Exact Solutions For Fractional Partial Differential Equations By A New Generalized Fractional Sub-equation Method

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Abstract: In this paper, combining with a fractional complex transformation, we propose a new generalized fractional sub-equation method named fractional $(\frac{D^\alpha G}{G})$ method to seek exact solutions for fractional partial differential equations. This method is the fractional version of the improved (G'/G) method. Based on this method, some new exact solutions for the space-time fractional (2+1)-dimensional breaking soliton equations and the space-time fractional Fokas equation are successfully found. Under some special cases, we get solitary wave solutions for them.

Key–Words: Fractional sub-equation method; Fractional partial differential equations; Exact solutions; Fractional complex transformation; Fractional breaking soliton equations; Fractional Fokas equation.

1 Introduction

Fractional differential equations are generalizations of classical differential equations of integer order. The main advantage of fractional differential equations in comparison with classical differential equations of integer order mainly lies in that fractional derivative is more useful in describing the memory and hereditary properties of materials and processes. In recent decades, fractional differential equations have gained much attention as they are widely used to describe various complex phenomena in many fields such as the fluid flow, signal processing, control theory, systems identification, biology and other areas. The mathematical modeling and simulation of systems and processes leads to the research of the theory for fractional differential equations. Many authors have investigated some aspects of fractional differential equations so far. Among these investigations for fractional differential equations, research for seeking numerical solutions and exact solutions of fractional differential equations has been a hot topic, which can also provide valuable reference for other related research. Many powerful and efficient methods have been proposed to obtain numerical solutions and exact solutions of fractional differential equations so far. For example, these methods include the (G'/G) method [1-3], the Homo-Separation of Variables method [4], the variational iterative method [5-7], the Adomian decomposition method [8,9], the fractional sub-equation method [10-12], the homotopy perturbation method [13-15] and so on. Based on these methods, a variety of fractional differential equations have been investigated.

In this paper, using a new fractional sub-equation, we propose a new generalized fractional sub-equation method named fractional $(\frac{D^\alpha G}{G})$ method to seek exact solutions for fractional partial differential equations in the sense of modified Riemann-Liouville derivative. This method can be seen as the fractional version of the improved (G'/G) method [16].

The modified Riemann-Liouville fractional derivative, defined by Jumarie in [17-20], has many excellent characters in handling with many fractional calculus problems. Many authors have investigated various applications of the modified Riemann-Liouville fractional derivative (for example, see [21-23]). We now list the definition for it as follows.

Definition 1 *The modified Riemann-Liouville derivative of order α is defined by the following expression:*

$$D_t^\alpha f(t) = \begin{cases} \frac{d}{dt}\int_0^t \frac{(t-\xi)^{-\alpha}}{\Gamma(1-\alpha)}(f(\xi)-f(0))d\xi, \\ \quad 0<\alpha<1, \\ (f^{(n)}(t))^{(\alpha-n)},\ n\le\alpha<n+1, \\ \quad n\ge 1. \end{cases}$$

Definition 2 *The Riemann-Liouville fractional integral of order α on the interval $[0,t]$ is defined by*

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds.$$

Some important properties for the modified Riemann-Liouville derivative and fractional integral are listed as follows (see [17, Eqs. (3.10)-(3.13)], and see also [10-12,21-23]) (the interval concerned below is always defined by $[0, t]$):

$$D_t^{\alpha} t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}; \tag{1}$$

$$D_t^{\alpha}(f(t)g(t)) = g(t)D_t^{\alpha}f(t) + f(t)D_t^{\alpha}g(t); \tag{2}$$

$$\begin{aligned} D_t^{\alpha}f[g(t)] &= f_g'[g(t)]D_t^{\alpha}g(t) \\ &= D_g^{\alpha}f[g(t)](g'(t))^{\alpha}; \end{aligned} \tag{3}$$

$$I^{\alpha}(D_t^{\alpha}f(t)) = f(t) - f(0) \tag{4}$$

$$\begin{aligned} I^{\alpha}(g(t)D_t^{\alpha}f(t)) &= f(t)g(t) - f(0)g(0) \\ &\quad -I^{\alpha}(f(t)D_t^{\alpha}g(t)). \end{aligned} \tag{5}$$

The rest of this paper is organized as follows. In Section 2, we give the description of the fractional $(\frac{D^{\alpha}G}{G})$ method for solving fractional partial differential equations. Then in Section 3 we apply this method to establish exact solutions for the space-time fractional (2+1)-dimensional breaking soliton equations and the space-time fractional Fokas equation. Some conclusions are presented at the end of the paper.

2 Description of the fractional $(\frac{D^{\alpha}G}{G})$ method

In this section we give the description of the $(\frac{D^{\alpha}G}{G})$ method for solving fractional partial differential equations.

Suppose that a fractional partial differential equation, say in the independent variables $t, x_1, x_2, ..., x_n$, is given by

$$\begin{aligned} &P(u_1, ..u_k, D_t^{\alpha}u_1, .., D_t^{\alpha}u_k, D_{x_1}^{\alpha}u_1, .., D_{x_1}^{\alpha}u_k, \ldots \\ &D_{x_n}^{\alpha}u_1, .., D_{x_n}^{\alpha}u_k, D_t^{2\alpha}u_1, ..., D_t^{2\alpha}u_k, D_{x_1}^{2\alpha}u_1, ...) \\ &= 0, \end{aligned} \tag{6}$$

where $u_i = u_i(t, x_1, x_2, ..., x_n),\ i = 1, ..., k$ are unknown functions, P is a polynomial in u_i and their various partial derivatives including fractional derivatives.

