Well-posedness of l-set optimization problem under variable order structure

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Abstract: The concepts of well-posedness of l-set optimization problem under variable order structure are introduced, the metric characterizations and sufficient criteria of well-posedness of a l-set optimization problem are proposed, and the equivalent relations between the well-posedness of l-set optimization problem and that of a scalarization minimization problem are established. Finally, by discussing the lower semi-continuity and convexity of gap functions of l-set convex optimization problems, their well-posedness of are investigated.

*Keywords:*Well-posedness, l-set optimization, gap function, variable order structure

1 Introduction

In the last several decades, two types of criteria of optimization in terms of a set-valued mapping $f : X \to 2^Y$ are considered, where X is a nonempty set and Y is a real topological vector ordered by a closed convex cone $D \subset Y$. The most overwhelmingly popular criterion is looking for an efficient point of the set $f(X) = \cup_{x\in X} f(x)$. In 1999, Kuroiwa [1] introduced *set optimization criterion*. This corresponding criterion is seeking a minimal or maximal set of the whole image set $\mathcal{A} = \{f(y) : y \in X\}$. Hence various aspects of set optimization problem have been subsequently studied by many authors (see [2-12] and the reference therein in detail) forasmuch as its wide applications in economics, optimal control and differential inclusion, etc. Refer to the literatures [13-15].

It is easy to consider that a problem can be solved by approximating method. In other word, we can construct several kinds of iterative sequences to approximate its solution, e.g. [16, 17]. Since it is difficult to solve many practical problems directly, so we can use the solutions of their approximate problems to approximating some solution of the original problem. The key issue is whether the approximating solution sequences converge some solution of the original problem. Thus the notion of well-posedness is introduced. At present, there are a large number of articles investigating the well-posedness for many problems, such as [8, 11, 12, 18].

As we know, there are a few articles concerning well-posedness of set optimization problems. Specifically, three kinds of k_0-well-posedness and three kinds of B-well-posedness of a set optimization problem were discussed by Zhang-Li-Teo [8] and Long-Peng [12], respectively. The k_0-well-posedness at a minimizer introduced in [8] was clarified and dealt with in the setting of set optimization problems by Gutiérrez-Miglierina-Molho-Novod [11].

It is worth noting that the ordering structures considered above were always defined by constant cones. In view of this fact, a cone mapping is introduced to define a variable ordering structure and the notions of well-posedness for set optimization problems are discussed under the defined ordering structure in this paper.

Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ be the sets of real numbers, nonnegative real numbers and positive integers, respectively, and let $\mathcal{N}(*)$ be the collection of open neighborhoods of $*$, where $*$ is a point or a set. A set-valued mapping $f : X \to 2^Y$ is said to be *strict* if $f(x) \neq \emptyset$ for each $x \in X$, where X and Y are nonempty sets.

A subset D of a real topology vector space Y is called a *cone*, if $\lambda x \in D$ for all $x \in D$ and $\lambda > 0$. A cone D in Y is said to be *proper* if $D \neq Y$; to be pointed if $D \cap (-D) = \{0_Y\}$. Let $S(Y)$ represent the collections of all nonempty subsets of Y. A set $A \in S(Y)$ is said to be *D-proper* if $A + D \neq Y$; to be *D-closed* ([19]) if $A+\mathrm{cl}D$ is closed, where $\mathrm{cl}D$ represents the closure of D; to be *D-bounded* ([19]) if for each neighborhood $U \in \mathcal{N}(0_Y)$, there exists $\lambda > 0$ such that $A \subset \lambda U + D$. The families of

*This author has another address:College of Applied Science, Beijing University of Technology,Beijing 100124 all D-proper, D-bounded and D-closed subsets of Y are denoted by $S_D(Y)$, $S_D^b(Y)$ and $S_D^c(Y)$, respectively. It follows that each nonempty compact set of Y is both D-bounded and D-closed (see Lemma 3.3 in [19]) and also D-proper if further D is proper. For every $A,\ B \in S(Y)$ and $\lambda \in \mathbb{R}$, we write $A + B = \{x + y :\ x \in A,\ y \in B\}$, $\lambda A = \{\lambda x :\ x \in A\}$ and $x + A = \{x\} + A$.

Let Y be a real topological vector space ordered by a convex closed cone $D \subset Y$ with nonempty interior. For any $x,\ y \in Y$, write $x \leq_D y$ if $y - x \in D$ and $x \ll_D y$ if $y - x \in$int$D$. For any $A,\ B \in S(Y)$, denote $A \leq_D^l B$ and $A \ll_D^l B$ by $B \subset A + D$ and $B \subset A+$int$D$, respectively. For any $\mathcal{A} \subset S(Y)$, the l-strong minimal set [10] (resp., l-strong maximal set) of $\mathcal{A}$ is defined as

$$l - s\mathrm{Min}_D\mathcal{A} = \{A \in \mathcal{A} :\ A \leq_D^l B,\ \forall\, B \in \mathcal{A}\}$$

(resp.,

$$l - s\mathrm{Max}_D\mathcal{A} = \{A \in \mathcal{A} :\ B \leq_D^l A,\ \forall\, B \in \mathcal{A}\}.)$$

In the sequel, we always let X be a nonempty closed subset of a Hausdorff topological space $\mathfrak{X}$ and Y be a real Hausdorff topological vector space, let $C : X \to 2^Y$ be a set-valued mapping. The so-called cone mapping is that $C(x)$ is a closed convex cone with nonempty interior for each $x \in X$ and $e : X \to Y$ a vector-valued mapping satisfying that $e(x) \in -$int$C(x)$ for each $x \in X$.

