Integro-Differential Splines of Two Variables

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Abstract: Here we construct the basic splines of two variables which can be used for approximation functions. The approximation can be constructed in every elementary rectangular separately if the values of the function in nodes and the values of the integrals over elementary rectangles are known. The purpose of the article is to describe representation of surfaces using the local basic splines of two variables. We discuss the construction of surfaces with given accuracy. As a result we present examples and suggest directions for further investigations.

Key–Words: Polynomial splines, Exponential splines, Integro-Differential Splines, Interpolation.

1 Introduction

Nowadays there are many splines for solving different problems [1–10]. Polynomial integro-differential splines were first used by Kireev V.I. Local splines can be used to solve different problems of mathematical physics and to plot surfaces.

The problem of construction images was regarded in [11–14]. Integro-differential nonpolynomial approximations of one variable were suggested in [15, 16]. We create the surface if the values of the function in nodes and the values of the integrals are known.

2 Construction of the approximation

Let n, m be integer numbers, such that $n \geq 2$, $m \geq 1$, Supposing a, b, c, d are real numbers. Let us consider a rectangular domain $\bar{\Omega} = \Omega \bigcup \Gamma$ where

$$\Omega = \{(x, y)|a < x < b, c < y < d\}$$

and Γ is the boundary of Ω. We introduce Δ_x : $a = x_0 < x_1 < \ldots < x_{n+1} = b$, Δ_y : $c = y_0 < y_1 < \ldots < y_{m+1} = d$, and a mesh of lines on $\bar{\Omega}$ which divides the domain $\bar{\Omega}$ into the rectangles $\bar{\Omega}_{j,k} = \Omega_{j,k} \bigcup \Gamma_{j,k}$,

$$\Omega_{j,k} = \{(x, y)|x \in (x_j, x_{j+1}), y \in (y_k, y_{k+1})\},$$

$\Gamma_{j,k}$ is the boundary of $\Omega_{j,k}$, $j = 0, \ldots, n$, $k = 0, \ldots, m$, $h_j = x_{j+1} - x_j$, $h_k = y_{k+1} - y_k$.

We use $u_{j,k}$ to denote $u(x_j, y_k)$. We associate the mesh $\Delta_x \times \Delta_y$ with the data: $(x_j, y_k, u_{j,k})$, $j = 0, 1, \ldots, n + 1$, $k = 0, 1, \ldots, m + 1$. It is supposed that we know

$$I^{<0>}_{j,k} = \iint\limits_{\bar{\Omega}_{j,k}} u(x, y)dxdy,$$

$$I^{<-1>}_{j,k} = \int_{x_{j-1}}^{x_j} \int_{y_k}^{y_{k+1}} u(x, y)dxdy.$$

Let us take the approximation $\tilde{u}(x, y)$ of $u(x, y)$ in $\bar{\Omega}_{j,k}$ in the form:

$$\tilde{u}(x, y) = u_{j,k}W_1(x, y) + u_{j+1,k}W_2(x, y)+$$
$$+u_{j,k+1}W_3(x, y) + u_{j+1,k+1}W_4(x, y)+$$
$$+I^{<0>}_{j,k}W_5(x, y) + I^{<-1>}_{j,k}W_6(x, y), \qquad (1)$$

where we obtain the basic splines $W_i(x, y)$ from:

$$\tilde{u}(x, y) = u(x, y) \text{ for } u(x, y) = 1, x, y, xy, x^2, y^2. \qquad (2)$$

Using Taylor's formula

$$u(x, y) = u(x_j, y_k) + (x - x_j)u'_x(x_j, y_k)+$$
$$+(y - y_k)u'_y(x_j, y_k) + \frac{1}{2!}\{(x - x_j)^2 u''_{xx}(x_j, y_k)+$$
$$2(x-x_j)(y-y_k)u''_{xy}(x_j, y_k)+(y-y_k)^2u''_{yy}(x_j, y_k)\}+r,$$
$$r = \frac{1}{3!}\{(x-x_j)^3u'''_{xxx}(s)+3(x-x_j)^2(y-y_k)u'''_{xxy}(s)+$$
$$+3(x - x_j)(y - y_k)^2u'''_{xyy}(s) + (y - y_k)^3u'''_{yyy}(s)\},$$
$$s = (x_j+\tau(x_{j+1}-x_j), y_k+\tau(y_{k+1}-y_k)),\ \tau \in [0, 1],$$

we can obtain $W_i(x, y)$ from the system of equations:

$$W_1(x, y) + W_2(x, y) + W_3(x, y)+$$
$$+W_4(x, y) + I^{<0>}_{j,k}W_5(x, y) + I^{<-1>}_{j,k}W_6(x, y) = 1,$$
$$h_jW_2(x, y)+h_jW_4(x, y)+\iint\limits_{\bar{\Omega}_{j,k}}(x-x_j)dxdyW_5(x, y)+$$

$$+\int\limits_{x_{j-1}}^{x_j}\int\limits_{y_k}^{y_{k+1}}(x-x_j)dxdyW_6(x,y)=x-x_j,$$

$$h_kW_3(x,y)+h_kW_4(x,y)+\iint\limits_{\bar{\Omega}_{j,k}}(y-y_k)dxdy\times$$

$$\times W_5(x,y)+\int\limits_{x_{j-1}}^{x_j}\int\limits_{y_k}^{y_{k+1}}(y-y_k)dxdyW_6(x,y)=y-y_k,$$

$$h_j^2W_2(x,y)+h_j^2W_4(x,y)+\iint\limits_{\bar{\Omega}_{j,k}}(x-x_j)^2dxdyW_5(x,y)+$$

$$+\int\limits_{x_{j-1}}^{x_j}\int\limits_{y_k}^{y_{k+1}}(x-x_j)^2dxdy\,W_6(x,y)=(x-x_j)^2,$$

$$h_jh_kW_3(x,y)+h_jh_kW_4(x,y)+$$

$$+\iint\limits_{\bar{\Omega}_{j,k}}(x-x_j)(y-y_k)dxdy\,W_5(x,y)+$$

$$+\int\limits_{x_{j-1}}^{x_j}\int\limits_{y_k}^{y_{k+1}}(x-x_j)(y-y_k)dxdyW_6(x,y)=(x-x_j)(y-y_k),$$

$$h_k^2W_3(x,y)+h_k^2W_4(x,y)+\iint\limits_{\bar{\Omega}_{j,k}}(y-y_k)^2dxdy\,W_5(x,y)+$$

$$+\int\limits_{x_{j-1}}^{x_j}\int\limits_{y_k}^{y_{k+1}}(y-y_k)^2dxdy\,W_6(x,y)=(y-y_k)^2.$$