Step 1. Suppose that

$$\begin{aligned} u_i(t, x_1, x_2, ..., x_n) &= U_i(\xi), \\ xi = ct + k_1x_1 + k_2x_2 &+ ... + k_nx_n + \xi_0. \end{aligned} \tag{7}$$

Then by the second equality in Eq. (3), Eq. (7) can be turned into the following fractional ordinary differential equation with respect to the variable ξ:

$$\begin{aligned} &\widetilde{P}(U_1, ..., U_k, c^{\alpha}D_{\xi}^{\alpha}U_1, ..., c^{\alpha}D_{\xi}^{\alpha}U_k, k_1^{\alpha}D_{\xi}^{\alpha}U_1, ..., \\ &k_1^{\alpha}D_{\xi}^{\alpha}U_k, ..., k_n^{\alpha}D_{\xi}^{\alpha}U_1, ..., k_n^{\alpha}D_{\xi}^{\alpha}U_k, c^{2\alpha}D_{\xi}^{2\alpha}U_1, \\ &..., c^{2\alpha}D_{\xi}^{2\alpha}U_k, k_1^{2\alpha}D_{\xi}^{2\alpha}U_1, ...) = 0. \end{aligned} \tag{8}$$

Step 2. Suppose that the solution of (8) can be expressed by a polynomial in $(\frac{D_{\xi}^{\alpha}G}{G})$ as follows:

$$U_j(\xi) = \sum_{i=0}^{m_j} a_{j,i}(\frac{D_{\xi}^{\alpha}G}{G})^i,\ j = 1, 2, ..., k, \tag{9}$$

where $a_{j,i},\ i = 0, 1, ..., m,\ j = 1, 2, ..., k$ are constants to be determined later with $a_{j,m} \neq 0$, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (8), $G = G(\xi)$ satisfies the following fractional ordinary differential equation:

$$\begin{aligned} &AGD_{\xi}^{2\alpha}G(\xi) - BGD_{\xi}^{\alpha}G(\xi) \\ &-C(D_{\xi}^{\alpha}G(\xi))^2 - EG^2(\xi) = 0, \end{aligned} \tag{10}$$

where $D_{\xi}^{\alpha}G(\xi)$ denotes the modified Riemann-Liouville derivative of order α for $G(\xi)$ with respect to ξ, and $A,\ B,\ C,\ E$ are real parameters.

In order to obtain the general solutions for Eq. (10), we suppose $G(\xi) = H(\eta)$, and a nonlinear fractional complex transformation $\eta = \frac{\xi^{\alpha}}{\Gamma(1+\alpha)}$. Then by Eq. (1) and the first equality in Eq. (3), Eq. (10) can be turned into the following second ordinary differential equation:

$$\begin{aligned} &AHH''(\eta) - BHH'(\eta) \\ &-C(H'(\eta))^2 - EH^2(\eta) = 0. \end{aligned} \tag{11}$$

Denote $\Delta_1 = B^2 + 4E(A-C),\ \Delta_2 = E(A-C)$. By the general solutions of Eq. (11) [16] we have the following expressions for $\frac{H'(\eta)}{H(\eta)}$:

When $B \neq 0,\ \Delta_1 > 0$:

$$\begin{aligned} \frac{H'(\eta)}{H(\eta)} &= \frac{B}{2(A-C)} + \frac{\sqrt{\Delta_1}}{2(A-C)} \\ &[\frac{C_1 \sinh \frac{\sqrt{\Delta_1}\eta}{2(A-C)} + C_2 \cosh \frac{\sqrt{\Delta_1}\eta}{2(A-C)}}{C_1 \cosh \frac{\sqrt{\Delta_1}\eta}{2(A-C)} + C_2 \sinh \frac{\sqrt{\Delta_1}\eta}{2(A-C)}}], \end{aligned} \tag{12}$$

where $C_1,\ C_2$ are arbitrary constants.

When $B \neq 0,\ \Delta_1 < 0$:

$$\begin{aligned} \frac{H'(\eta)}{H(\eta)} &= \frac{B}{2(A-C)} + \frac{\sqrt{-\Delta_1}}{2(A-C)} \\ &[\frac{-C_1 \sin \frac{\sqrt{-\Delta_1}\eta}{2(A-C)} + C_2 \cos \frac{\sqrt{-\Delta_1}\eta}{2(A-C)}}{C_1 \cos \frac{\sqrt{-\Delta_1}\eta}{2(A-C)} + C_2 \sin \frac{\sqrt{-\Delta_1}\eta}{2(A-C)}}], \end{aligned} \tag{13}$$

where $C_1,\ C_2$ are arbitrary constants.

When $B \neq 0,\ \Delta_1 = 0$:

$$\frac{H'(\eta)}{H(\eta)} = \frac{B}{2(A-C)} + \frac{C_2}{C_1 + C_2\eta}, \tag{14}$$

where $C_1,\ C_2$ are arbitrary constants.

When $B=0,\ \Delta_2>0$:

$$\frac{H'(\eta)}{H(\eta)}=\frac{\sqrt{\Delta_2}}{(A-C)}[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\eta}{(A-C)}+C_2\cosh\frac{\sqrt{\Delta_2}\eta}{(A-C)}}{C_1\cosh\frac{\sqrt{\Delta_2}\eta}{(A-C)}+C_2\sinh\frac{\sqrt{\Delta_2}\eta}{(A-C)}}], \quad (15)$$

where $C_1,\ C_2$ are arbitrary constants.

When $B=0,\ \Delta_2<0$:

$$\frac{H'(\eta)}{H(\eta)}=\frac{\sqrt{-\Delta_2}}{(A-C)}[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\eta}{(A-C)}+C_2\cos\frac{\sqrt{-\Delta_2}\eta}{(A-C)}}{C_1\cos\frac{\sqrt{-\Delta_2}\eta}{(A-C)}+C_2\sin\frac{\sqrt{-\Delta_2}\eta}{(A-C)}}], \quad (16)$$

where $C_1,\ C_2$ are arbitrary constants.