Let $f : X \to 2^Y$ be a strict set-valued mapping. The l-set minimization problem and the l-set maximization problem are given as follows:

$$(lP)\ \begin{cases} l - \mathrm{Minimize}_C f(x) \\ \text{subject to } x \in X, \end{cases}$$

and

$$(lQ)\ \begin{cases} l - \mathrm{Maximize}_C f(x) \\ \text{subject to } x \in X, \end{cases}$$

respectively. $\bar{x} \in X$ is called an l-strong minimal solution of (lP) (resp., l-strong maximal solution of (lQ)) with respect to C if

$$\bar{x} \in \mathcal{F}^l = \{x \in X :\ f(x) \in l - s\mathrm{Min}_{C(x)}\mathcal{A}\}$$

(resp.,

$$\bar{x} \in \mathcal{G}^l = \{x \in X :\ f(x) \in l - s\mathrm{Max}_{C(x)}\mathcal{A}\}),$$

where $\mathcal{A} = \{f(y) :\ y \in X\}$. If $C(x) = D$ for all $x \in X$, then the conception of l-strong minimal solution of (lP) with respect to C reduces to the notion of strong optimal solution with respect to the pre-order $\leq_D^l$ defined by Definition 5.1 (ii) in [10].

The rest is organized as follows: In Section 2, some preliminaries are provided. In Sections 3, the metric characterizations and sufficient criteria of (lP) and (lQ) are proposed. The equivalent relations between the well-posedness of (lP) and of (S) and between well-posedness of (lQ) and of (S) are established in Section 4, where (S) represents a scalarization minimizing problem with objective function $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is described as follows:

$$(S)\ \begin{cases} \text{Minimize } \phi(x) \\ \text{subject to } x \in X. \end{cases}$$

It is worth mentioning that ϕ is just a gap function of (lP) or (lQ). The optimal set and optimal value of (S) are denoted by argminϕ and $\tilde{u}$, respectively. Finally, by discussing the lower semi-continuity and convexity of the gap functions of l-set convex optimization problems, their well-posedness are investigated in Section 5.

2 Preliminaries

Let $\mathfrak{X}$ and Y be topological spaces. A function $g : \mathfrak{X} \to \mathbb{R} \cup \{+\infty\}$ is said to be *upper semi-continuous* (resp., *lower semi-continuous*) on $\mathfrak{X}$, if $\{x \in \mathfrak{X} :\ g(x) < \lambda\}$ (resp., $\{x \in \mathfrak{X} :\ g(x) > \lambda\}$) is open for each $\lambda \in \mathbb{R}$; to be *level-compact* on $\mathfrak{X}$, if $\{x \in \mathfrak{X} :\ g(x) \leq \lambda\}$ is compact for $\lambda \in \mathbb{R}$.

The following conceptions of continuity for a set-valued mapping can be found in [20].

A set-valued mapping $f : \mathfrak{X} \to 2^Y$ is said to be *upper semi-continuous at* $x_0 \in \mathfrak{X}$, if for any $N \in \mathcal{N}(f(x_0))$, there exists $B \in \mathcal{N}(x_0)$ such that $f(x) \subset N$ for all $x \in B$; to be *lower semi-continuous at* $x_0 \in \mathfrak{X}$, if for any $y_0 \in f(x_0)$ and any $N \in \mathcal{N}(y_0)$, there exists $B \in \mathcal{N}(x_0)$ such that $f(x) \cap N \neq \emptyset$ for all $x \in B$; to be *upper semi-continuous* (resp., *lower semi-continuous*) *on* X, if f is upper semi-continuous (resp., lower semi-continuous) at each $x \in \mathfrak{X}$; to be *continuous at* $x_0 \in \mathfrak{X}$ (resp., *on* $\mathfrak{X}$), if f is both upper semi-continuous and lower semi-continuous at x_0 (resp., on $\mathfrak{X}$); to be *closed*, if its graph $Graphf = \{(x, y) \in \mathfrak{X} \times Y :\ y \in f(x)\}$ is closed in $\mathfrak{X} \times Y$.

Suppose that $(\mathfrak{X}, \|\cdot\|)$ is a finite dimension normed linear space and X is a nonempty subset of $\mathfrak{X}$. $g :\ X \to \mathbb{R} \cup \{+\infty\}$ is said to be *level-bounded on* X, if X is bounded or

$$\lim_{x \in \mathfrak{X},\ \|x\| \to +\infty} g(x) = +\infty.$$

Lemma 1. *[21] (i) (See [20]) Suppose that $\mathfrak{X}$ and Y are Hausdorff topological spaces. If a set-valued mapping $f : \mathfrak{X} \to 2^Y$ is upper semi-continuous on $\mathfrak{X}$ with closed-values, then f is closed.*

(ii) Let $\mathfrak{X}$ be a Hausdorff topological space and Y a real Hausdorff topological vector space, and let $f,\ g : \mathfrak{X} \to 2^Y$ be two set-valued mappings. If both f and g are upper semi-continuous on $\mathfrak{X}$, then so is $f + g$.

Proof. (ii) This argument is analogous to that of Theorem 5.1.3 in [21]. □

Let (X, d) be a metric space and $A,\ B \subset X$ nonempty subsets. *The excess $\tilde{e}(A, B)$ of A to B* and *the Hausdoraff distance $H(A, B)$ of A and B* are defined as

$$\tilde{e}(A, B) = \sup\{d(x, B) :\ x \in A\},$$
$$H(A, B) = \max\{\tilde{e}(A, B),\ \tilde{e}(B, A)\},$$

respectively, where $d(x, B)$ is the distance from x to B.

Let A be a nonempty bounded subset of a complete metric space (X, d). *The Kuratowski noncompactness measure* [22] of A is defined as

$$\alpha(A) = \inf\left\{\varepsilon > 0 \left|\ \begin{array}{l} \exists\, n \in \mathbb{N}, \text{ s.t.} \\ A \subset \cup_{i=1}^{n} A_i, \\ diamA_i < \varepsilon,\ \forall\, i \in [1, n] \end{array}\right.\right\}$$

where $diamA = \sup\{d(a, b) :\ a,\ b \in A\}$ is the diameter of A. It follows from [22] that

(i) $\alpha(A) = 0$ if A is compact;

(ii) For $\varepsilon > 0$, $B = \{a \in X : d(a, A) < \varepsilon\}$, then $\alpha(B) \le \alpha(A) + 2\varepsilon$;

(iii) $\alpha(A) = \alpha(\mathrm{cl}A)$.

