|)}{|gradu|} dx \\ &= \int_\Omega \frac{\partial}{\partial x}(u_{xx_s}) u_{x_s} \frac{S_\eta(|gradu|)}{|gradu|} dx \\ &= \int_\Omega \frac{\partial}{\partial x}(u_{xx_s}) \frac{\partial}{\partial \xi_s} I_\eta(|gradu|) dx \\ &= \int_{\partial\Omega} u_{xx_s} n_1 \frac{\partial}{\partial \xi_s} I_\eta(|gradu|) d\sigma \\ &- \int_\Omega \frac{\partial^2 I_\eta(|gradu|)}{\partial \xi_s \partial \xi_p} u_{x_sx} u_{x_px} dx. \end{aligned} \tag{26}$$

where $\xi_s = u_{x_s}$, $d\sigma$ is the surface integrable unit.

$$\begin{aligned} &\varepsilon \int_\Omega \triangle u_{x_s} u_{x_s} \frac{S_\eta(|gradu|)}{|gradu|} dx \\ &= \varepsilon \int_{\partial\Omega} \frac{\partial I_\eta(|gradu|)}{\partial x_i} n_i d\sigma \\ &- \varepsilon \int_\Omega \frac{\partial^2 I_\eta(|gradu|)}{\partial \xi_s \partial \xi_p} u_{x_sx_i} u_{x_px_i} dx. \end{aligned} \tag{27}$$

$$\begin{aligned} &- \int_\Omega \frac{\partial (uu_y)}{\partial x_s} u_{x_s} \frac{S_\eta(|gradu|)}{|gradu|} dx \\ &= - \int_\Omega \frac{\partial u}{\partial y} \left[|gradu| S_\eta(|gradu|) - I_\eta(|gradu|) \right] dx \\ &- \int_{\partial\Omega} u I_\eta(|gradu|) n_2 d\sigma \\ &= - \int_\Omega \frac{\partial u}{\partial y} \left[|gradu| S_\eta(|gradu|) - I_\eta(|gradu|) \right] dx. \end{aligned} \tag{28}$$

By the assumption of that f_t, f_x, f_y are bounded, and f_u is bounded due to $|u| \le c$, then

$$\int_\Omega \frac{\partial f(x,y,t,u)}{\partial x_s} u_{x_s} \frac{S_\eta(|gradu|)}{|gradu|} dx \le c \int_\Omega |gradu| dx. \tag{29}$$

From (25)-(29), we have

$$\begin{aligned} \frac{d}{dt} \int_\Omega I_\eta(|gradu|) dx &= - \int_\Omega \frac{\partial^2 I_\eta(|gradu|)}{\partial \xi_s \partial \xi_p} u_{x_sx} u_{x_px} dx \\ &- \varepsilon \int_\Omega \frac{\partial^2 I_\eta(|gradu|)}{\partial \xi_s \partial \xi_p} u_{x_sx_i} u_{x_px_i} dx \\ &- \int_\Omega \frac{\partial u}{\partial y} \left[|gradu| S_\eta(|gradu|) - I_\eta(|gradu|) \right] dx \\ &+ \int_{\partial\Omega} \frac{\partial I_\eta(|gradu|)}{\partial x_1} n_1 d\sigma \\ &+ \varepsilon \int_{\partial\Omega} \frac{\partial I_\eta(|gradu|)}{\partial x_i} n_i d\sigma \\ &+ \int_\Omega \frac{\partial f(t,x,y,u)}{\partial x_s} u_{x_s} \frac{S_\eta(|gradu|)}{|gradu|} dx. \end{aligned} \tag{30}$$

Observing that on $\Sigma = \partial\Omega \times [0, T)$,

$$u = 0, u_{x_3}|_\Sigma = u_t|_\Sigma = 0,$$

and so we have

$$\partial_{xx} u \mid_\Sigma + \varepsilon \triangle u \mid_\Sigma = f(x, y, t, 0). \tag{31}$$

then the surface integral in (30) just remains the following term

$$S = \varepsilon \int_{\partial\Omega} \frac{\partial I_\eta(|gradu|)}{\partial x_i} n_i d\sigma + \int_{\partial\Omega} \frac{\partial I_\eta(|gradu|)}{\partial x_1} n_1 d\sigma$$

By Lemma 4, using (31), it is able to deduce that $\lim_{\eta\to 0} S$ can be estimated by $|gradu|_{L_1(\Omega)}$, one can refer to [18] for details.