Since $D_\xi^\alpha G(\xi)=D_\xi^\alpha H(\eta)=H'(\eta)D_\xi^\alpha\eta=H'(\eta)$, we obtain the following expressions for $\frac{D_\xi^\alpha G(\xi)}{G(\xi)}$:

When $B\neq0,\ \Delta_1>0$:

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)}=\frac{B}{2(A-C)}+\frac{\sqrt{\Delta_1}}{2(A-C)} [\frac{C_1\sinh\frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}+C_2\cosh\frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}+C_2\sinh\frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}], \quad (17)$$

where $C_1,\ C_2$ are arbitrary constants.

When $B\neq0,\ \Delta_1<0$:

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)}=\frac{B}{2(A-C)}+\frac{\sqrt{-\Delta_1}}{2(A-C)} [\frac{-C_1\sin\frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}+C_2\cos\frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}{C_1\cos\frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}+C_2\sin\frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}], \quad (18)$$

where $C_1,\ C_2$ are arbitrary constants.

When $B\neq0,\ \Delta_1=0$:

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)}=\frac{B}{2(A-C)}+\frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}, \quad (19)$$

where $C_1,\ C_2$ are arbitrary constants.

When $B=0,\ \Delta_2>0$:

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)}=\frac{\sqrt{\Delta_2}}{(A-C)} [\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}], \quad (20)$$

where $C_1,\ C_2$ are arbitrary constants.

When $B=0,\ \Delta_2<0$:

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)}=\frac{\sqrt{-\Delta_2}}{(A-C)} [\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}], \quad (21)$$

where $C_1,\ C_2$ are arbitrary constants.

Step 3. Substituting (9) into (8) and using (10), collecting all terms with the same order of $(\frac{D_\xi^\alpha G}{G})$ together, the left-hand side of (8) is converted into another polynomial in $(\frac{D_\xi^\alpha G}{G})$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_{j,i},\ i=0,1,...,m,\ j=1,2,...,k$.

Step 4. Solving the equations system in Step 3, and using (17)-(21), we can construct a variety of exact solutions for Eq. (6).

Remark 3 *If we set $\alpha=1$ in Eq. (10), then it becomes $AGG''(\xi)-BGG'(\xi)-C(G'(\xi))^2-EG^2(\xi)=0$, which is the foundation of the improved (G'/G) method [16] for solving partial differential equations (PDEs). So the fractional $(\frac{D_\xi^\alpha G}{G})$ method is the extension of the improved (G'/G) method to fractional case.*

3 Applications of the fractional $(\frac{D^\alpha G}{G})$ method to some fractional partial differential equations

3.1 Space-time fractional (2+1)-dimensional breaking soliton equations

Consider the space-time fractional (2+1)-dimensional breaking soliton equations [24]

$$\begin{cases}\frac{\partial^\alpha u}{\partial t^\alpha}+a\frac{\partial^{3\alpha}u}{\partial x^{2\alpha}y^\alpha}+4au\frac{\partial^\alpha v}{\partial x^\alpha}+4a\frac{\partial^\alpha u}{\partial x^\alpha}v=0,\\ \frac{\partial^\alpha u}{\partial y^\alpha}=\frac{\partial^\alpha v}{\partial x^\alpha},\end{cases}, \quad (22)$$

where $0<\alpha\leq1$. In [24], the authors obtained some new exact solutions for Eqs. (22) by use of a fractional sub-equation method, which is based on the following fractional sub-equation:

$$D_\xi^{2\alpha}G(\xi)+\lambda D_\xi^\alpha G(\xi)+\mu G(\xi)=0, \quad (23)$$

In the following, we will apply the fractional $(\frac{D^\alpha G}{G})$ method described in Section 3 to solve Eqs. (22). To begin with, we suppose $u(x,y,t)=U(\xi),\ v(x,y,t)=V(\xi)$, where $\xi=k_1x+k_2y+ct+\xi_0,k_1,\ k_2,\ c,\ \xi_0$ are all constants with $k_1,\ k_2,\ c\neq0$. Then by use of the second equality in Eq. (4), Eqs. (22) can be turned into

$$\begin{cases}c^\alpha D_\xi^\alpha U+ak_1^{2\alpha}k_2^\alpha D_\xi^{3\alpha}U\\ \quad+4ak_1^\alpha UD_\xi^\alpha V+4ak_1^\alpha VD_\xi^\alpha U=0,\\ k_2^\alpha D_\xi^\alpha U=k_1^\alpha D_\xi^\alpha V.\end{cases} \quad (24)$$

Suppose that the solution of Eqs. (24) can be expressed by

$$\begin{cases}U(\xi)=\sum_{i=0}^{m_1}a_i(\frac{D_\xi^\alpha G}{G})^i,\\ V(\xi)=\sum_{i=0}^{m_2}b_i(\frac{D_\xi^\alpha G}{G})^i.\end{cases} \quad (25)$$

Balancing the order of $D_\xi^{3\alpha}U$ and $UD_\xi^{\alpha}V$, $D_\xi^{\alpha}U$ and $D_\xi^{\alpha}V$ in (24) we have $m_1 = m_2 = 2$. So

$$\begin{cases} U(\xi) = a_0 + a_1(\frac{D_\xi^\alpha G}{G})^1 + a_2(\frac{D_\xi^\alpha G}{G})^2, \\ V(\xi) = b_0 + b_1(\frac{D_\xi^\alpha G}{G})^1 + b_2(\frac{D_\xi^\alpha G}{G})^2. \end{cases} \tag{26}$$