Now we introduce and discuss the scalarization functions involving set-valued mappings under variable order structure in order to establish the gap functions of (lP) and (lQ). In the rest of this section, assume that Y is a real topological vector space ordered by a proper, closed and convex cone D with $\mathrm{int}D \ne \emptyset$. Set $d_0 \in -\mathrm{int}D$ and $a \in Y$. A mapping $G_{d_0,D,a} : Y \to \mathbb{R}$ defined by, for $\forall y \in Y$,

$$G_{d_0,D,a}(y) = \min\{\lambda \in \mathbb{R} :\ y \in \lambda d_0 + a + D\}$$

is called the Gerstewizt's function. The Gerstewizt's function with $a = 0_Y$ is studied in [19, 23]. If further D is pointed, it is discussed in [24].

Using a set $A \in S(Y)$ substitute a, a function $G_{d_0,D,A} : Y \to \mathbb{R} \cup \{-\infty\}$ is defined as, for $\forall\, y \in Y$,

$$G_{d_0,D,A}(y) = \inf\{\lambda \in \mathbb{R} :\ y \in \lambda d_0 + A + D\}.$$

Obviously, $G_{d_0,D,A}(y) = \inf\{G_{d_0,D,a}(y) :\ a \in A\}$.

Definition 2. *Define a scalarization function $G_{d_0,D} : S_D(Y) \times S_D(Y) \to \mathbb{R} \cup \{\pm\infty\}$ by*

$$G_{d_0,D}(A, B) = \sup\{G_{d_0,D,A}(b) :\ b \in B\},$$
$$\forall\, (A, B) \in S_D(Y) \times S_D(Y).$$

When D is pointed, both $G_{d_0,D,A}$ and $G_{d_0,D}$ have been studied in [6, 11]. By inspecting carefully, it is easy to see that the arguments of Lemmas 2.16, 2.17, 3.5, Proposition 3.2 and Theorems 3.6, 3.10 in [6] do not require the assumption that D is pointed. Based on this fact, we give the following consequences.

Lemma 3. *(i) Let $A \in S(Y)$. Then*

$$A \in S_D(Y) \Longleftrightarrow G_{d_0,D,A}(y) > -\infty,\ \forall\, y \in Y.$$

(ii) For any $A \in S_D(Y)$, $y \in Y$ and $\lambda \in \mathbb{R}$, we have

(a) $G_{d_0,D,A}(y) < \lambda \Longleftrightarrow \lambda d_0 + A \ll_D^l y$;

(b) $G_{d_0,D,A}(y) \le \lambda \Longleftrightarrow y \in \lambda d_0 + cl(A + D)$.

Moreover, if $A \in S_D^c(Y)$, then

$$G_{d_0,D,A}(y) \le \lambda \Longleftrightarrow \lambda d_0 + A \le_D^l y.$$

Proof: (i) The sufficiency is clear. Now we show its necessity. Let $R(\lambda) = \lambda d_0 + A + D$ and $r(\lambda) = \lambda d_0 + A + \mathrm{int}D$. Argue it by contradiction. Suppose that $G_{d_0,D,A}(y_0) = -\infty$ for some $y_0 \in Y$. Then $y_0 \in \lambda d_0 + A + D,\ \forall\, \lambda \in \mathbb{R}$. It is easy to obtain $r(\lambda) \subset R(\lambda) \subset r(\mu) \subset R(\mu)$ for any $\lambda < \mu$. So $y_0 \in \lambda d_0 + A + \mathrm{int}D,\ \forall\, \lambda \in \mathbb{R}$, namely,

$$y_0 - \lambda d_0 \in A + \mathrm{int}D,\ \forall\, \lambda \in \mathbb{R}. \tag{1}$$

For any $x \in Y$, taking $\mu \in \mathbb{R}$ such that $x \in -\mu d_0 + \mathrm{int}D$, we have $x \in -y_0 + y_0 - \mu d_0 + \mathrm{int}D \subset -y_0 + A + \mathrm{int}D$ by (1). Thus $Y \subset -y_0 + A + \mathrm{int}D$, which contradicts to $A \in S_D(Y)$.

(ii) This proof is similar to that of Theorem 2.1 in [23]. □

Lemma 4. *(i) For any $A \in S_D(Y) \cap S_D^c(Y)$, $B \in S_D(Y)$ and $\lambda \in \mathbb{R}$,*

$$G_{d_0,D}(A, B) \le \lambda \Longleftrightarrow \lambda d_0 + A \le_D^l B.$$

(ii) $G_{d_0,D}(A, A) = 0,\ \forall\, A \in S_D(Y) \cap S_D^c(Y)$.

(iii) If $A \in S_D(Y) \cap S_D^b(Y) \cap S_D^c(Y)$ and $B \in S_D(Y) \cap S_D^c(Y)$, then $G_{d_0,D}(A, B)$ is real-valued and

$$G_{d_0,D}(A, B) = \min\{\lambda \in \mathbb{R} :\ \lambda d_0 + A \le_D^l B\}.$$

Proof: (i) Clearly, this follows from Lemma 3 (ii). (ii) We proceed analogously to the proof of Theorem 3.10 (i) in [6]. (iii) This consequence is shown by referring to the arguments of Proposition 3.2 and Theorem 3.6 in [6]. □

Let

$$S_C(Y) = \cap\{S_{C(x)}(Y) :\ x \in X\},$$
$$S_C^b(Y) = \cap\{S_{C(x)}^b(Y) :\ x \in X\},$$
$$S_C^c(Y) = \cap\{S_{C(x)}^c(Y) :\ x \in X\}.$$

Two important assumptions are list as follows:

(A) $C(x)$ is proper for each $x \in X$;

(B) $f(x) \in S_C(Y) \cap S_C^b(Y) \cap S_C^c(Y)$ for each $x \in X$.

Obviously, both $S_C^b(Y)$ and $S_C^c(Y)$ are nonempty. If further (A) holds, then $S_C(Y)$ is nonempty. By the way, the following exemplifies that (A) and (B) can be satisfied even if C is not a constant mapping.

Example 5. *Let $\mathfrak{X} = \mathbb{R}$, $Y = \mathbb{R}^2$ and $X = \mathbb{R}_+$ and let $C : X \to 2^Y$ be defined as*

$$C(x) = \{(u, v) : \ v \geq -xu \text{ and } v \geq 0\}, \quad \forall\, x \in X. \tag{2}$$

Evidently, $\mathbb{R}^2_+ \subset C(x) \subset \{(u, v) : \ v \geq 0\}$ and $C(x)$ is proper for each $x \in X$. So (A) is satisfied. For any compact-valued mapping $f : X \to 2^Y$, (B) is also satisfied.