Thus, by (30), letting $\eta \to 0$, and noticing that

$$\lim_{\eta\to 0} \left[|gradu| S_\eta(|gradu|) - I_\eta(|gradu|) \right] = 0,$$

we have

$$\frac{d}{dt} \int_\Omega |gradu| dx \le c_1 + c_2 \int_\Omega |gradu| dx, \tag{32}$$

by the well-known Gronwall Lemma, we have

$$\int_\Omega |gradu| dx \le c, \tag{33}$$

where c is constant independent of t. By (33), using the equation (21), it is easy to show that

$$\int\int_{Q_T} |u_{x_1}|^2 dx_1 dx_2 dt = \int\int_{Q_T} |u_x|^2 dxdydt \leq c. \tag{34}$$

Now, we denote back that u_ε is the solution of (21). Thus by Kolmogoroff's theorem, there exists a subsequence $\{u_{\varepsilon_n}\}$ of u_ε and a function $u \in BV(Q_T) \cap L^\infty(Q_T)$ such that u_{ε_n} is strongly convergent to u, and so $u_{\varepsilon_n} \to u$ a.e. on Q_T. By (34), there exist functions $g^1 \in L^2(Q_T)$ and a subsequence of $\{\varepsilon\}$, we can simply denote this subsequence as ε itself, such that when $\varepsilon \to 0$,

$$\frac{\partial u_\varepsilon}{\partial x} \rightharpoonup g^1, \ in \ L^2(Q_T).$$

We now prove that u is a generalized solution of the original problem (1)-(2)-(4). Let $\varphi \in C^2(\bar{Q}_T), \ \varphi_1 \geq 0$, supp$\varphi \subset \bar{\Omega} \times (0,T)$, $\varphi \mid_{\Sigma \setminus \Sigma_3} = 0$, $\varphi_{1x} \mid_{\partial\Omega} = 0$. Multiply (21) by $\varphi_1 S_\eta(u_\varepsilon - k)$, and integrate over Q_T, obtain

$$\int\int_{Q_T} \frac{\partial u_\varepsilon}{\partial t} \varphi_1 S_\eta(u_\varepsilon - k)dxdydt$$
$$= \int\int_{Q_T} \frac{\partial}{\partial x}(\frac{\partial u_\varepsilon}{\partial x})\varphi_1 S_\eta(u_\varepsilon - k)dxdydt$$
$$+\varepsilon \int\int_{Q_T} \Delta u_\varepsilon \varphi_1 S_\eta(u_\varepsilon - k)dxdydt$$
$$+\int\int_{Q_T} u_\varepsilon u_{\varepsilon y} \varphi_1 S_\eta(u_\varepsilon - k)dxdydt$$
$$+\int\int_{Q_T} f(x,y,t,u_\varepsilon)\varphi_1 S_\eta(u_\varepsilon - k)dxdydt. \tag{35}$$

Let's calculate every term in (35) by the part integral method.

$$\int\int_{Q_T} \frac{\partial u_\varepsilon}{\partial t} \varphi_1 S_\eta(u_\varepsilon - k)dxdydt$$
$$= -\int\int_{Q_T} I_\eta(u_\varepsilon - k)\varphi_{1t} dxdydt. \tag{36}$$

$$\varepsilon \int\int_{Q_T} \Delta u_\varepsilon \varphi_1 S_\eta(u_\epsilon - k)dxdydt$$
$$= \varepsilon \int_0^T \int_{\partial\Omega} \nabla u_\varepsilon \cdot \overrightarrow{n} \varphi_1 S_\eta(u_\epsilon - k)dtd\sigma$$
$$-\varepsilon \int\int_{Q_T} \nabla u_\varepsilon(S_\eta(u_\varepsilon - k)\nabla\varphi_1 + \varphi_1 S'_\eta(u_\varepsilon - k)\nabla u_\varepsilon)dxdydt$$
$$= -\varepsilon S_\eta(k) \int_0^T \int_{\partial\Omega} \nabla u_\varepsilon \cdot \overrightarrow{n} \varphi_1 dtd\sigma$$
$$-\varepsilon \int\int_{Q_T} \nabla u_\varepsilon S_\eta(u_\varepsilon - k)\nabla\varphi_1 dxdydt$$
$$-\varepsilon \int\int_{Q_T} \mid \nabla u_\varepsilon \mid^2 S'_\eta(u_\varepsilon - k)\varphi_1 dxdydt, \tag{37}$$