The value of the determinant of the system is the following: $D=-(1/6)h_j^6h_k^6$. We can obtain for $x,y\in\bar{\Omega}_{j,k}$:

$W_1(x,y)=-1/(2h_k^2h_j^2)\cdot(-4h_j^2y^2-4h_j^2h_k^2k^2+6h_kyh_j^2-6h_k^2kh_j^2+xh_jh_k^2-jh_j^2h_k^2+x^2h_k^2+j^2h_j^2h_k^2+8h_j^2h_kyk-2h_jh_kxy+2h_jh_k^2xk+2h_j^2h_kjy-2h_j^2h_k^2jk-2xjh_jh_k^2-2h_j^2h_k^2)$,

$W_2(x,y)=1/(2h_k^2h_j^2)\cdot(xh_jh_k^2-jh_j^2h_k^2+x^2h_k^2-2xjh_jh_k^2+j^2h_j^2h_k^2-2h_jh_kxy+2h_jh_k^2xk+2h_j^2h_kjy-2h_j^2h_k^2jk+2h_j^2y^2-4h_j^2h_kyk+2h_j^2h_k^2k^2-2h_kyh_j^2+2h_k^2kh_j^2)$,

$W_3(x,y)=-1/(2h_k^2h_j^2)\cdot(x^2h_k^2-2xjh_jh_k^2+j^2h_j^2h_k^2-4h_j^2y^2+8h_j^2h_kyk-4h_j^2h_k^2k^2-xh_jh_k^2+jh_j^2h_k^2+2h_kyh_j^2-2h_k^2kh_j^2+2h_jh_kxy-2h_jh_k^2xk-2h_j^2h_kjy+2h_j^2h_k^2jk)$,

$W_4(x,y)=1/(2h_k^2h_j^2)\cdot(x^2h_k^2-2xjh_jh_k^2+j^2h_j^2h_k^2+2h_j^2y^2-4h_j^2h_kyk+2h_j^2h_k^2k^2-2h_kyh_j^2+2h_k^2kh_j^2-xh_jh_k^2+jh_j^2h_k^2+2h_jh_kxy-2h_jh_k^2xk-2h_j^2h_kjy+2h_j^2h_k^2jk)$,

$W_5(x,y)=-1/(h_j^3h_k^3)\cdot(5h_j^2y^2+5h_j^2h_k^2k^2-10h_j^2h_kyk-2xjh_jh_k^2-5h_kyh_j^2+5h_k^2kh_j^2+x^2h_k^2+j^2h_j^2h_k^2-xh_jh_k^2+jh_j^2h_k^2)$,

$W_6(x,y)=1/(h_j^3h_k^3)\cdot(-h_j^2y^2-h_j^2h_k^2k^2+2h_j^2h_kyk-2xjh_jh_k^2+h_kyh_j^2-h_k^2kh_j^2+x^2h_k^2+j^2h_j^2h_k^2-xh_jh_k^2+jh_j^2h_k^2)$.

If $h_k=h_j=h$, $x=x_j+th, y=x_k+t_1h$, $t,t_1\in[0,1]$, then we get $W_1(x_j+th,y_k+t_1h)=-(1/2)t^2+2t_1^2-3t_1-(1/2)t+tt_1+1$,

$W_2(x_j+th,y_k+t_1h)=-tt_1+t_1^2-t_1+(1/2)t^2+(1/2)t$,

$W_3(x_j+th,y_k+t_1h)=-tt_1+2t_1^2-t_1-(1/2)t^2+(1/2)t$,

$W_4(x_j+th,y_k+t_1h)=tt_1+t_1^2-t_1+(1/2)t^2-(1/2)t$,

$W_5(x_j+th,y_k+t_1h)=-(1/h^2)(5t_1^2-5t_1+t^2-t)$,

$W_6(x_j+th,y_k+t_1h)=(1/h^2)(-t_1^2+t_1+t^2-t)$.

Figure 1 shows the basic functions $W_1(x,y)$ (left), $W_2(x,y)$ (right). Figure 2 shows the basic functions $W_3(x,y)$ (left), $W_4(x,y)$ (right). Figure 3 shows the basic functions $W_5(x,y)$ (left), $W_6(x,y)$(right).

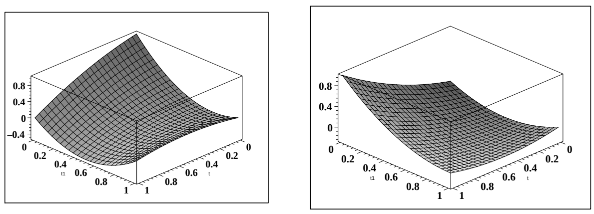

Figure 1: Plots of $W_1(x,y)$ (left), $W_2(x,y)$ (right)

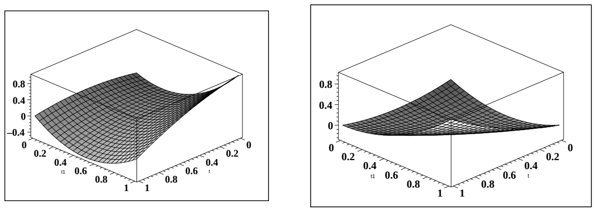

Figure 2: Plots of $W_3(x,y)$ (left), $W_4(x,y)$ (right)

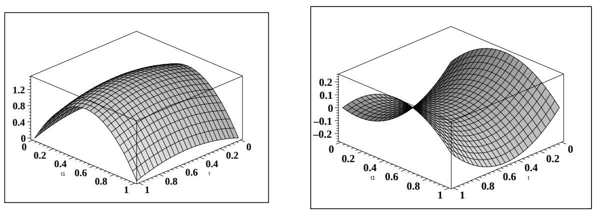

Figure 3: Plots of $W_5(x,y)$ (left), $W_6(x,y)$ (right)

Let us denote $\bar{\Omega}_h = \{(x,y) \,|\, a+h \le x \le b, c \le y \le d\}$,

Theorem 1. Let function $u(x,y)$ be such that $u \in C^3(\bar{\Omega})$, We assume that (1), (2) are fulfilled, $h_j = h_k = h$.