The following conceptions of scalarization functions involving set-valued mappings under variable order structure are well-defined according to Lemma 4 (iii).

Definition 6. *Assume that (A) and (B) hold. Define two* scalarization functions $\xi_{e,C},\ \eta_{e,C} : X \times X \to \mathbb{R}$, *respectively, as*

$$\xi_{e,C}(x, y) = \min\{\lambda : \ \lambda e(x) + f(x) \leq^l_{C(x)} f(y)\},$$

and

$$\eta_{e,C}(x, y) = \min\{\lambda : \ \lambda e(x) + f(y) \leq^l_{C(x)} f(x)\}.$$

Lemma 7. *Under (A) and (B), it yields that for each $x,\ y \in X$ and $\lambda \in \mathbb{R}$,*

(ξ_1): $\xi_{e,C}(x, y) \leq \lambda \iff \lambda e(x) + f(x) \leq^l_{C(x)} f(y)$;

(ξ_2): $\xi_{e,C}(x, y) < \lambda \iff \lambda e(x) + f(x) \ll^l_{C(x)} f(y)$;

(ξ_3): $\xi_{e,C}(x, x) = 0$.

(η_1): $\eta_{e,C}(x, y) \leq \lambda \iff \lambda e(x) + f(y) \leq^l_{C(x)} f(x)$;

(η_2): $\eta_{e,C}(x, y) < \lambda \iff \lambda e(x) + f(y) \ll^l_{C(x)} f(x)$;

(η_3): $\eta_{e,C}(x, x) = 0$.

Proof: We arrive at these conclusions in view of Lemmas 3 (ii) and 4 (ii). □

3 Well-posedness of (lP) and (lQ)

For each $\varepsilon \geq 0$, a *ε-l-minimizing set* of (lP) (resp., *ε-l-maximizing set* of (lQ)) is defined as

$$\mathcal{F}^l(\varepsilon) = \{x \in X : \ f(x) + \varepsilon e(x) \leq^l_{C(x)} f(y),\ \forall\, y \in X\}$$

(resp.,

$$\mathcal{G}^l(\varepsilon) = \{x \in X : \ f(y) \leq^l_{C(x)} f(x) - \varepsilon e(x),\ \forall\, y \in X\}).$$

Clearly, $\mathcal{F}^l(0) = \mathcal{F}^l$, $\mathcal{G}^l(0) = \mathcal{G}^l$, $\mathcal{F}^l \subset \mathcal{F}^l(\varepsilon)$ and $\mathcal{G}^l \subset \mathcal{G}^l(\varepsilon)$ for each $\varepsilon > 0$. In addition, for any $0 < \varepsilon < \varepsilon'$, $\mathcal{F}^l(\varepsilon) \subset \mathcal{F}^l(\varepsilon')$ and $\mathcal{G}^l(\varepsilon) \subset \mathcal{G}^l(\varepsilon')$.

Definition 8. *(i) A sequence $\{x_n\}$ is called an* l-minimizing sequence *of (lP), if there exists $\{\varepsilon_n\} \subset \mathbb{R}_+$ with $\varepsilon_n \to 0$ such that $x_n \in \mathcal{F}^l(\varepsilon_n)$.*

(ii) (lP) is said to be well-posed, *if $\mathcal{F}^l \neq \emptyset$ and for any l-minimizing sequence $\{x_n\}$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to \bar{x} \in \mathcal{F}^l$ as $i \to +\infty$.*

Definition 9. *(i) A sequence $\{x_n\}$ is called an* l-maximizing sequence *of (lQ), if there exists $\{\varepsilon_n\} \subset \mathbb{R}_+$ with $\varepsilon_n \to 0$ such that $x_n \in \mathcal{G}^l(\varepsilon_n)$.*

(ii) (lQ) is said to be well-posed, *if $\mathcal{G}^l \neq \emptyset$ and each l-maximizing sequence $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to \bar{x} \in \mathcal{G}^l$ as $i \to +\infty$.*

Remark 10. *Further assume that $(\mathfrak{X}, d)$ is a metric space.*

(i) (lP) is well-posed if and only if $\mathcal{F}^l$ is a nonempty compact set and for its any l-minimizing sequence $\{x_n\}$, $d(x_n, \mathcal{F}^l) \to 0$.

(ii) The well-posedness of (lQ) is equivalent to the fact that $\mathcal{G}^l$ is nonempty compact and for its any l-maximizing sequence $\{x_n\}$, $d(x_n, \mathcal{G}^l) \to 0$.

Lemma 11. *Suppose that*

(a1) f is upper semi-continuous on X and $f(x)$ is $C(x)$-closed for each $x \in X$;

(a2) e is continuous on X;

(a3) C is upper semi-continuous on X.

Then (i) $\mathcal{F}^l(\varepsilon)$ is closed for each $\varepsilon > 0$;

(ii) $\mathcal{F}^l = \cap_{\varepsilon > 0} \mathcal{F}^l(\varepsilon)$.

Proof: (i) For each $\varepsilon > 0$, let $\{x_n\} \subset \mathcal{F}^l(\varepsilon)$ with $x_n \to \bar{x} \in X$. Then $f(x_n) + \varepsilon e(x_n) \leq^l_{C(x_n)} f(y)$, $\forall\, y \in X$, and so

$$z \in f(x_n) + \varepsilon e(x_n) + C(x_n), \quad \forall\, z \in f(y),\ \forall\, y \in X. \tag{3}$$

The upper semi-continuity of f and C implies the upper semi-continuity of $f + C$ by Lemma 1 (ii). Since $f + C$ is closed-valued in virtue of (a1), $f + C$ is closed on X by Lemma 1 (i). This, together with (3) and the continuity of e, implies

$$z \in f(\bar{x}) + \varepsilon e(\bar{x}) + C(\bar{x}) \quad \forall\, z \in f(y),\ \forall\, y \in X, \tag{4}$$

namely, $f(\bar{x}) + \varepsilon e(\bar{x}) \leq^l_{C(\bar{x})} f(y),\ \forall\, y \in X$. Thus, $\bar{x} \in \mathcal{F}^l(\varepsilon)$ and the closeness $\mathcal{F}^l(\varepsilon)$ is shown.