$$\int\int_{Q_T} \partial_{xx} u_\varepsilon \varphi_1 S_\eta(u_\varepsilon - k)dxdydt$$
$$= -S_\eta(k) \int_0^T \int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial x} n_1 \varphi_1 dtd\sigma$$
$$-\int\int_{Q_T} \frac{\partial u_\varepsilon}{\partial x}(S_\eta(u_\varepsilon - k)\varphi_{1x} + \varphi_1 S'_\eta(u_\varepsilon - k)u_{\varepsilon x})dxdydt$$
$$= -S_\eta(k) \int_0^T \int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial x} n_1 \varphi_1 dtd\sigma$$
$$-\int\int_{Q_T} \frac{\partial u_\varepsilon}{\partial x} S_\eta(u_\varepsilon - k)\varphi_{1x} dxdydt$$
$$-\int\int_{Q_T} u^2_{\varepsilon x} S'_\eta(u_\varepsilon - k)\varphi_1 dxdydt, \tag{38}$$

$$-\int\int_{Q_T} \frac{\partial u_\varepsilon}{\partial x} S_\eta(u_\varepsilon - k)\varphi_{1x} dxdydt$$
$$= \int\int_{Q_T} I_\eta(u_\varepsilon - k)\varphi_{1xx} dxdydt$$
$$-\int_0^T \int_{\partial\Omega} \varphi_{1x} n_1 I_\eta(u_\varepsilon - k)dtd\sigma, \tag{39}$$

$$\int\int_{Q_T} u_\varepsilon \frac{\partial u_\varepsilon}{\partial y} \varphi_1 S_\eta(u_\varepsilon - k)dxdydt$$
$$= \frac{1}{2} \int_0^T \int_{\partial\Omega} (u^2_\varepsilon - k^2)n_2 \varphi_1 S_\eta(u_\varepsilon - k)dtd\sigma$$
$$-\frac{1}{2} \int\int_{Q_T} [u^2_\varepsilon - k^2][\frac{\partial \varphi_1}{\partial y} S_\eta(u_\varepsilon - k)$$
$$+\varphi_1 S'_\eta(u_\varepsilon - k)\frac{\partial u_\varepsilon}{\partial y}]dxdydt$$
$$= -\int\int_{Q_T} B_\eta(u_\varepsilon, k)\varphi_{1y} dxdydt$$
$$-\frac{1}{2} \int_0^T \int_{\partial\Omega} \int_k^0 s^2 S_\eta(s - k)ds \varphi_1 n_2 dxd\sigma. \tag{40}$$

From (35)-(40), we have

$$\int\int_{Q_T} I_\eta(u_\varepsilon - k)\varphi_{1t} dxdydt$$
$$+\int\int_{Q_T} S_\eta(u_\varepsilon, k)\varphi_{1xx} dxdydt$$
$$-\int\int_{Q_T} B_\eta(u_\varepsilon, k)\varphi_{1y} dxdydt$$
$$-\varepsilon \int\int_{Q_T} \nabla u_\varepsilon \cdot \nabla\varphi_1 S_\eta(u_\varepsilon - k)dxdydt$$