For $(x,y) \in \bar{\Omega}_{j,k} \subset \bar{\Omega}_h$ we have

$$|\tilde{u}(x,y) - u(x,y)| \le h^3 K \|u'''\|_{\bar{\Omega}},\ K = 1.$$

Proof follows from Taylor's formula

$$f(z+\xi) = \sum_{|\beta|=0}^{n} \frac{\xi^\beta}{\beta!} f^{(\beta)}(z) + \varrho,$$

$$\varrho = (n+1)\int_0^1 \sum_{|\beta|=n+1} \frac{\xi^\beta}{\beta!} f^{(\beta)}(z+t\xi)(1-t)^n dt, \tag{3}$$

where $n = 2$, $z = (x,y)$, and the relations: $|W_1| \le 1$, $|W_2| \le 1$, $|W_3| \le 1$, $|W_4| \le 1$, $|W_5| \le 1.5/h^2$, $|W_6| \le 0.25/h^2$.

Example 1. Let us take $\bar{\Omega} = [-0.5, 0.4] \times [-0.5, 0.4]$, $h = 0.1$. Table 1 shows the error of the approximation: $\max_{j,k} |\widetilde{u}(s_{j,k}) - u(s_{j,k})|$, $s_{j,k} = (x_j + 0.05, y_k + 0.05)$. Table 2 shows the error of the approximation: $\max_{j,k} |\widetilde{u}(s_{j,k}) - u(s_{j,k})|$, $s_{j,k} = (x_j + 0.05, y_k)$. Calculations were made in Maple with Digits=10.

Table 1.

N	$u(x,y)$	$\max_{j,k} \lvert\widetilde{u}(s_{j,k}) - u(s_{j,k})\rvert$
1	x^2y^2	$0.208334e-5$
2	$(\sin(x)\cos(y))^2$	$0.247076e-5$
3	x^4y^4	$0.319319e-5$
4	$\sin(3x+3y)$	$0.502999e-4$
5	$\sin(5x+5y)$	$0.385416e-3$

Table 2.

N	$u(x,y)$	$\max_{j,k} \lvert\widetilde{u}(s_{j,k}) - u(s_{j,k})\rvert$
1	x^2y^2	$0.154167e-3$
2	$(\sin(x)\cos(y))^2$	$0.913953e-3$
3	x^4y^4	$0.612051e-4$
4	$\sin(3x+3y)$	$0.112559e-2$
5	$\sin(5x+5y)$	$0.521490e-2$

Figure 4 shows $\widetilde{u}(s_{j,k})$ and $\widetilde{u}(s_{j,k}) - u(s_{j,k})$ in $\bar{\Omega} = [-0.5, 0.5] \times [-0.5, 0.5]$ if $u(x,y) = x^2y^2$, $h = 0.05$. Figure 5 shows $\widetilde{u}(s_{j,k})$ and $\widetilde{u}(s_{j,k}) - u(s_{j,k})$ in $\bar{\Omega} = [-0.5, 0.5] \times [-0.5, 0.5]$ if $u(x,y) = \sin(5x + 5y)$, $h = 0.05$. Figure 6 shows $\widetilde{u}(s_{j,k}) - u(s_{j,k})$ in $\bar{\Omega} = [-0.5, 0.5] \times [-0.5, 0.5]$ if $u(x,y) = (\sin(x)\cos(y))^2$, $u(x,y) = \sin(3x+3y)$, $h = 0.05$.

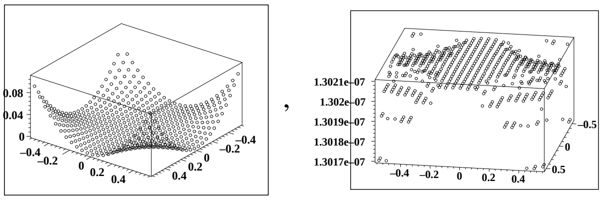

Figure 4: Plots of $\widetilde{u}(s_{j,k})$ (left) and $\widetilde{u}(s_{j,k}) - u(s_{j,k})$ (right), $u(x,y) = x^2y^2$

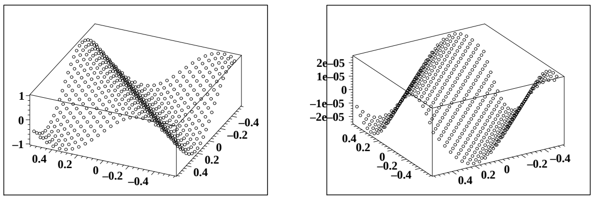

Figure 5: Plots of $\widetilde{u}(s_{j,k})$ (left) and $\widetilde{u}(s_{j,k}) - u(s_{j,k})$ (right), $u(x,y) = \sin(5x + 5y)$

3 Second order approximation

Now it is supposed that we know

$$I_{j,k}^{<0>} = \iint\limits_{\bar{\Omega}_{j,k}} u(x,y)\,dxdy.$$

1) We construct an approximation $\tilde{u}(x,y)$ of $u(x,y)$ in $\bar{\Omega}_{jk}$ in the form:

$$\tilde{u}(x,y) = u(x_j, y_k)W_{1,j,k}(x,y) +$$

$$+ u(x_{j+1}, y_k)W_{2,j,k}(x,y) + I_{j,k}^{<0>} W_{j,k}^{<0>}(x,y), \tag{4}$$

where basic splines $W_{1,j,k}(x,y)$, $W_{2,j,k}(x,y)$, $W_{j,k}^{<0>}(x,y)$ we obtain from the relations:

$$\tilde{u}(x,y) = u(x,y) \text{ for } u(x,y) = 1, x, y. \tag{5}$$

Using Taylor's formula

$$u(x,y) = u(x_j, y_k) + (x - x_j)u'_x(x_j, y_k) +$$

$$+ (y - y_k)u'_y(x_j, y_k) + r,$$

where

$r = \frac{1}{2!}\{(x - x_j)^2 u''_{xx}(s) + 2(x - x_j)(y - y_k)u''_{xy}(s) + (y - y_k)^2 u''_{yy}(s)\}$,

$s = (x_j + \tau(x_{j+1} - x_j), y_k + \tau(y_{k+1} - y_k))$, $\tau \in [0,1]$.