Substituting (26) into (24), using the properties (1)-(3) and Eq. (10), collecting all the terms with the same power of $(\frac{D_\xi^\alpha G}{G})$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations yields:

$$\begin{aligned} &a_0 = a_0, \\ &a_1 = \tfrac{3(A-C)Bk_1^{2\alpha}}{2A^2}, \\ &a_2 = -\tfrac{3(C^2+A^2-2CA)k_1^{2\alpha}}{2A^2}, \\ &b_0 = -\tfrac{ak_1^{2\alpha}k_2^\alpha B^2+8ak_1^{2\alpha}k_2^\alpha aE(C-A)+c^\alpha A^2+4aa_0A^2k_2^\alpha}{4aA^2k_1^\alpha}, \\ &b_1 = \tfrac{3(A-C)Bk_1^\alpha k_2^\alpha}{2A^2}, \\ &b_2 = -\tfrac{3(C^2+A^2-2CA)k_1^\alpha k_2^\alpha}{2A^2}, \end{aligned}$$

where a_0 is an arbitrary constant.

Substituting the result above into Eqs. (26), and combining with (17)-(21) we can obtain the following exact solutions to Eqs. (22).

Family 1: when $B \neq 0,\ \Delta_1 > 0$:

$$\begin{aligned} &u_1(x,y,t) = a_0 + \\ &\tfrac{3(A-C)Bk_1^{2\alpha}}{2A^2} \times \{\tfrac{B}{2(A-C)} + \tfrac{\sqrt{\Delta_1}}{2(A-C)} \\ &[\tfrac{C_1 \sinh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \cosh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}{C_1 \cosh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \sinh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}]\} \\ &-\tfrac{3(C^2+A^2-2CA)k_1^{2\alpha}}{2A^2}\{\tfrac{B}{2(A-C)} + \tfrac{\sqrt{\Delta_1}}{2(A-C)} \\ &[\tfrac{C_1 \sinh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \cosh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}{C_1 \cosh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \sinh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}]\}^2, \end{aligned} \tag{27}$$

$$\begin{aligned} &v_1(x,y,t) = - \\ &\tfrac{ak_1^{2\alpha}k_2^\alpha B^2+8ak_1^{2\alpha}k_2^\alpha aE(C-A)+c^\alpha A^2+4aa_0A^2k_2^\alpha}{4aA^2k_1^\alpha} \\ &+\tfrac{3(A-C)Bk_1^\alpha k_2^\alpha}{2A^2}\{\tfrac{B}{2(A-C)} + \tfrac{\sqrt{\Delta_1}}{2(A-C)} \\ &[\tfrac{C_1 \sinh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \cosh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}{C_1 \cosh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \sinh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}]\} \\ &-\tfrac{3(C^2+A^2-2CA)k_1^\alpha k_2^\alpha}{2A^2}\{\tfrac{B}{2(A-C)} + \tfrac{\sqrt{\Delta_1}}{2(A-C)} \\ &[\tfrac{C_1 \sinh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \cosh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}{C_1 \cosh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \sinh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}]\}^2, \end{aligned} \tag{28}$$

where $C_1,\ C_2$ are arbitrary constants, $\xi = k_1x + k_2y + ct + \xi_0$. In particular, if we set $C_2 = 0$ in (27)-(28), then we obtain the following solitary solutions:

$$\begin{aligned} &u_2(x,y,t) = a_0 + \tfrac{3(A-C)Bk_1^{2\alpha}}{2A^2} \\ &\{\tfrac{B}{2(A-C)} + \tfrac{\sqrt{\Delta_1}}{2(A-C)} \tanh \tfrac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}\} \\ &-\tfrac{3(C^2+A^2-2CA)k_1^{2\alpha}}{2A^2} \\ &\{\tfrac{B}{2(A-C)} + \tfrac{\sqrt{\Delta_1}}{2(A-C)} \tanh \tfrac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}\}^2, \end{aligned} \tag{29}$$

$$\begin{aligned} &v_2(x,y,t) = \\ &-\tfrac{ak_1^{2\alpha}k_2^\alpha B^2+8ak_1^{2\alpha}k_2^\alpha aE(C-A)+c^\alpha A^2+4aa_0A^2k_2^\alpha}{4aA^2k_1^\alpha} \\ &+\tfrac{3(A-C)Bk_1^\alpha k_2^\alpha}{2A^2} \\ &\{\tfrac{B}{2(A-C)} + \tfrac{\sqrt{\Delta_1}}{2(A-C)} \tanh \tfrac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}\} \\ &-\tfrac{3(C^2+A^2-2CA)k_1^\alpha k_2^\alpha}{2A^2} \\ &\{\tfrac{B}{2(A-C)} + \tfrac{\sqrt{\Delta_1}}{2(A-C)} \tanh \tfrac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}\}^2. \end{aligned} \tag{30}$$

Family 2: when $B \neq 0,\ \Delta_1 < 0$:

$$\begin{aligned} &u_3(x,y,t) = a_0 + \tfrac{3(A-C)Bk_1^{2\alpha}}{2A^2}\{\tfrac{B}{2(A-C)} + \tfrac{\sqrt{-\Delta_1}}{2(A-C)} \\ &[\tfrac{-C_1 \sin \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \cos \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}{C_1 \cos \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \sin \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}]\} \\ &-\tfrac{3(C^2+A^2-2CA)k_1^{2\alpha}}{2A^2}\{\tfrac{B}{2(A-C)} + \tfrac{\sqrt{-\Delta_1}}{2(A-C)} \\ &[\tfrac{-C_1 \sin \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \cos \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}{C_1 \cos \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \sin \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}]\}^2, \end{aligned} \tag{31}$$