(ii) It is sufficient to verify $\cap_{\varepsilon>0}\mathcal{F}^l(\varepsilon) \subset \mathcal{F}^l$. If $\bar{x} \in \cap_{\varepsilon>0}\mathcal{F}^l(\varepsilon)$, then (4) holds for each $\varepsilon > 0$. By letting $\varepsilon \to 0$ in (4), the closeness of $f(\bar{x}) + C(\bar{x})$ implies that $z \in f(\bar{x}) + C(\bar{x}),\ \forall\, z \in f(y),\ \forall\, y \in X$, and so $f(\bar{x}) \leq^l_{C(\bar{x})} f(y),\ \forall\, y \in X$, that is, $\bar{x} \in \mathcal{F}^l$. □

Lemma 12. *If (a2)-(a3) and the following are fulfilled:*

(a4) f is lower semi-continuous on X with compact-values,

then

(i) $\mathcal{G}^l(\varepsilon)$ is closed for each $\varepsilon > 0$;

(ii) $\mathcal{G}^l = \cap_{\varepsilon>0}\mathcal{G}^l(\varepsilon)$.

Proof: (i) For each $\varepsilon > 0$, by taking $\{x_n\} \subset \mathcal{G}^l(\varepsilon)$ with $x_n \to \bar{x} \in X$, it follows that $f(y) \leq^l_{C(x_n)} f(x_n) - \varepsilon e(x_n),\ \forall\, y \in X$. In view of the lower semi-continuity of f, for each $\bar{u} \in f(\bar{x})$, $\hat{u}_n \in f(x_n)$ can be chosen to satisfy $\hat{u}_n \to \bar{u}$ by the equivalent statement of lower semi-continuity (See [20]). Then for each $y \in X$,

$$\hat{u}_n - \varepsilon e(x_n) \in \hat{v}_n + C(x_n) \text{ for some } \hat{v}_n \in f(y). \tag{5}$$

Assume that $\hat{v}_n \to \bar{v} \in f(y)$ with loss of generality. To all appearances, C is closed by Lemma 1 (i). Letting $n \to +\infty$ in (5), we have $\bar{u} - \varepsilon e(\bar{x}) \in \bar{v} + C(\bar{x})$, and so $f(\bar{x}) - \varepsilon e(\bar{x}) \subset f(y) + C(\bar{x})$. Thereby, $\bar{x} \in \mathcal{G}^l(\varepsilon)$ and $\mathcal{G}^l(\varepsilon)$ is closed.

(ii) Now testify $\cap_{\varepsilon>0}\mathcal{G}^l(\varepsilon) \subset \mathcal{G}^l$ for each $\varepsilon > 0$. Taking $\bar{x} \in \cap_{\varepsilon>0}\mathcal{G}^l(\varepsilon)$, we have $u - \varepsilon e(\bar{x}) \subset f(y) + C(\bar{x}),\ \forall\, u \in f(\bar{x})$. It is obvious that $f(y)$ is $C(\bar{x})$-closed due to its compactness. Hence $f(\bar{x}) \subset f(y) + C(\bar{x}),\ \forall\, y \in X$, and so $\bar{x} \in \mathcal{G}^l$. □

Theorem 13. *Suppose that $(\mathfrak{X}, d)$ is a Hausdorff complete metric space and X is a nonempty closed bounded subset.*

(i) If (lP) is well-posed, then

$$\mathcal{F}^l(\varepsilon) \neq \emptyset,\ \forall\, \varepsilon \geq 0 \text{ and } \lim_{\varepsilon\to 0} \alpha(\mathcal{F}^l(\varepsilon)) = 0. \tag{6}$$

(ii) If (a1)-(a3) hold, then (6) implies the well-posedness of (lP).

Proof: (i) The well-posedness of (lP) implies that $\mathcal{F}^l$ is nonempty compact. Since $\mathcal{F}^l \subset \mathcal{F}^l(\varepsilon),\ \forall\, \varepsilon > 0$, $\alpha(\mathcal{F}^l) = 0$ and $\mathcal{F}^l(\varepsilon) \neq \emptyset$ for all $\varepsilon > 0$, which deduces that $\alpha(\mathcal{F}^l(\varepsilon)) \leq 2\tilde{e}(\mathcal{F}^l(\varepsilon), \mathcal{F}^l) + \alpha(\mathcal{F}^l) = 2\tilde{e}(\mathcal{F}^l(\varepsilon), \mathcal{F}^l)$. Now it suffices to testify that $\lim\limits_{\varepsilon\to 0} \tilde{e}(\mathcal{F}^l(\varepsilon), \mathcal{F}^l) = 0$. Or else, there exist $r > 0$, $\varepsilon_n \downarrow 0$ and $x_n \in \mathcal{F}^l(\varepsilon_n)$ such that

$$d(x_n, \mathcal{F}^l) \geq r \text{ for all } n \in \mathbb{N}. \tag{7}$$

Clearly, $\{x_n\}$ is an l-minimizing sequence of (lP) and satisfies $d(x_n, \mathcal{F}^l) \to 0$ by Remark 10 (i), which contradicts to (7).

(ii) For any l-minimizing sequence of (lP), take $\{\varepsilon_n\} \subset \mathbb{R}_+$ with $\varepsilon_n \to 0$ such that $x_n \in \mathcal{F}^l(\varepsilon_n)$. By Lemma 11 and the boundedness of X, $\mathcal{F}^l(\varepsilon_n)$ is a nonempty bounded and closed, and $\lim_{\varepsilon\to 0} \mathcal{F}^l(\varepsilon) = \mathcal{F}^l$. Since $\mathcal{F}^l(\varepsilon) \subset \mathcal{F}^l(\varepsilon')$ for any $0 < \varepsilon < \varepsilon'$, and $\lim_{\varepsilon\to 0} \alpha(\mathcal{F}^l(\varepsilon)) = 0$, $\mathcal{F}^l$ is nonempty compact and $\lim_{\varepsilon\to 0} H(\mathcal{F}^l(\varepsilon), \mathcal{F}^l) = 0$ by Kuratowski Theorem [22]. Thereby, $\lim_{\varepsilon\to 0} d(x_n, \mathcal{F}^l) = 0$ and so the well-posedness of (lP) is testified by Remark 10 (i). □

A similar argument of the proof in Theorem 13 covers the case where the metric characterization of (lQ) is obtained according to Lemma 12.