We obtain $W_{1,j,k}(x,y)$, $W_{2,j,k}(x,y)$, $W_{j,k}^{<0>}(x,y)$ from the system of equations:

$$W_{1,j,k}(x,y) + W_{2,j,k}(x,y) + I_{j,k}^{<0>} W_{j,k}^{<0>}(x,y) = 1,$$

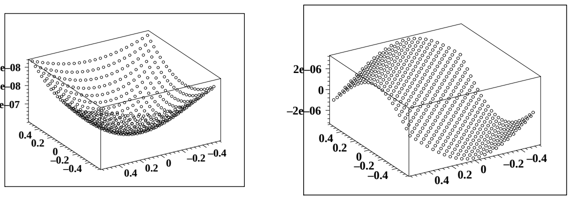

Figure 6: Plots of $\widetilde{u} - u$, $u(x, y) = (\sin(x)\cos(y))^2$ (left), $u(x, y) = \sin(3x + 3y)$(right)

$$h_j W_{2,j,k}(x,y) + \iint_{\bar{\Omega}_{j,k}} (x - x_j) dxdy W_{j,k}^{<0>}(x,y) = x - x_j,$$

$$\iint_{\bar{\Omega}_{j,k}} (y - y_k) dxdy W_{j,k}^{<0>}(x,y) = y - y_k.$$

The value of the determinant of the system is: $D = h_j^2 h_k^2/2$.

We can obtain:

$W_{1,j,k}(x, y) = (h_j h_k - h_k x + h_j h_k j - h_j y + h_k h_j k)/(h_j h_k)$,

$W_{2,j,k}(x, y) = -(-h_k x + h_j h_k j + h_j y - h_k h_j k)/(h_j h_k)$,

$W_{j,k}^{<0>}(x, y) = -2(-y + k h_k)/(h_k^2 h_j)$,

If $h_j = h_k = h$, and we put $x = x_j + th, y = y_k + t_1 h, t, t_1 \in [0, 1]$ then we get $W_{1,j,k}(x_j + th, y_k + t_1 h) = 1 - t - t_1$,

$W_{2,j,k}(x_j + th, y_k + t_1 h) = t - t_1$,

$W_{j,k}^{<0>}(x_j + th, y_k + t_1 h) = 2t_1/h^2$.

Theorem 2. Let function $u(x, y)$ be such that $u \in C^2(\bar{\Omega})$. We assume that (4), (5) are fulfilled and $h_j = h_k = h$.

For $(x, y) \in \bar{\Omega}_{j,k} \subset \bar{\Omega}$ we obtain

$$|\tilde{u}(x, y) - u(x, y)| \le h^2 K \|u''\|_{\bar{\Omega}},\ K = 1.$$

Proof. The above follows from Taylor's formula (3) where $n = 1$, $z = (x, y)$ and the relations: $|W_{1,j,k}(x, y)| \le 1$, $|W_{2,j,k}(x, y)| \le 1$, $|W_{j,k}^{<0>}(x, y)| \le 0.2/h^2$.

2) We construct an approximation $\tilde{\tilde{u}}(x, y)$ of $u(x, y)$ in Ω_{jk} in the form:

$$\tilde{\tilde{u}}(x, y) = u(x_j, y_k)W_{1,j,k}(x, y) +$$

$$+ u(x_{j+1}, y_k)W_{2,j,k}(x, y) + I_{j,k}^{<0>} W_{j,k}^{<0>}(x, y),$$

where we obtain the basic splines $W_{1,j,k}(x, y)$, $W_{2,j,k}(x, y)$, $W_{j,k}^{<0>}(x, y)$ from the relations:

$$\tilde{u}(x, y) = u(x, y) \text{ for } u(x, y) = 1, e^x, e^y.$$

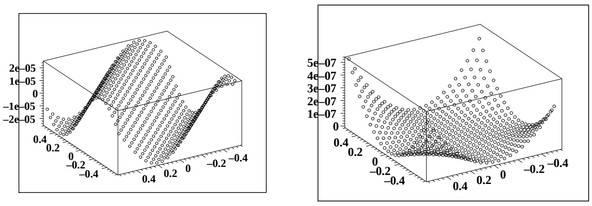

Figure 7: Plots of $\widetilde{u} - u$, $u(x, y) = \sin(5x + 5y)$ (left), $u(x, y) = x^4 y^4$ (right)

Example 2. Let us take $\bar{\Omega} = [-0.5, 0.4] \times [-0.5, 0.4]$, $h = 0.1$. Figure 7 shows $\widetilde{u}(s_{j,k}) - u(s_{j,k})$ in Ω if $u(x, y) = \sin(5x + 5y)$, $u(x, y) = x^4 y^4$.

Table 3 shows the errors of approximations $\max_{j,k} |\widetilde{u}(s_{j,k}) - u(s_{j,k})|$ and $\max_{j,k} |\widetilde{\widetilde{u}}(s_{j,k}) - u(s_{j,k})|$ of the function $u(x, y)$, $s_{j,k} = (x_j + 0.05, y_k + 0.05)$, $\bar{\Omega} = [-0.5, 0.4] \times [-0.5, 0.4]$, $h = 0.1$.

Table 3.