$$\begin{aligned} &v_3(x,y,t) = \\ &-\tfrac{ak_1^{2\alpha}k_2^\alpha B^2+8ak_1^{2\alpha}k_2^\alpha aE(C-A)+c^\alpha A^2+4aa_0A^2k_2^\alpha}{4aA^2k_1^\alpha} \\ &+\tfrac{3(A-C)Bk_1^\alpha k_2^\alpha}{2A^2}\{\tfrac{B}{2(A-C)} + \tfrac{\sqrt{-\Delta_1}}{2(A-C)} \\ &[\tfrac{-C_1 \sin \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \cos \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}{C_1 \cos \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \sin \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}]\} \\ &-\tfrac{3(C^2+A^2-2CA)k_1^\alpha k_2^\alpha}{2A^2}\{\tfrac{B}{2(A-C)} + \tfrac{\sqrt{-\Delta_1}}{2(A-C)} \\ &[\tfrac{-C_1 \sin \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \cos \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}{C_1 \cos \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \sin \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}]\}^2, \end{aligned} \tag{32}$$

where C_1, C_2 are arbitrary constants, $\xi = k_1x + k_2y + ct + \xi_0$.

Family 3: when $B \neq 0,\ \Delta_1 = 0$:

$$\begin{aligned} &u_4(x,y,t) = a_0 + \tfrac{3(A-C)Bk_1^{2\alpha}}{2A^2} \\ &\{\tfrac{B}{2(A-C)} + \tfrac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\} \\ &-\tfrac{3(C^2+A^2-2CA)k_1^{2\alpha}}{2A^2} \\ &\{\tfrac{B}{2(A-C)} + \tfrac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\}^2, \end{aligned} \tag{33}$$

$$v_4(x,y,t)= -\frac{ak_1^{2\alpha}k_2^\alpha B^2+8ak_1^{2\alpha}k_2^\alpha aE(C-A)+c^\alpha A^2+4aa_0A^2k_2^\alpha}{4aA^2k_1^\alpha} +\frac{3(A-C)Bk_1^\alpha k_2^\alpha}{2A^2}\{\frac{B}{2(A-C)}+\frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\} -\frac{3(C^2+A^2-2CA)k_1^\alpha k_2^\alpha}{2A^2} \{\frac{B}{2(A-C)}+\frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\xi^\alpha}\}^2, \tag{34}$$

where $C_1,\ C_2$ are arbitrary constants, and $\xi=k_1x+k_2y+ct+\xi_0$.

Family 4: when $B=0,\ \Delta_2>0$:

$$u_5(x,y,t)=a_0+\frac{3(A-C)Bk_1^{2\alpha}}{2A^2}\{\frac{\sqrt{\Delta_2}}{(A-C)} [\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}]\} -\frac{3(C^2+A^2-2CA)k_1^{2\alpha}}{2A^2}\{\frac{\sqrt{\Delta_2}}{(A-C)} [\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}]\}^2, \tag{35}$$

$$v_5(x,y,t)= -\frac{ak_1^{2\alpha}k_2^\alpha B^2+8ak_1^{2\alpha}k_2^\alpha aE(C-A)+c^\alpha A^2+4aa_0A^2k_2^\alpha}{4aA^2k_1^\alpha} +\frac{3(A-C)Bk_1^\alpha k_2^\alpha}{2A^2}\{\frac{\sqrt{\Delta_2}}{(A-C)} [\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}]\} -\frac{3(C^2+A^2-2CA)k_1^\alpha k_2^\alpha}{2A^2}\{\frac{\sqrt{\Delta_2}}{(A-C)} [\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\cosh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\sinh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}]\}^2, \tag{36}$$

where $C_1,\ C_2$ are arbitrary constants, $\xi=k_1x+k_2y+ct+\xi_0$.

Family 5: when $B=0,\ \Delta_2<0$:

$$u_6(x,y,t)=a_0+\frac{3(A-C)Bk_1^{2\alpha}}{2A^2}\{\frac{\sqrt{-\Delta_2}}{(A-C)} [\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}]\} -\frac{3(C^2+A^2-2CA)k_1^{2\alpha}}{2A^2}\{\frac{\sqrt{-\Delta_2}}{(A-C)} [\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}]\}^2, \tag{37}$$

$$v_6(x,y,t)= -\frac{ak_1^{2\alpha}k_2^\alpha B^2+8ak_1^{2\alpha}k_2^\alpha aE(C-A)+c^\alpha A^2+4aa_0A^2k_2^\alpha}{4aA^2k_1^\alpha} +\frac{3(A-C)Bk_1^\alpha k_2^\alpha}{2A^2}\{\frac{\sqrt{-\Delta_2}}{(A-C)} [\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}]\} -\frac{3(C^2+A^2-2CA)k_1^\alpha k_2^\alpha}{2A^2}\{\frac{\sqrt{-\Delta_2}}{(A-C)} [\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\cos\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}+C_2\sin\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}]\}^2, \tag{38}$$

where $C_1,\ C_2$ are arbitrary constants, $\xi=k_1x+k_2y+ct+\xi_0$.

3.2 Space-time fractional Fokas equation

Consider the space-time fractional Fokas equation

$$4\frac{\partial^{2\alpha}q}{\partial t^\alpha\partial x_1^\alpha}-\frac{\partial^{4\alpha}q}{\partial x_1^{3\alpha}\partial x_2^\alpha}+\frac{\partial^{4\alpha}q}{\partial x_2^{3\alpha}\partial x_1^\alpha}+12\frac{\partial^\alpha q}{\partial x_1^\alpha}\frac{\partial^\alpha q}{\partial x_2^\alpha} +12q\frac{\partial^{2\alpha}q}{\partial x_1^\alpha\partial x_2^\alpha}-6\frac{\partial^{2\alpha}q}{\partial y_1^\alpha\partial y_2^\alpha}=0,\ 0<\alpha\le 1. \tag{39}$$

In [25], the authors solved Eq. (37) by a fractional Riccati sub-equation method. Based on the this method, some exact solutions for it were obtained. Now we will apply the fractional $(\frac{D^\alpha G}{G})$ method described in Section 3 to solve Eq. (37).