$$-\varepsilon\int\int_{Q_T}|\nabla u_\varepsilon|^2 S'_\eta(u_\varepsilon-k)\varphi_1 dxdydt$$
$$-\int\int_{Q_T}u_{\varepsilon x}^2 S'_\eta(u_\varepsilon-k)\varphi_1 dxdydt$$
$$-\int\int_{Q_T}f(x,y,t,u)\varphi_1 S_\eta(u_\varepsilon-k)dxdydt$$
$$-\varepsilon S_\eta(k)\int_0^T\int_{\partial\Omega}\nabla u_\varepsilon\cdot\overrightarrow{n}\varphi_1 dtd\sigma$$
$$-S_\eta(k)\int_0^T\int_{\partial\Omega}\frac{\partial u_\varepsilon}{\partial x}n_1\varphi_1 dtd\sigma$$
$$-S_\eta(k)\int_0^T\int_{\partial\Omega}\varphi_{1x}n_1 dtd\sigma$$
$$-\frac{1}{2}\int_0^T\int_{\partial\Omega}\int_k^0 s^2 S_\eta(s-k)ds\varphi_1 n_2 dxd\sigma=0. \quad (41)$$

Taking $\varphi_2\in C^2(\bar{Q}_T)$,

$$\varphi_1\mid_{\partial\Omega\times[0,T]}=\varphi_2\mid_{\partial\Omega\times[0,T]},$$

and supp$\varphi_2\subset\bar{\Omega}\times(0,T)$,

$$-S_\eta(k)\int_0^T\int_{\partial\Omega}\frac{\partial u_\varepsilon}{\partial x}n_1\varphi_1 dtd\sigma$$
$$-\varepsilon S_\eta(k)\int_0^T\int_{\partial\Omega}\nabla u_\varepsilon\cdot\overrightarrow{n}\varphi_1 dtd\sigma$$
$$=S_\eta(k)[-\varepsilon\int\int_{Q_T}\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial\varphi_2}{\partial x_i}dxdydt$$
$$-\int\int_{Q_T}\frac{\partial u_\varepsilon}{\partial x}\varphi_{2x}dxdydt$$
$$-\frac{1}{2}\int\int_{Q_T}u_\varepsilon^2\frac{\partial\varphi_2}{\partial y}dxdydt+\frac{1}{2}\int_0^T\int_{\partial\Omega}u_\varepsilon^2\varphi_2 n_2 dtd\sigma$$
$$+\int\int_{Q_T}u_\varepsilon\frac{\partial\varphi_2}{\partial t}dxdydt], \quad (42)$$

$$\int\int_{Q_T}\frac{\partial u_\varepsilon}{\partial x}\varphi_{2x}dxdydt=\int_0^T\int_{\partial\Omega}u_\varepsilon\frac{\partial\varphi_2}{\partial x}n_1 dtd\sigma$$
$$-\int\int_{Q_T}u_\varepsilon\varphi_{2xx}dxdydt. \quad (43)$$

Since $\varphi_{1x}\mid_{\partial\Omega}=0$, $\varphi_1|_{\Sigma\setminus\Sigma_3}=0$, by (41)-(43), and by the homogeneous boundary value condition (22), we have

$$\int\int_{Q_T}I_\eta(u_\varepsilon-k)\varphi_{1t}dxdydt$$
$$+\int\int_{Q_T}I_\eta(u_\varepsilon-k)\triangle\varphi_1 dxdydt$$
$$-\int\int_{Q_T}B_\eta(u_\varepsilon,k)\varphi_{1y}dxdydt$$
$$+S_\eta(k)[-\varepsilon\int\int_{Q_T}\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial\varphi_2}{\partial x_i}dxdydt$$
$$+\int\int_{Q_T}u_\varepsilon\varphi_{2xx}dxdydt-\frac{1}{2}\int\int_{Q_T}u_\varepsilon^2\frac{\partial\varphi_2}{\partial y}dxdydt$$
$$+\int\int_{Q_T}u_\varepsilon\frac{\partial\varphi_2}{\partial t}dxdydt]$$
$$-\varepsilon\int\int_{Q_T}\nabla u_\varepsilon\cdot\nabla\varphi_1 S_\eta(u_\varepsilon-k)dxdydt$$
$$-\int\int_{Q_T}u_{\varepsilon x}^2 S'_\eta(u_\varepsilon-k)\varphi_1 dxdydt$$
$$-\frac{1}{2}\int_0^T\int_{\sum_3}\int_k^0 s^2 S_\eta(s-k)ds\varphi_1 n_2 dxd\sigma$$
$$-\int\int_{Q_T}f(x,y,t,u)\varphi_1 S_\eta(u_\varepsilon-k)dxdt\geq 0. \quad (44)$$