N	$u(x,y)$	$\max\|\widetilde{u} - u\|$	$\max\|\widetilde{\widetilde{u}} - u\|$
1	$x^2 y^2$	$0.33819e{-3}$	$0.24576e{-2}$
2	$\sin^2(x)\cos^2(y)$	$0.82393e{-3}$	$0.91366e{-3}$
3	$x^4 y^4$	$0.16815e{-2}$	$0.82013e{-1}$
4	$\sin(3x + 3y)$	$0.74588e{-2}$	$0.81701e{-2}$
5	$\sin(5x + 5y)$	$0.20609e{-1}$	$0.21895e{-1}$
6	$e^x e^y$	$0.20504e{-2}$	$0.49988e{-4}$
7	$e^x + e^y$	$0.13071e{-2}$	$0.$

4 Continuous approximation

In this part we suggest a way to construct a continuous approximation with integro-differential splines on the line parallel to axe x using interpolation. We can use cubic polynomial interpolation. We divide h into 4 parts, put $h_1 = h/4$ and we introduce additional nodes X_{j-1}, X_j, X_{j+1}, X_{j+2}. Now we can construct a continuous approximation $\widetilde{\widetilde{u}} \in C[x_j, x_{j+2}]$ in the form:

$$\widetilde{\widetilde{u}}(x) = \widetilde{u}(X_{j-1})w_{j-1}(x) + \widetilde{u}(X_j)w_j(x) +$$

$$\widetilde{u}(X_{j+1})w_{j+1}(x) + \widetilde{u}(X_{j+2})w_{j+2}(x),$$

where $x \in [X_j, X_{j+1}] \subset [x_{j+1} - h_1, x_{j+1} + h_1]$, $\widetilde{\widetilde{u}}(x) = \widetilde{u}(x), x \in [x_j, x_{j+1} - h_1] \cup [x_{j+1} + h_1, x_{j+2}]$,

$w_{j-1}(x) = (x - X_j)/(X_{j-1} - X_j) \cdot (x - X_{j+1})/(X_{j-1} - X_{j+1}) \cdot (x - X_{j+2})/(X_{j-1} - X_{j+2})$,

$w_j(x) = (x - X_{j-1})/(X_j - X_{j-1}) \cdot (x - X_{j+1})/(X_j - X_{j+1}) \cdot (x - X_{j+2})/(X_j - X_{j+2})$,

$w_{j+1}(x) = (x - X_{j-1})/(X_{j+1} - X_{j-1}) \cdot (x - X_j)/(X_{j+1} - X_j) \cdot (x - X_{j+2})/(X_{j+1} - X_{j+2})$,

$w_{j+2}(x) = (x - X_{j-1})/(X_{j+2} - X_{j-1}) \cdot (x - X_j)/(X_{j+2} - X_j) \cdot (x - X_{j+1})/(X_{j+2} - X_{j+1})$.

We take nodes so that $X_{j-1}, X_j \in [x_j + h_1, x_{j+1} - h_1]$, $X_{j+1}, X_{j+2} \in [x_{j+1} + h_1, x_{j+2} - h_1]$. We can use $X_{j-1} = x_j + h_1$, $X_j = x_{j+1} - h_1$, $X_{j+1} = x_{j+1} + h_1$, $X_{j+2} = x_{j+2} - h_1$.

Now we apply this approach to construct the continuous approximation. Let us take $x \in [0, 1]$, $y = 0.05$, $h = 0.1$ and assume that $\widetilde{u}^a$ is the approximation with the additional interpolation.

Plot of the error $\widetilde{u} - u$ of the approximation of the function $u(x, y) = 1/((1 + 25x^2)(1 + 25y^2))$ without the additional interpolation is on graph 8 (left). Plot of the error $\widetilde{u}^a - u$ with the additional interpolation is on graph 8 (right).

Plot of $\widetilde{u} - u$, where $u(x, y) = 1/((1 + 25x^2)(1 + 25y^2))$, $y = 0.0$, without the additional interpolation is on graph 9 (left), plot of $\widetilde{u}^a - u$ with the additional interpolation is on graph 9 (right).

Plots of $\widetilde{u}(x, y)$ and $u(x, y)$, where $u(x, y) = 1/((1 + 25x^2)(1 + 25y^2))$, $y = 0.05$, are on graph 10 (left), plots of the the error of the approximation with and without the additional interpolation are on graph 10 (right).

Plots of the error of the approximation of the function $u(x, y) = 1/((1 + 25x^2)(1 + 25y^2))$ with and without the additional interpolation for $y = 0.01$ are on graph 11 (left), for $y = 0.0$ are on graph 11 (right).

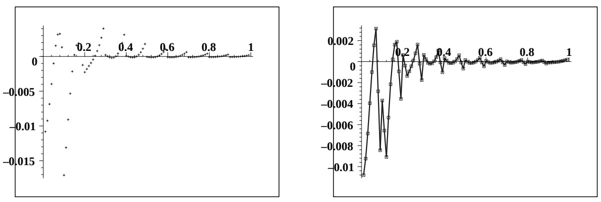

Figure 8: Plots of $\widetilde{u} - u$ (left), $\widetilde{u}^a - u$ (right), here $u(x, y) = 1/((1 + 25x^2)(1 + 25y^2))$, $y = 0.05$, $h = 0.1$

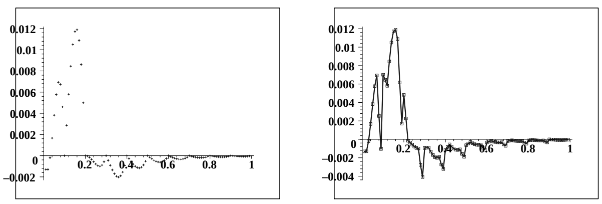

Figure 9: Plots of $\widetilde{u} - u$ (left), $\widetilde{u}^a - u$ (right), here $u(x, y) = 1/((1 + 25x^2)(1 + 25y^2))$, $y = 0.0$

Table 4 shows the error of approximation functions without additional interpolation and the error of approximation with additional interpolation, $h = 0.1$, $x \in [0, 1]$, $y = 0.05$.