Suppose $q(t,x_1,x_2,y_1,y_2)=U(\xi)$, where $\xi=ct+k_1x_1+k_2x_2+l_1y_1+l_2y_2+\xi_0$, $k_1,\ k_2,\ l_1,\ l_2,\ c,\ \xi_0$ are all constants with $k_1,\ k_2,\ l_1,\ l_2,\ c\neq 0$. Then by use of the second equality in Eq. (3), Eq. (37) can be turned into

$$4c^\alpha k_1^\alpha D_\xi^{2\alpha}U-k_1^{3\alpha}k_2^\alpha D_\xi^{4\alpha}U+k_2^{3\alpha}k_1^\alpha D_\xi^{4\alpha}U+ 12k_1^\alpha k_2^\alpha(D_\xi^\alpha U)^2+12k_1^\alpha k_2^\alpha UD_\xi^{2\alpha}U -6l_1^\alpha l_2^\alpha D_\xi^{2\alpha}U=0. \tag{40}$$

Suppose that the solution of Eq. (38) can be expressed by

$$U(\xi)=\sum_{i=0}^m a_i(\frac{D_\xi^\alpha G}{G})^i, \tag{41}$$

where $G=G(\xi)$ satisfies Eq. (10). By Balancing the order between the highest order derivative term and nonlinear term in Eq. (38), we can obtain $m=2$. So we have

$$U(\xi)=a_0+a_1(\frac{D_\xi^\alpha G}{G})+a_2(\frac{D_\xi^\alpha G}{G})^2. \tag{42}$$

Substituting (40) into (38), using the properties (1)-(3) and Eq. (10), collecting all the terms with the same power of $(\frac{D_\xi^\alpha G}{G})$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations, yields:

$$\begin{aligned}
a_0 &= -\frac{4A^2c^\alpha k_1^\alpha - 6A^2 l_1^\alpha l_2^\alpha + 8AEk_2^\alpha k_1^{3\alpha} - 8AEk_2^{3\alpha} k_1^\alpha}{12k_1^\alpha k_2^\alpha A^2} \\
&\quad - \frac{-k_1^{3\alpha} k_2^\alpha B^2 - 8k_1^{3\alpha} k_2^\alpha CE + B^2 k_2^{3\alpha} k_1^\alpha + 8Ek_2^{3\alpha} k_1^\alpha C}{12k_1^\alpha k_2^\alpha A^2}, \\
a_1 &= -\frac{(Ak_1^{3\alpha} k_2^\alpha - Ak_2^{3\alpha} k_1^\alpha - k_1^{3\alpha} k_2^\alpha C + k_2^{3\alpha} k_1^\alpha C)B}{k_1^\alpha k_2^\alpha A^2}, \\
a_2 &= \frac{A^2 k_1^{3\alpha} k_2^\alpha - A^2 k_2^{3\alpha} k_1^\alpha - 2Ak_1^{3\alpha} k_2^\alpha C}{k_1^\alpha k_2^\alpha A^2} \\
&\quad + \frac{2Ak_2^{3\alpha} k_1^\alpha C + k_1^{3\alpha} k_2^\alpha C^2 - k_2^{3\alpha} k_1^\alpha C^2}{k_1^\alpha k_2^\alpha A^2}.
\end{aligned}$$

Substituting the result above into Eq. (40), and combining with (17)-(21) we can obtain the following exact solutions to Eq. (37).

Family 1: when $B \neq 0,\ \Delta_1 > 0$:

$$\begin{aligned}
&q_1(t, x_1, x_2, y_1, y_2) = \\
&-\frac{4A^2c^\alpha k_1^\alpha - 6A^2 l_1^\alpha l_2^\alpha + 8AEk_2^\alpha k_1^{3\alpha} - 8AEk_2^{3\alpha} k_1^\alpha}{12k_1^\alpha k_2^\alpha A^2} \\
&-\frac{-k_1^{3\alpha} k_2^\alpha B^2 - 8k_1^{3\alpha} k_2^\alpha CE + B^2 k_2^{3\alpha} k_1^\alpha + 8Ek_2^{3\alpha} k_1^\alpha C}{12k_1^\alpha k_2^\alpha A^2} \\
&-\frac{(Ak_1^{3\alpha} k_2^\alpha - Ak_2^{3\alpha} k_1^\alpha - k_1^{3\alpha} k_2^\alpha C + k_2^{3\alpha} k_1^\alpha C)B}{k_1^\alpha k_2^\alpha A^2} \times \\
&\{\frac{B}{2(A-C)} + \frac{\sqrt{\Delta_1}}{2(A-C)} \\
&[\frac{C_1 \sinh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \cosh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}{C_1 \cosh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \sinh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}]\} \\
&+\{\frac{A^2 k_1^{3\alpha} k_2^\alpha - A^2 k_2^{3\alpha} k_1^\alpha - 2Ak_1^{3\alpha} k_2^\alpha C}{k_1^\alpha k_2^\alpha A^2} \\
&+\frac{2Ak_2^{3\alpha} k_1^\alpha C + k_1^{3\alpha} k_2^\alpha C^2 - k_2^{3\alpha} k_1^\alpha C^2}{k_1^\alpha k_2^\alpha A^2}\} \times \\
&\{\frac{B}{2(A-C)} + \frac{\sqrt{\Delta_1}}{2(A-C)} \\
&[\frac{C_1 \sinh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \cosh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}{C_1 \cosh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \sinh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}]\}^2,
\end{aligned} \tag{43}$$