By Lemma 3,

$$\liminf_{\varepsilon\to 0}\int\int_{Q_T}S'_\eta(u_\varepsilon-k)\frac{\partial u_\varepsilon}{\partial x}\frac{\partial u_\varepsilon}{\partial x}\varphi_1 dxdydt$$
$$\geq\int\int_{Q_T}|g^1|^2 S'_\eta(u-k)\varphi_1 dxdydt. \quad (45)$$

Let $\varepsilon\to 0$ in (44). By (45), we get (19) and the boundary value condition is naturally concealed in the limiting process.

The proof of (20) is similar to that in [2], [6], we omit details here.

5 The stability of the solutions

In the last section of the paper, we give the stability of the solutions in the sense of Definition 1. The proof is similar as [17], so we only give the outlines.

Lemma 6 [6] *Let u be a solution of (1)-(2)-(3). Then*

$$(u^+-u^-)v_1=0,\ a.e.\ (x,y,t)\ on\ \Gamma_u. \quad (46)$$

which is true in the sense of Hausdorff measure $H_2(\Gamma_u)$.

The lemma can be proved in a similar way as the proof of Lemma 6.1 in [6], we omit the details here.

Let u,v be two entropy solutions of (1) with initial values

$$u(x,y,0)=u_0(x,y),\ v(x,y,0)=v_0(x,y). \quad (47)$$

and with the homogeneous boundary value $u(x,y,t)=v(x,y,t)=0$ when $(x,y,t)\in\Sigma_3$.

For simplicity, we denote the spatial variables (x, y) as (x_1, x_2) or (y_1, y_2) in what follows, and correspondingly, $dx = dx_1dx_2, dy = dy_1dy_2$.

Let $\varphi_1 = \varphi_2 = \varphi(x,t) \in C_0^\infty(Q_T)$. Then we have

$$\int\int_{Q_T}[I_\eta(u-k)\varphi_t - B_\eta(u,k)\varphi_{x_2} + I_\eta(u-k)\varphi_{x_1x_1}$$

$$-S'_\eta(u-k) \mid g^1(u) \mid^2 \varphi - f(\cdot,u)S_\eta(u-k)\varphi]dxdt \geq 0, \tag{48}$$

$$\int\int_{Q_T}[I_\eta(v-l)\varphi_\tau - B_\eta(v,l)\varphi_{y_2} + I_\eta(v-l)\varphi_{y_1y_1}$$

$$-S'_\eta(v-l) \mid g^1(v) \mid^2 \varphi - f(\cdot,v)S_\eta(v-l)\varphi]dyd\tau \geq 0. \tag{49}$$

Let $\psi(x,t,y,\tau) = \phi(x,t)j_h(x-y,t-\tau)$, where $\phi(x,t) \geq 0,\ \phi(x,t) \in C_0^\infty(Q_T)$, and

$$j_h(x-y,t-\tau) = \omega_h(t-\tau)\Pi_{i=1}^2\omega_h(x_i-y_i), \tag{50}$$

$$\omega_h(s) = \frac{1}{h}\omega(\frac{s}{h}), \omega(s) \in C_0^\infty(R),$$

$$\omega(s) \geq 0,\ \omega(s) = 0\ if\ \mid s \mid > 1,$$

$$\int_{-\infty}^{\infty}\omega(s)ds = 1. \tag{51}$$

We choose $k = v(y,\tau),\ l = u(x,t),\ \varphi = \psi(x,t,y,\tau)$ in (48) (49), integrate over Q_T, plus them together, then we get

$$\int\int_{Q_T}\int\int_{Q_T}\{[I_\eta(u-v)(\psi_t+\psi_\tau)$$

$$-(B_\eta(u,v)\psi_{x_2} + B_\eta(v,u)\psi_{y_2})$$

$$+I_\eta(u-v)\psi_{x_1x_1} + I_\eta(v-u)\psi_{y_1y_1}]$$

$$-S'_\eta(u-v)\left(\mid g^1(u) \mid^2 + \mid g^1(v) \mid^2\right)$$

$$-[f(\cdot,u)S_\eta(u-v) + f(\cdot,v)S_\eta(v-u)]\}\varphi dxdtdyd\tau. \tag{52}$$