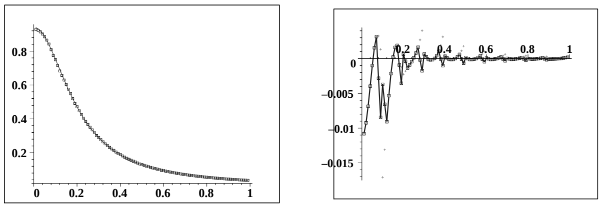

Figure 10: Plots of $\widetilde{u}(x, y)$, $u(x, y)$ (left), $\widetilde{u}^a - u$ and $\widetilde{u}(x, y) - u(x, y)$ (right), here $u(x, y) = 1/((1 + 25x^2)(1 + 25y^2))$, $y = 0.05$

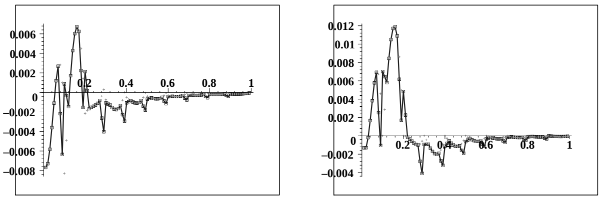

Figure 11: Plots of $\widetilde{u}^a(x, y) - u(x, y)$ and $\widetilde{u}(x, y) - u(x, y)$, $y = 0.01$ (left), $y = 0.0$ (right), here $u(x, y) = 1/((1 + 25x^2)(1 + 25y^2))$.

Table 4.

N	$u(x,y)$	$\max\lvert\widetilde{u}-u\rvert$	$\max\lvert\widetilde{u}^a-u\rvert$
1	x^2y^2	$0.2399e{-3}$	$0.1697e{-3}$
2	$\sin^2(x)\cos^2(y)$	$0.2472e{-3}$	$0.2782e{-2}$
3	x^4y^4	$0.6556e{-5}$	$0.5997e{-5}$
4	$\sin(3x+3y)$	$0.2195e{-2}$	$0.7603e{-2}$
5	$\sin(5x+5y)$	$0.9963e{-2}$	$0.1282e{-1}$
6	e^xe^y	$0.2235e{-3}$	$0.6852e{-2}$
7	e^x+e^y	$0.1047e{-3}$	$0.6502e{-2}$

Table 5 shows the error of approximation functions without additional interpolation and with additional interpolation, $h = 0.1$, $x \in [0, 1]$, $y = 0.01$.

Table 5.

N	$u(x,y)$	$\max\lvert\widetilde{u}-u\rvert$	$\max\lvert\widetilde{u}^a-u\rvert$
1	x^2y^2	$0.7899e{-4}$	$0.7864e{-4}$
2	$\sin^2(x)\cos^2(y)$	$0.1073e{-3}$	$0.2715e{-2}$
3	x^4y^4	$0.7286e{-5}$	$0.7282e{-5}$
4	$\sin(3x+3y)$	$0.1435e{-2}$	$0.8257e{-2}$
5	$\sin(5x+5y)$	$0.6588e{-2}$	$0.1693e{-1}$
6	e^xe^y	$0.1462e{-3}$	$0.6507e{-2}$
7	e^x+e^y	$0.6674e{-4}$	$0.6553e{-2}$

5 On interpolation splines

Now we interpolate the function $u(x, y)$ in $\bar{\Omega}_{jk}$ in the following form:

$$\tilde{u}(x, y) = u_{j,k}w_{j,k}(x, y) + u_{j+1,k}w_{j+1,k}(x, y)+$$

$+u_{j,k+1}w_{j,k+1}(x,y) + u_{j+1,k+1}w_{j+1,k+1}(x,y)+$
$+u_{j-1,k}w_{j-1,k}(x,y) + u_{j-1,k-1}w_{j-1,k-1}(x,y).$

From the relation

$$\tilde{u}(x,y) = u(x,y) \text{ for } u(x,y) = 1, x, y, xy, x^2, y^2,$$

we have the value of the system determinant:

$D = -4h_j^4 h_k^4$

and

$w_{j,k}(x,y) = -1/(h_j^2\, h_k^2) \cdot (j^2\, h_j^2\, h_k^2 - h_j^2\, h_k^2 + j\, h_j^2\, y\, h_k - j\, h_j^2\, k\, h_k^2 - 2\, x\, j\, h_j\, h_k^2 - 2\, y\, k\, h_k\, h_j^2 - h_j\, x\, y\, h_k + y^2\, h_j^2 + k^2\, h_k^2\, h_j^2 + x^2\, h_k^2 + h_j\, x\, k\, h_k^2),$

$w_{j+1,k}(x,y) = 1/(2h_j^2 h_k^2) \cdot (x\, h_j\, h_k^2 - j\, h_j^2\, h_k^2 + x^2\, h_k^2 - 2\, x\, j\, h_j\, h_k^2 + j^2\, h_j^2\, h_k^2 - 2\, h_j\, x\, y\, h_k + 2\, h_j\, x\, k\, h_k^2 + 2\, j\, h_j^2\, y\, h_k - 2\, j\, h_j^2\, k\, h_k^2 + y^2\, h_j^2 - 2\, y\, k\, h_k\, h_j^2 + k^2\, h_k^2\, h_j^2 - y\, h_j^2\, h_k + k\, h_k^2\, h_j^2),$

$w_{j,k+1}(x,y) = -1/(h_j h_k^2) \cdot (x\, y\, h_k - x\, k\, h_k^2 - j\, h_j\, y\, h_k + j\, h_j\, k\, h_k^2 - y^2\, h_j + 2\, y\, k\, h_k\, h_j - k^2\, h_k^2\, h_j),$

$w_{j+1,k+1}(x,y) = 1/(2h_j h_k^2) \cdot (2\, x\, y\, h_k - 2\, x\, k\, h_k^2 - 2\, j\, h_j\, y\, h_k + 2\, j\, h_j\, k\, h_k^2 - y^2\, h_j + 2\, y\, k\, h_k\, h_j - k^2\, h_k^2\, h_j + y\, h_j\, h_k - k\, h_k^2\, h_j),$