where $C_1,\ C_2$ are arbitrary constants, $\xi = ct + k_1x_1 + k_2x_2 + l_1y_1 + l_2y_2 + \xi_0$. In particular, if we set $C_2 = 0$ in (41), then we obtain the following solitary solutions:

$$\begin{aligned}
&q_2(t, x_1, x_2, y_1, y_2) = \\
&-\frac{4A^2c^\alpha k_1^\alpha - 6A^2 l_1^\alpha l_2^\alpha + 8AEk_2^\alpha k_1^{3\alpha} - 8AEk_2^{3\alpha} k_1^\alpha}{12k_1^\alpha k_2^\alpha A^2} \\
&-\frac{-k_1^{3\alpha} k_2^\alpha B^2 - 8k_1^{3\alpha} k_2^\alpha CE + B^2 k_2^{3\alpha} k_1^\alpha + 8Ek_2^{3\alpha} k_1^\alpha C}{12k_1^\alpha k_2^\alpha A^2} \\
&-\frac{(Ak_1^{3\alpha} k_2^\alpha - Ak_2^{3\alpha} k_1^\alpha - k_1^{3\alpha} k_2^\alpha C + k_2^{3\alpha} k_1^\alpha C)B}{k_1^\alpha k_2^\alpha A^2} \\
&\{\frac{B}{2(A-C)} + \frac{\sqrt{\Delta_1}}{2(A-C)} \tanh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}\} \\
&+\{\frac{A^2 k_1^{3\alpha} k_2^\alpha - A^2 k_2^{3\alpha} k_1^\alpha - 2Ak_1^{3\alpha} k_2^\alpha C}{k_1^\alpha k_2^\alpha A^2} \\
&+\frac{2Ak_2^{3\alpha} k_1^\alpha C + k_1^{3\alpha} k_2^\alpha C^2 - k_2^{3\alpha} k_1^\alpha C^2}{k_1^\alpha k_2^\alpha A^2}\} \times \\
&\{\frac{B}{2(A-C)} + \frac{\sqrt{\Delta_1}}{2(A-C)} \tanh \frac{\sqrt{\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}\}^2.
\end{aligned}$$

Family 2: when $B \neq 0,\ \Delta_1 < 0$:

$$\begin{aligned}
&q_3(t, x_1, x_2, y_1, y_2) = \\
&-\frac{4A^2c^\alpha k_1^\alpha - 6A^2 l_1^\alpha l_2^\alpha + 8AEk_2^\alpha k_1^{3\alpha} - 8AEk_2^{3\alpha} k_1^\alpha}{12k_1^\alpha k_2^\alpha A^2} \\
&-\frac{-k_1^{3\alpha} k_2^\alpha B^2 - 8k_1^{3\alpha} k_2^\alpha CE + B^2 k_2^{3\alpha} k_1^\alpha + 8Ek_2^{3\alpha} k_1^\alpha C}{12k_1^\alpha k_2^\alpha A^2} \\
&-\frac{(Ak_1^{3\alpha} k_2^\alpha - Ak_2^{3\alpha} k_1^\alpha - k_1^{3\alpha} k_2^\alpha C + k_2^{3\alpha} k_1^\alpha C)B}{k_1^\alpha k_2^\alpha A^2} \times \\
&\{\frac{B}{2(A-C)} + \frac{\sqrt{-\Delta_1}}{2(A-C)} \\
&[\frac{-C_1 \sin \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \cos \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}{C_1 \cos \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \sin \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}]\} \\
&+\{\frac{A^2 k_1^{3\alpha} k_2^\alpha - A^2 k_2^{3\alpha} k_1^\alpha - 2Ak_1^{3\alpha} k_2^\alpha C}{k_1^\alpha k_2^\alpha A^2} \\
&+\frac{2Ak_2^{3\alpha} k_1^\alpha C + k_1^{3\alpha} k_2^\alpha C^2 - k_2^{3\alpha} k_1^\alpha C^2}{k_1^\alpha k_2^\alpha A^2}\} \times \\
&\{\frac{B}{2(A-C)} + \frac{\sqrt{-\Delta_1}}{2(A-C)} \\
&[\frac{-C_1 \sin \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \cos \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}{C_1 \cos \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)} + C_2 \sin \frac{\sqrt{-\Delta_1}\xi^\alpha}{2(A-C)\Gamma(1+\alpha)}}]\}^2,
\end{aligned} \tag{44}$$

where $C_1,\ C_2$ are arbitrary constants, $\xi = ct + k_1x_1 + k_2x_2 + l_1y_1 + l_2y_2 + \xi_0$.

Family 3: when $B \neq 0,\ \Delta_1 = 0$:

$$\begin{aligned}
&q_4(t, x_1, x_2, y_1, y_2) = \\
&-\frac{4A^2c^\alpha k_1^\alpha - 6A^2 l_1^\alpha l_2^\alpha + 8AEk_2^\alpha k_1^{3\alpha} - 8AEk_2^{3\alpha} k_1^\alpha}{12k_1^\alpha k_2^\alpha A^2} \\
&-\frac{-k_1^{3\alpha} k_2^\alpha B^2 - 8k_1^{3\alpha} k_2^\alpha CE + B^2 k_2^{3\alpha} k_1^\alpha + 8Ek_2^{3\alpha} k_1^\alpha C}{12k_1^\alpha k_2^\alpha A^2} \\
&-\frac{(Ak_1^{3\alpha} k_2^\alpha - Ak_2^{3\alpha} k_1^\alpha - k_1^{3\alpha} k_2^\alpha C + k_2^{3\alpha} k_1^\alpha C)B}{k_1^\alpha k_2^\alpha A^2} \times \\
&\{\frac{B}{2(A-C)} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha) + C_2\xi^\alpha}\} \\
&+\{\frac{A^2 k_1^{3\alpha} k_2^\alpha - A^2 k_2^{3\alpha} k_1^\alpha - 2Ak_1^{3\alpha} k_2^\alpha C}{k_1^\alpha k_2^\alpha A^2} \\
&+\frac{2Ak_2^{3\alpha} k_1^\alpha C + k_1^{3\alpha} k_2^\alpha C^2 - k_2^{3\alpha} k_1^\alpha C^2}{k_1^\alpha k_2^\alpha A^2}\} \times \\
&\{\frac{B}{2(A-C)} + \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha) + C_2\xi^\alpha}\}^2,
\end{aligned} \tag{45}$$

where C_1, C_2 are arbitrary constants, and $\xi = ct + k_1x_1 + k_2x_2 + l_1y_1 + l_2y_2 + \xi_0$.