Theorem 14. *Let $(\mathfrak{X}, d)$ be a Hausdorff complete metric space and X a nonempty closed bounded subset.*

(i) If (lQ) is well-posed, then

$$\mathcal{G}^l(\varepsilon) \neq \emptyset,\ \forall\, \varepsilon \geq 0 \text{ and } \lim_{\varepsilon\to 0} \alpha(\mathcal{G}^l(\varepsilon)) = 0. \tag{8}$$

(ii) If (a2)-(a4) are imposed, then (8) implies the well-posedness of (lQ).

Now we pay attention to the sufficient conditions of well-posedness of (lP) and (lQ).

Theorem 15. *Assume that (a1)-(a3) hold and $\mathcal{F}^l \neq \emptyset$. Then (lP) is well-posed if*

(b1) $\mathcal{F}^l(\varepsilon_0)$ is compact for some $\varepsilon_0 > 0$.

Proof: For any l-minimizing sequence $\{x_n\}$ of (lP), let $\{\varepsilon_n\} \subset \mathbb{R}_+$ with $\varepsilon_n \to 0$ satisfying $x_n \in \mathcal{F}^l(\varepsilon_n)$. Under (b1), assume that $x_n \to \bar{x} \in X$ without loss of generality since $x_n \in \mathcal{F}^l(\varepsilon_0)$ for enough large $n \in \mathbb{N}$. Then $f(x_n) + \varepsilon_n e(x_n) \leq^l_{C(x_n)} f(y),\ \forall\, y \in X$. By the similar argument of the proof in Lemma 11 (i), we see that $f + C$ is closed, and so $f(\bar{x}) \leq^l_{C(\bar{x})} f(y),\ \forall\, y \in X$, namely, $\bar{x} \in \mathcal{F}^l$. Consequently, (lP) is well-posed. □

Theorem 16. *Let (a2)-(a4) hold and $\mathcal{G}^l \neq \emptyset$. Then (lQ) is well-posed if*

(b2) $\mathcal{G}^l(\varepsilon_0)$ is compact for some $\varepsilon_0 > 0$.

Proof: For any given l-maximizing sequence $\{x_n\}$ of (lQ), let $\{\varepsilon_n\} \subset \mathbb{R}_+$ with $\varepsilon_n \to 0$ such that $x_n \in \mathcal{G}^l(\varepsilon_n)$. As a result of (b2), $\{x_n\}$ has a subsequence, still denoted by $\{x_n\}$, such that $x_n \to \bar{x} \in X$.

This means $f(y)+\varepsilon_n e(x_n) \preceq^l_{C(x_n)} f(x_n),\ \forall\, y \in X$. Furthermore, we have $f(y) \preceq^l_{C(\bar{x})} f(\bar{x}),\ \forall\, y \in X$ by the analogous argument of the proof in Lemma 12 (i), and so $\bar{x} \in \mathcal{G}^l$. Therefore, the well-posedness of (lQ) is shown. □

Note that the upper semi-continuity (resp., lower semi-continuity) of f is necessary to guarantee the well-posedness of (lP) (resp., (lQ)) in Theorem 15 (resp., Theorem 16). See the following examples.

Example 17. *Let $\mathfrak{X} = \mathbb{R}$, $Y = \mathbb{R}^2$ and $X = [0,1] \cup [2,3]$, let C, e and f be defined by*

$$C(x) = \begin{cases} \mathbb{R}^2_+, & x \in [0,1], \\ -\mathbb{R}^2_+, & x \in [2,3], \end{cases} \tag{9}$$

$$e(x) = \begin{cases} (-1,-1), & x \in [0,1], \\ (1,1), & x \in [2,3], \end{cases} \tag{10}$$

$$f(x) = \begin{cases} \{(0,0)\}, & x = 0, \\ \mathbb{R}^2_+, & x \in (0,1] \cup [2,3]. \end{cases}$$

By observing simply, (lP) is ill-posed (i.e., not well-posed) by Remark 10 (i) since $\mathcal{F}^l = (0,1] \cup [2,3]$ is noncompact. Apparently, the conditions except the upper semi-continuity of f in Theorem 15 are satisfied (Indeed, f is not upper semi-continuous at $x = 0$).

Example 18. *Let $\mathfrak{X} = \mathbb{R}$, $Y = \mathbb{R}^2$ and $X = [0,1] \cup [2,3]$. Define C by (9), e by (10), and f by*

$$f(x) = \begin{cases} \{(0,0)\}, & x = 0, \\ [1,2] \times [1,2], & x \in (0,1] \cup [2,3]. \end{cases}$$

Lightly, $\mathcal{G}^l = (0,1] \cup [2,3]$, which implies the ill-posedness of (lQ) by Remark 10 (ii). It is easy to show that the conditions in Theorem 16 are satisfied excluding the lower semi-continuity of f (Actually, f fails to be lower semi-continuous at $x = 0$).

By Remark 10 and Theorems 15-16, the following implications hold:

well-posedness of (lP)
$\Longrightarrow$ $\mathcal{F}^l$ is nonempty compact,
well-posedness of (lQ)
$\Longrightarrow$ $\mathcal{G}^l$ is nonempty compact,
(b1) $\overset{\text{(a1)}-\text{(a3)}}{\Longrightarrow}$ well-posedness of (lP),
(b2) $\overset{\text{(a2)}-\text{(a4)}}{\Longrightarrow}$ well-posedness of (lQ).

In the following results, we will see that the reciprocal statements are true in a Hausdorff locally compact metric space (X,d).