No, by Lemma 6, by Kruzkov's bi-variables method (c.f.[17]), we are able to obtain the following inequality

$$\int\int_{Q_T}\{|u(x,t)-v(x,t)|\phi_t + \mid u-v \mid \phi_{x_1x_1}$$

$$-\frac{1}{2}sgn(u-v)(u^2-v^2)\phi_{x_2}$$

$$-[f(\cdot,u)-f(\cdot,v)]sgn(u-v)\phi\}dxdt \geq 0. \tag{53}$$

Theorem 7 *Let $0 \leq u \leq 1, 0 \leq v \leq 1$ be two solutions of equation (1) with the homogeneous boundary value $\gamma u \mid_{\sum_3} = \gamma v \mid_{\sum_3} = 0$, and with the different initial values $u_0(x_1,x_2)$, $v_0(x_1,x_2) \in L^\infty(\Omega)$ respectively. Suppose that $|f_u(\cdot,u)| \leq c$, and suppose the distance function $d(x) = dist(x,\partial\Omega)$ satisfies that*

$$|d_{x_1x_1}| \leq c, \tag{54}$$

near the boundary, then for any $t \in (0,T)$,

$$\int_\Omega \mid u(x_1,x_2,t) - v(x_1,x_2,t) \mid dx_1dx_2$$

$$\leq \int_\Omega \mid u_0(x_1,x_2) - v_0(x_1,x_2) \mid dx_1dx_2$$

$$+c \cdot ess\sup \mid u-v \mid_{(x,t)\in\Sigma'_3\times(0,T)}. \tag{55}$$

where $\Sigma'_3 = \partial\Omega \setminus \Sigma_3$.

Proof Now, we can choose ϕ in (53) by

$$\phi(x,t) = \omega_\lambda(x)\eta(t),$$

where $\eta(t) \in C_0^\infty(0,T)$, $\omega_\lambda(x) \in C^2(\Omega)$ is defined as follows. For any given small enough $0 < \lambda$, $0 \leq \omega_\lambda \leq 1, \omega|_{\partial\Omega} = 0$ and

$$\omega_\lambda(d) = 1, if\ d(x) = dist(x,\partial\Omega) \geq \lambda.$$

when $0 \leq d(x) \leq \lambda$,

$$\omega_\lambda(d(x)) = 1 - \frac{(d(x)-\lambda)^2}{\lambda^2},$$

Then

$$|\omega'_\lambda(d)| \leq \frac{c}{\lambda}. \tag{56}$$

$$\omega''_\lambda(d) = -\frac{2}{\lambda^2},\ if\ 0 < d < \lambda. \tag{57}$$

Now,

$$\begin{aligned}\phi_{x_1x_1} &= \eta(t)(\omega_\lambda(d(x)))_{x_1x_1}\\ &= \eta(t)(\omega'_\lambda(d)d_{x_1})_{x_1}\\ &= \eta(t)[\omega''_\lambda(d)d_{x_1}^2 + \omega'_\lambda(d)d_{x_1x_1}].\end{aligned} \tag{58}$$

By the condition (54), $|d_{x_1x_1}| \leq c$, and using the fact of that $|d_{x_i}| \leq |\nabla d| = 1, i = 1,2$, $0 \leq f_u(\cdot,u) \leq c$, $0 \leq u, v \leq c$, from (53), we have

$$\int\int_{Q_T}|u(x,t)-v(x,t)|\phi_t dxdt$$

$$+c\int_0^T\int_{\Omega_\lambda}\eta(t)|\omega'_\lambda(d)| \mid u-v \mid dxdt$$

$$+c\int\int_{Q_T}|u(x,t)-v(x,t)|\phi dxdt \geq 0. \tag{59}$$

Here $\Omega_\lambda = \{x : d(x) = dist(x, \partial\Omega) < \lambda\}$. By (56),