Family 4: when $B = 0,\ \Delta_2 > 0$:

$$q_5(t, x_1, x_2, y_1, y_2) = -\frac{4A^2c^\alpha k_1^\alpha - 6A^2 l_1^\alpha l_2^\alpha + 8AEk_2^\alpha k_1^{3\alpha} - 8AEk_2^{3\alpha}k_1^\alpha}{12k_1^\alpha k_2^\alpha A^2} - \frac{-k_1^{3\alpha}k_2^\alpha B^2 - 8k_1^{3\alpha}k_2^\alpha CE + B^2k_2^{3\alpha}k_1^\alpha + 8Ek_2^{3\alpha}k_1^\alpha C}{12k_1^\alpha k_2^\alpha A^2} - \frac{(Ak_1^{3\alpha}k_2^\alpha - Ak_2^{3\alpha}k_1^\alpha - k_1^{3\alpha}k_2^\alpha C + k_2^{3\alpha}k_1^\alpha C)B}{k_1^\alpha k_2^\alpha A^2}\left\{\frac{\sqrt{\Delta_2}}{(A-C)}\left[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)} + C_2\cosh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)} + C_2\sinh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}\right]\right\} + \left\{\frac{A^2k_1^{3\alpha}k_2^\alpha - A^2k_2^{3\alpha}k_1^\alpha - 2Ak_1^{3\alpha}k_2^\alpha C}{k_1^\alpha k_2^\alpha A^2} + \frac{2Ak_2^{3\alpha}k_1^\alpha C + k_1^{3\alpha}k_2^\alpha C^2 - k_2^{3\alpha}k_1^\alpha C^2}{k_1^\alpha k_2^\alpha A^2}\right\}\left\{\frac{\sqrt{\Delta_2}}{(A-C)}\left[\frac{C_1\sinh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)} + C_2\cosh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}{C_1\cosh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)} + C_2\sinh\frac{\sqrt{\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}\right]\right\}^2, \tag{46}$$

where $C_1,\ C_2$ are arbitrary constants, and $\xi = ct + k_1x_1 + k_2x_2 + l_1y_1 + l_2y_2 + \xi_0$.

Family 5: when $B = 0,\ \Delta_2 < 0$:

$$q_6(t, x_1, x_2, y_1, y_2) = -\frac{4A^2c^\alpha k_1^\alpha - 6A^2 l_1^\alpha l_2^\alpha + 8AEk_2^\alpha k_1^{3\alpha} - 8AEk_2^{3\alpha}k_1^\alpha}{12k_1^\alpha k_2^\alpha A^2} - \frac{-k_1^{3\alpha}k_2^\alpha B^2 - 8k_1^{3\alpha}k_2^\alpha CE + B^2k_2^{3\alpha}k_1^\alpha + 8Ek_2^{3\alpha}k_1^\alpha C}{12k_1^\alpha k_2^\alpha A^2} - \frac{(Ak_1^{3\alpha}k_2^\alpha - Ak_2^{3\alpha}k_1^\alpha - k_1^{3\alpha}k_2^\alpha C + k_2^{3\alpha}k_1^\alpha C)B}{k_1^\alpha k_2^\alpha A^2}\left\{\frac{\sqrt{-\Delta_2}}{(A-C)}\left[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)} + C_2\cos\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)} + C_2\sin\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}\right]\right\} + \left\{\frac{A^2k_1^{3\alpha}k_2^\alpha - A^2k_2^{3\alpha}k_1^\alpha - 2Ak_1^{3\alpha}k_2^\alpha C}{k_1^\alpha k_2^\alpha A^2} + \frac{2Ak_2^{3\alpha}k_1^\alpha C + k_1^{3\alpha}k_2^\alpha C^2 - k_2^{3\alpha}k_1^\alpha C^2}{k_1^\alpha k_2^\alpha A^2}\right\}\left\{\frac{\sqrt{-\Delta_2}}{(A-C)}\left[\frac{-C_1\sin\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)} + C_2\cos\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}{C_1\cos\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)} + C_2\sin\frac{\sqrt{-\Delta_2}\xi^\alpha}{(A-C)\Gamma(1+\alpha)}}\right]\right\}^2, \tag{47}$$

where $C_1,\ C_2$ are arbitrary constants, $\xi = ct + k_1x_1 + k_2x_2 + l_1y_1 + l_2y_2 + \xi_0$.

4 Conclusions

By use of a new fractional sub-equation, we have proposed a new fractional $(\frac{D^\alpha G}{G})$ method to seek exact solutions for fractional partial differential equations. As for applications, we apply this method to solve the space-time fractional (2+1)-dimensional breaking soliton equations and the space-time fractional Fokas equation. Abundant new exact solutions including hyperbolic function solutions, trigonometric function solutions, and rational function solutions for the two equations have been successfully found. This method is based on the homogeneous balancing principle and the fractional complex transformation. Being concise and powerful, it can be applied to solve other fractional partial differential equations.

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