$w_{j-1,k}(x,y) = 1/(2h_j^2 h_k^2) \cdot (-2\, x\, j\, h_j\, h_k^2 + j^2\, h_j^2\, h_k^2 + x^2\, h_k^2 + 2\, y\, k\, h_k\, h_j^2 - y^2\, h_j^2 - k^2\, h_k^2\, h_j^2 + y\, h_j^2\, h_k - k\, h_k^2\, h_j^2 - x\, h_j\, h_k^2 + j\, h_j^2\, h_k^2),$

$w_{j-1,k-1}(x,y) = 1/(2h_k^2) \cdot (-y + k\, h_k)\,(h_k - y + k\, h_k).$

If we use $h_j = h_k = h, x = x_j + th, y = y_k + t_1 h$, then we have the resulting formulas:

$w_{j,k}(x,y) = 1 + t\, t_1 - t^2 - t_1^2,$

$w_{j+1,k}(x,y) = -t\, t_1 + (1/2)\, t^2 + (1/2)\, t_1^2 + (1/2)\, t - (1/2)\, t_1,$

$w_{j,k+1}(x,y) = -t_1\,(t - t_1),$

$w_{j+1,k+1}(x,y) = (1/2)\, t_1\,(2\, t - t_1 + 1),$

$w_{j-1,k}(x,y) = (1/2)\, t^2 - (1/2)\, t_1^2 + (1/2)\, t_1 - (1/2)\, t,$

$w_{j-1,k-1}(x,y) = (1/2)\, t_1\,(-1 + t_1).$

Thus we get:

$|w_{j-1,k}(x,y)| < 0.15, |w_{j,k}(x,y)| < 1,$
$|w_{j+1,k}(x,y)| < 1, |w_{j,k+1}(x,y)| < 1,$
$|w_{j-1,k-1}(x,y)| < 0.125.$

Table 6 shows the errors of approximation $\max_{j,k} |\widetilde{u}(s_{j,k}) - u(s_{j,k})|$ by interpolating spines. Here we take $h = 0.1$, $s_{j,k} = (x_j + 0.05, y_k + 0.05)$.

Table 6.

N	$u(x,y)$	$\max_{j,k} \lvert\widetilde{u}(s_{j,k}) - u(s_{j,k})\rvert$
1	x^2y^2	$0.731250e-3$
2	$(\sin(x)\cos(y))^2$	$0.541150e-3$
3	x^4y^4	$0.605585e-3$
4	$\sin(3x+3y)$	$0.131990e-1$
5	$\sin(5x+5y)$	$0.591601e-1$

Now construct a continuous approximation with interpolating splines on the line parallel axe x using the additional interpolation.

The error of the approximation of the function $u(x,y) = 1/((1+25x^2)\cdot(1+25y^2))$, $h = 0.1$, $y = 0.05$ are on graph 12 (left) and with the additional interpolation are on graph 12 (right).

The error of the approximation of the function $u(x,y) = 1/((1+25x^2)\cdot(1+25y^2))$, $h = 0.1$, $y = 0.01$ are on graph 13 (left) and $\widetilde{u}(x,y)$, $u(x,y)$ are on graph 13 (right).

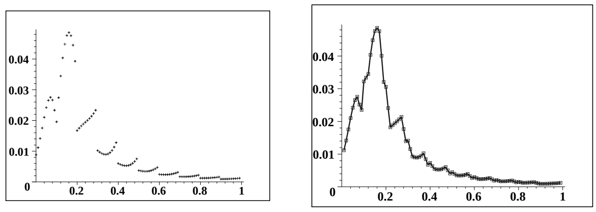

Figure 12: Plots of $\widetilde{u} - u$ (left), with the additional interpolation (right), $u(x,y) = 1/((1+25x^2)(1+25y^2))$, $y = 0.05$

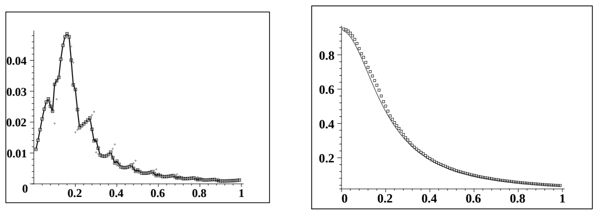

Figure 13: Plots of $\widetilde{u}(x,y)$, $u(x,y)$ (right), $\widetilde{u}(x,y) - u(x,y)$, with the additional interpolation and without the additional interpolation (left), $u(x,y) = 1/((1+25x^2)(1+25y^2))$, $y = 0.05$

6 Calculation of the integral

Assuming that functions $f(x)$ and $v(y)$ are as follows $f \in C^4[a,b]$, $v \in C^4[c,d]$, besides $\{X_j\}$ is a set of nodes on $[a,b]$, $\{Y_k\}$ is the set of nodes on $[c,d]$.

The approximation $\tilde{f}(x)$ of $f(x)$, $x \in [X_j, X_{j+1}]$ we construct in the form:

$\tilde{f}(x) = f(X_{j-1})\omega_{j-1}(x) + f(X_j)\omega_j(x) + f(X_{j+1})\omega_{j+1}(x) + f(X_{j+2})\omega_{j+2}(x).$

Here $\omega_{j-1}, \omega_j, \omega_{j+1}, \omega_{j+2}$ we obtain from

$\tilde{f}(x) = f(x)$ for $f = 1, x, x^2, x^3$.

We get

$\omega_{j-1}(x) = (x - X_j)/(X_{j-1} - X_j) \cdot (x - X_{j+1})/(X_{j-1} - X_{j+1}) \cdot (x - X_{j+2})/(X_{j-1} - X_{j+2}),$

$\omega_j(x) = (x - X_{j-1})/(X_j - X_{j-1}) \cdot (x - X_{j+1})/(X_j - X_{j+1}) \cdot (x - X_{j+2})/(X_j - X_{j+2}),$

$\omega_{j+1}(x) = (x - X_{j-1})/(X_{j+1} - X_{j-1}) \cdot (x - X_j)/(X_{j+1} - X_j) \cdot (x - X_{j+2})/(X_{j+1} - X_{j+2}),$

$\omega_{j+2}(x) = (x - X_{j-1})/(X_{j+2} - X_{j-1}) \cdot (x - X_j)/(X_{j+2} - X_j) \cdot (x - X_{j+1})/(X_{j+2} - X_{j+1})$.