Theorem 19. *Assume that (X,d) is a Hausdorff locally compact metric space and $\mathcal{F}^l(\varepsilon)$ is a connected subset for each $\varepsilon \geq 0$. If (a1)-(a3) hold, then the following are equivalent:*

(i) $\mathcal{F}^l \neq \emptyset$ and (b1) holds;
(ii) (lP) is well-posed;
(iii) $\mathcal{F}^l$ is nonempty compact.

Proof: The implications (i) ⇒ (ii) and (ii) ⇒ (iii) are Theorem 15 and Remark 10 (i), respectively.

(iii) ⇒ (i) According to the local compactness of $\mathfrak{X}$, there exists $\epsilon > 0$ such that $B_\epsilon = \{x : d(x,\mathcal{F}^l) < \epsilon\}$ has compact closure. Write $S_\epsilon = \{x : d(x,\mathcal{F}^l) = \epsilon\}$. If (b1) fails, then $\mathcal{F}^l(\frac{1}{n}) \cap (B_\epsilon)^c \neq \emptyset$ for all $n \in \mathbb{N}$. Otherwise, $\mathcal{F}^l(\frac{1}{n}) \subset B_\epsilon \subset clB_\epsilon$ and so $\mathcal{F}^l(\frac{1}{n})$ is compact for all $n \in \mathbb{N}$ by Lemma 11 (i), which is absurd by the assumption that (b1) fails. Since $\emptyset \neq \mathcal{F}^l \subset \mathcal{F}^l(\frac{1}{n}) \cap B_\epsilon$, x_n can be selected in $\mathcal{F}^l(\frac{1}{n}) \cap S_\epsilon$ by the connection of $\mathcal{F}^l(\frac{1}{n})$ for each $n \in \mathbb{N}$. Without loss of generality, let $x_n \to \bar{x} \in S_\epsilon$ by the compactness of S_ϵ. Clearly, $f(x_n) + \frac{1}{n}e(x_n) \preceq^l_{C(x_n)} f(y),\ \forall\, y \in X$. By the similar argument of (4), we have $f(\bar{x}) \preceq^l_{C(\bar{x})} f(y),\ \forall\, y \in X$ and so $\bar{x} \in \mathcal{F}^l$. An obvious contradiction arises. Thus, (b1) holds. □

Theorem 20. *Suppose that $(\mathfrak{X},d)$ is a Hausdorff locally compact metric space and $\mathcal{G}^l(\varepsilon)$ is a connected subset for each $\varepsilon \geq 0$. If (a2)-(a4) are fulfilled, then the following are equivalent:*

(i) $\mathcal{G}^l \neq \emptyset$ and (b2) holds;
(ii) (lQ) is well-posed;
(iii) $\mathcal{G}^l$ is nonempty compact.

Proof: Similar to the proof in the preceding Theorem.

4 The relations between well-posedness of (lP)/(lQ) and that of (S)

Now the equivalent relations between the well-posedness of (lP) and that of (S) and between well-posedness of (lQ) and that of (S) are studied, where we regard the gap functions of (lP) and (lQ) as the objective function ϕ of (S).

Definition 21. *(i) $\{x_n\}$ is called a* minimizing sequence *of (S), if $\lim_{n\to+\infty} \phi(x_n) = \tilde{u}$.*

(ii) (S) is said to be well-posed*, if argmin $\phi \neq \emptyset$ and any minimizing sequence $\{x_n\}$ of (S) has a subsequence $\{x_{n_i}\}$ satisfying $x_{n_i} \to \bar{x} \in$ argminϕ as $i \to +\infty$.*

Definition 22. *A function $g : X \to \mathbb{R} \cup \{+\infty\}$ is called a* gap function *of (lP) (resp., (lQ)), if*

(i) $g(x) \geq 0$ for all $x \in X$;
(ii) $g(x) = 0$ if and only if $x \in \mathcal{F}^l$ (resp., $x \in \mathcal{G}^l$.)

Lemma 23. *Let the criteria (A) and (B) hold and define $F,\ G : X \to \mathbb{R} \cup \{+\infty\}$ as follows:*

$$F(x) = \sup_{y\in X} \xi_{e,C}(x,y),\ \forall\, x \in X, \tag{11}$$

$$G(x) = \sup_{y\in X} \eta_{e,C}(x,y),\ \forall\, x \in X, \tag{12}$$

respectively, where $\xi_{e,C}$ and $\eta_{e,C}$ are defined by Definition 6. Then F (resp., G) is a gap function of (lP) (resp., (lQ)).

Proof: For any $x \in X$, $\xi_{e,C}(x,x) = 0$ by Lemma 7 (ξ_3). Thus,

$$F(x) \geq 0,\ \forall\, x \in X. \tag{13}$$

Finally,

$$\begin{aligned} & F(x) = 0 \\ \Longleftrightarrow\ & \xi_{e,C}(x,y) \leq 0,\ \forall\, y \in X \text{ (By (13))} \\ \Longleftrightarrow\ & f(x) \leq^l_{C(x)} f(y),\ \forall\, y \in X \text{ (By } (\xi_1)) \\ \Longleftrightarrow\ & x \in \mathcal{F}^l. \end{aligned}$$

Thus F is a gap function of (lP).

It follows from the similar argument that

$$G(x) \geq 0,\ \forall\, x \in X \tag{14}$$

according to Lemma 7 (η_3), and that $G(x) = 0$ if and only if $x \in \mathcal{G}^l$ by Lemma 7 (η_1) and (14). This completes to verity the fact that G is a gap function of (lQ).

Theorem 24. *Under the terms (A) and (B), we have*

(i) (lP) is well-posed if and only if so is (S), where its objective mapping $\phi = F$ is defined by (11).

(ii) (lQ) is well-posed if and only if so is (S), where its objective mapping $\phi = G$ is defined by (12).