$$0 \leq \int\int_{Q_T} |u(x,t) - v(x,t)|\eta'(t)|\omega_\lambda(d)dxdt$$
$$+c\int_0^T \int_{\Omega_\lambda} \eta(t)|\omega'_\lambda(d)| \mid u - v \mid dxdt$$
$$+c\int\int_{Q_T} |u(x,t) - v(x,t)|\eta(t)\omega_\lambda(d)dxdt$$
$$\leq \int\int_{Q_T} |u(x,t) - v(x,t)|\eta'(t)|\omega_{\lambda\varepsilon}(d)dxdt$$
$$+c\int_0^T \eta(t)dt\frac{1}{\lambda}\int_{\Omega_\lambda} \mid u - v \mid dx$$
$$+c\int\int_{Q_T} |u(x,t) - v(x,t)|\eta(t)\omega_\lambda(d)dxdt.$$

As $\lambda \to 0$, according to the definition of the trace, by $\gamma u \mid_{\sum_3} = \gamma v \mid_{\sum_3} = 0$, we have

$$0 \leq \int\int_{Q_T} |u(x,t) - v(x,t)|\eta'(t)dxdt$$
$$+c\int_0^T \eta(t)| \mid u - v \mid_{\partial\Omega} dt$$
$$+c\int\int_{Q_T} |u(x,t) - v(x,t)|\eta(t)dxdt$$
$$= \int\int_{Q_T} |u(x,t) - v(x,t)|\eta'_t dxdt$$
$$+c\int_0^T \eta(t)| \mid u - v \mid_{\Sigma'_3\times(0,T)} dt$$
$$+c\int\int_{Q_T} |u(x,t) - v(x,t)|\eta(t)dxdt. \tag{60}$$

Let $0 < s < \tau < T$, and

$$\eta(t) = \int_{\tau-t}^{s-t} \alpha_\epsilon(\sigma)d\sigma, \ \ \epsilon < \min\{\tau, T - s\}.$$

Here $\alpha_\epsilon(t)$ is the kernel of mollifier with $\alpha_\epsilon(t) = 0$ for $t \notin (-\epsilon, \epsilon)$. Let $\epsilon \to 0$. Then

$$\int_\Omega |u(x,s) - v(x,s)|dx \leq \int_\Omega |u(x,\tau) - v(x,\tau)|dx$$
$$+c \cdot ess\sup| \mid u - v \mid_{\Sigma'_3\times(0,T)}$$
$$+\int_s^\tau \int_\Omega |u(x,t) - v(x,t)|dxdt$$

By Gronwall Lemma, the desired result follows by letting $s \to 0$,

$$|u(x,\tau) - v(x,\tau)|_{L^1(\Omega)}$$
$$\leq |u(x,0)-v(x,0)|_{L^1(\Omega)}+c\cdot ess\sup| \mid u-v \mid_{\Sigma'_3\times(0,T)} .$$

we have the conclusion.

Acknowledgements: The research was supported by Natural Science Foundation, no: 11371297; supported by Science Foundation of Xiamen University of Technology, China.

References:

[1] F. Antonelli, E. Barucci and M. E. Mancino, A comparison result for BFSDE's and applications to decisions theory, *Math. Methods Opwer. Res.* 54, 2001, pp. 407–423.

[2] M. Escobedo, J. L. Vazquez and E. Zuazua, Entropy solutions for diffusion-convection equations with partial diffusivity, *Trans. Amer. Math. Soc.* 343, 1994, pp. 829–842.

[3] M.G. Crandall, H. Ishii and P.L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)* 27, 1992, pp. 1–67.

[4] F. Antonelli and A. Pascucci, On the viscosity solutions of a stochastic differential utility problem, *J. Diff. Equations* 186, 2002, pp. 69–87.

[5] G. Citti, A. Pascucci and S. Polidoro, Regularity properties of viscosity solutions of a non-Hőrmander degenerate equation, *J. Math. Pures Appl.* 80, 2001, pp. 901–918.

[6] H. Zhan, The study of the Cauchy problem of a second order quasilinear degenerate parabolic equation and the parallelism of a Riemannian manifold, Doctor Thesis, Xiamen University, 2004.

[7] H. Zhan and J. Zhao, The stability of solutions for second quasilinear degenerate parabolic equation, *Acta Math. Sinica, Chinese Ser.* 50, 2007, pp. 615-628.