We construct the approximation $\tilde{v}(y)$ of $v(y)$, $y \in [Y_k, Y_{k+1}]$ in the form:

$\tilde{v}(y) = v(Y_{k-1})w_{k-1}(y) + v(Y_k)w_k(y) + v(Y_{k+1})w_{k+1}(y) + v(Y_{k+2})w_{k+2}(y)$,

where $w_i(y)$, $i = k-1, k, k+1, k+2$, can be obtained from $\tilde{v}(y) = v(y)$ for $v = 1, y, y^2, y^3$.

The functions $w_i(y)$, $i = k-1, k, k+1, k+2$, can be presented as:

$w_{k-1}(y) = (y - Y_k)/(Y_{k-1} - Y_k) \cdot (y - Y_{k+1})/(Y_{k-1} - Y_{k+1}) \cdot (y - Y_{k+2})/(Y_{k-1} - Y_{k+2})$,

$w_k(y) = (y - Y_{k-1})/(Y_k - Y_{k-1}) \cdot (y - Y_{k+1})/(Y_k - Y_{k+1}) \cdot (y - Y_{k+2})/(Y_k - Y_{k+2})$,

$w_{k+1}(y) = (y - Y_{k-1})/(Y_{k+1} - Y_{k-1}) \cdot (y - Y_k)/(Y_{k+1} - Y_k) \cdot (y - Y_{k+2})/(Y_{k+1} - Y_{k+2})$,

$w_{k+2}(y) = (y - Y_{k-1})/(Y_{k+2} - Y_{k-1}) \cdot (y - Y_k)/(Y_{k+2} - Y_k) \cdot (y - Y_{k+1})/(Y_{k+2} - Y_{k+1})$.

If $X_{j+1} - X_j = h$, $Y_{k+1} - Y_k = h$, then we get:

$$\int_{X_j}^{X_{j+1}} \omega_{j+2}(x)dx = \int_{Y_k}^{Y_{k+1}} w_{k+2}(y)dy = -(1/24)h,$$

$$\int_{X_j}^{X_{j+1}} \omega_{j+1}(x)dx = \int_{Y_k}^{Y_{k+1}} w_{k+1}(y)dy = (13/24)h,$$

$$\int_{X_j}^{X_{j+1}} \omega_j(x)dx = \int_{Y_k}^{Y_{k+1}} w_k(y)dy = (13/24)h,$$

$$\int_{X_j}^{X_{j+1}} \omega_{j-1}(x)dx = \int_{Y_k}^{Y_{k+1}} w_{k-1}(y)dy = -(1/24)h.$$

Now we have

$$\int_{X_j}^{X_{j+1}} \int_{Y_k}^{Y_{k+1}} U(x,y)dydx = \int_{X_j}^{X_{j+1}} F(x)dx,$$

where

$$F(x) = \int_{Y_k}^{Y_{k+1}} U(x,y)dy.$$

and

$$\int_{X_j}^{X_{j+1}} \int_{Y_k}^{Y_{k+1}} U(x,y)dydx \approx$$

$$\sum_{p=-1}^{2} \sum_{s=-1}^{2} U_{j+p,k+s} \int_{Y_k}^{Y_{k+1}} w_{k+s}(y)dy \int_{X_j}^{X_{j+1}} \omega_{j+p}(x)dx.$$

7 Application

It is well known that the RGB color model is an additive color model in which red, green, and blue light are added together in various ways to reproduce a broad array of colors. The RGB color specification requires three floating-point values for each color. The three values must each be between 0 and 1 and specify the amount of red, green, and blue light in the final color. For example, $COLOR(RGB, 1.0, 0.0, 0.0)$ is red, while $COLOR(RGB, 1.0, 1.0, 0.0)$ is yellow. So we can take $(x_j, y_k, u_{j,k})$, $j = 0, 1, \ldots, n+1$, $k = 0, 1, \ldots, m+1$, where (x_j, y_k) are the coordinates of the point in the plane and $u_{j,k}$ is the value of the color (red, green or blue), $0 \leq u_{j,k} \leq 1$. The application the integro-differential splines can be used to solve the problem of compressing the image and restoring it with the given accuracy.

Graph 14 shows the value of the function of the color given by the function $\sin(5x - 5y)\cos(5x - 5y)$ and the error of approximation. Graph 15 shows the value of the function of the color produced by the function $\sin(15x)\sin(15y)$ and the error of the approximation.

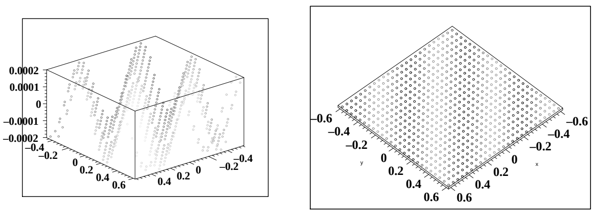

Figure 14: Plots of $\widetilde{u}(x,y) = \sin(5x - 5y)\cos(5x - 5y)$ (right), and $\widetilde{u}(x,y) - u(x,y)$ (left)

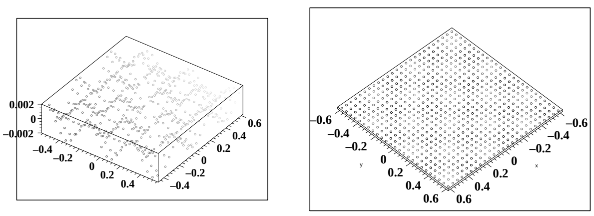

Figure 15: Plots of $u(x,y) = \sin(15x)\sin(15y)$ (right) and $\widetilde{u}(x,y) - u(x,y)$ (left)

8 Conclusion

Here we obtain the formulas for the approximation functions of two variables by integro-differential splines. We compare the results of the approximations with the interpolation splines of two variables and integro-differential splines of two variables. If the values of integrals are unknown we can use the cubature formulaes. To improve the application of integro-differential splines for image construction, detailed properties of constructing integro-differential splines will be studied in our future work.

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