[8] G. Q. Chen and B. Perthame, Well-Posedness for non-isotropic degenerate parabolic-hyperbolic equations, *Ann. I. H. Poincare-AN* 20, 2003, pp. 645-668.

[9] G. Q. Chen and E. DiBenedetto, Stability of entropy solutions to Cauchy problem for a class of nonlinear hyperbolic-parabolic equations, *SIAM J.Math. Anal.* 33, 2001, pp. 751-762.

[10] K.H. Karlsen and N.H. Risebro, On the uniqueness of entropy solutions of nonlinear degenerate parabolic equations with rough coefficient, *Discrete Contain. Dye. Sys.* 9, 2003, pp. 1081-1104.

[11] M. Bendahamane and K. H. Karlsen, Reharmonized entropy solutions for quasilinear anisotropic degenerate parabolic equations, *SIAM J.Math. Anal.* 36, 2004, pp. 405-422.

[12] J. Carrillo, Entropy solutions for nonlinear degenerate problems, *Arch.Rational Mech. Anal.* 147, 1999, pp. 269-361.

[13] H. Zhan, The solution of the Cauchy problem for a quasilinear degenerate parabolic equation, *Chinese Ann. of Math., Ser A.* 33, 2012, pp. 449-460.

[14] O.A. Oleiňk and V. N. Samokhin, Mathematical models in boundary layer theorem, Chapman and Hall/CRC, 1999.

[15] H. Zhan, On a quasilinear degenerate parabolic equation, *Chinese Ann. of Math., Ser. A* 27, 2006, pp. 731-740.

[16] H. Zhan, Oleinik line method and its application, *WSEAS Transactions on Mathematics* 13, 2014, pp. 768-779.

[17] H. Zhan, The Initial Boundary value problem of an equation from mathematics finance, accepted and to be appeared in *Chinese Ann. of Math., Ser. B*, 2016.

[18] Z. Wu and J. Zhao, The first boundary value problem for quasilinear degenerate parabolic equations of second order in several variables, *Chinse Ann. of Math. Ser. B* 4, 1983, pp. 319-358.

[19] G. Fichera, Sulle equazioni differenziatli lineari ellittico-paraboliche del secondo ordine, *Atti Accd, Naz. Lincei. Mem, CI. Sci. Fis. Mat. Nat. Sez.1* 5(8), 1956, pp. 1-30, MR 19 658. 1432.

[20] G. Fichera, On a unified theory of boundary value problems for elliptic-parabolic equations of second order, *in Boundary Problems, Differential Equations, Univ. of Wisconsin Press, Madison, Wis.*, 1960, pp. 97-120, MR 22, 2789.

[21] O. A. Oleiňk, A problem of Fichera, *Dokl.Akad. Nauk SSSR* 154, 1964, pp. 1297-1300. *Soviet Math. Dokl.* 5, 1964, pp. 1129-1133, MA 30, 1293.

[22] O. A. Oleiňk, Linear equations of second order with nonnegative characteristic form, *Math. Sb.* 69, 1966, pp. 111-140; English transl., *Amer. Math. Soc. Tranl.* 65(2), 1967, pp. 167-199, MR 33, 1603.

[23] A. I. Volpert and S.I. Hudjave, Analysis of class of discontinuous functions and the equations of mathematical physics (Russian), Izda. Nauka Moskwa, 1975.

[24] A.I. Volpert, BV space and quasilinear equations, *Mat.Sb.* 73, 1967, pp. 255-302.

[25] L. C. Evans, Weak convergence methods for nonlinear partial differential equations, Conference Board of the Mathematical Sciences, Regional Conferences Series in Mathematics Number 74, 1998.

[26] L. Gu, Second order parabolic partial differential equations, The Publishing Company of Xiamen University, 2002.

[27] H. Zhan, The boundary value condition of a degenerate parabolic equation, *Int. J. of Evo. Equ.*, 6, 2(2011), pp. 187–208.

[28] H. Zhan, Quasilinear degenerate parabolic equation from finance, *WSEAS Trans. On Math.*, 9, 2010, pp. 861–873.