Proof: (i) Clearly, $x \in \mathcal{F}^l$ if and only if $x \in$argmin$F$ and $\tilde{u} = 0$. In addition,

$\{x_n\}$ is an l-minimizing sequence of (lP)
$\Longleftrightarrow \exists\, \{\varepsilon_n\} \in \mathbb{R}_+$ with $\varepsilon_n \to 0$ s.t. $x_n \in \mathcal{F}^l(\varepsilon_n)$
$\Longleftrightarrow \exists\, \{\varepsilon_n\} \in \mathbb{R}_+$ with $\varepsilon_n \to 0$ s.t.
$\xi_{e,C}(x_n, y) \leq \varepsilon_n,\ \forall y \in X,$ (By (ξ_1))
$\Longleftrightarrow \exists\, \{\varepsilon_n\} \in \mathbb{R}_+$ with $\varepsilon_n \to 0$ s.t. $F(x_n) \leq \varepsilon_n$
$\Longleftrightarrow \lim_{n\to+\infty} F(x_n) = 0 = \tilde{u}$ (By (13))
$\Longleftrightarrow \{x_n\}$ is a minimizing sequence of (S).
Thereby, the conclusion of this theorem is proved.

(ii) Similarly, in the case of (lQ), the equivalent relation is verified by Lemma 7 (η_1) and (14). □

Further assume that $(\mathfrak{X}, d)$ is a metric space. By Theorem 24, the well-posedness of (lP) (resp., (lQ)) is equivalent to the fact that for any l-minimizing sequence $\{x_n\}$ of (lP) (resp., l-maximizing sequence $\{x_n\}$ of (lQ)), $F(x_n) \to \tilde{u}$ (resp., $G(x_n) \to \tilde{u}$) implies $d(x_n, \mathcal{F}^l) \to 0$ (resp., $d(x_n, \mathcal{G}^l) \to 0$. It's nature to consider how to estimate a bound below $|F(x) - \tilde{u}|$ (resp., $|G(x) - \tilde{u}|$) by $d(x, \mathcal{F}^l)$ (resp., $d(x, \mathcal{G}^l)$). A forcing function is defined for this purpose. A real-valued function $c : T \to \mathbb{R}_+$ is called a *forcing function* [25], if

$$\begin{aligned} & 0 \in T \subset \mathbb{R}_+,\ c(0) = 0, \\ & t_n \in T,\ c(t_n) \to 0 \Longrightarrow t_n \to 0. \end{aligned}$$

Theorem 25. *Let $(\mathfrak{X}, d)$ be a Hausdorff metric space. The following assertions are equivalent based on the assumptions of (A) and (B):*

(i) (lP) is well-posed;

(ii) $\mathcal{F}^l$ is nonempty compact and there exists a forcing function $c : T = \{d(x, \mathcal{F}^l) :\ x \in X\} \to \mathbb{R}_+$ satisfying

$$F(x) \geq c(d(x, \mathcal{F}^l)),\ \forall\, x \in X, \tag{15}$$

where F is defined by (11).

Proof: In virtue of Lemma 23, F is a gap function of (lP).

If (i) holds, then $\mathcal{F}^l$ is nonempty compact by Remark 10 (i). Now define $c : T = \{d(x, \mathcal{F}^l) :\ x \in X\} \to \mathbb{R}_+$ by

$$c(t) = \inf\{F(x) :\ d(x, \mathcal{F}^l) = t\},\ \forall\, t \in T.$$

Then $d(x, \mathcal{F}^l) = t = 0$ implies $x \in \mathcal{F}^l$ by the compactness of $\mathcal{F}^l$ and so $F(x) = 0$ according to Definition 22 (ii). Thus, $c(0) = 0$. Letting $t_n \in T$ with $c(t_n) \to 0$, we have $c(t_n) = \inf\{F(x) :\ d(x, \mathcal{F}^l) = t_n\}$. $\{x_n\} \subset X$ can be selected to satisfy $t_n = d(x_n, \mathcal{F}^l)$ and $F(x_n) \to 0$ by the definition of infimun. Obviously, $\{x_n\}$ is a minimizing sequence of (S) with objective mapping $\phi = F$, and also an l-minimizing sequence of (lP) by $\tilde{u} = 0$ and the argument of the proof of Theorem 24 (i). As a result, $t_n \to 0$ and (15) holds. Therefore, the assertion (ii) is true.

On the contrary, (ii) means for any l-minimizing sequence $\{x_n\}$ of (lP),(15) becomes $F(x_n) \geq c(d(x_n, \mathcal{F}^l)),\ \forall\ n \in \mathbb{N}$. By the same argument of the proof of Theorem 7 (i), $\{x_n\}$ is a minimizing sequence of (S) with $\phi = F$. So $F(x_n) \to 0$ and $c(t_n) \to 0$, where $t_n = d(x_n, \mathcal{F}^l)$. By (15), we have $t_n \to 0$. This, together with the assumption that $\mathcal{F}^l$ is nonempty compact, deduces the well-posedness of (lP) by Remark 10 (i).

Theorem 26. *Let $(\mathfrak{X}, d)$ be a Hausdorff metric space. The following are equivalent to each other under (A) and (B):*

(i) (lQ) is well-posed;

(ii) $\mathcal{G}^l$ is nonempty compact and there exists a forcing function $c : T = \{d(x, \mathcal{G}^l) :\ x \in X\} \to \mathbb{R}_+$ such that $G(x) \geq c(d(x, \mathcal{G}^l)),\ \forall\, x \in X$, where G is defined by (12).

Proof: In virtue of Lemma 23, G is a gap function of (lQ). The rest of this proof closely resembles that of Theorem 25 by referring to the proof of Theorem 24 (ii) instead of Theorem 24 (i) and replacing l-minimizing sequence by l-maximizing sequence. □

5 Well-posedness of l-set convex optimization problem

In general, the objective mapping ϕ of (S) is required to be lower semi-continuous. Also, Beer-Lucchetti [26] pointed out a main fact that all convex and lower semi-continuous problems (S) defined on a locally compact metric space with a unique minimizer are well-posed. Applying the convexity and lower semi-continuity of the constructed gap functions, we studied the well-posedness of l-set convex optimization problem.

Lemma 27. *Both the following conclusions hold in case of the hypotheses (A) and (B).*

(i) If (a1)-(a3) hold, then F defined by (11) is lower semi-continuous on X. If $\mathcal{F}^l \neq \emptyset$, then dom$F \neq \emptyset$.

(ii) If (a2)-(a4) hold, then G defined by (12) is lower semi-continuous on X. If $\mathcal{G}^l \neq \emptyset$, then dom$G \neq \emptyset$.

Proof: (i) In order to show the lower semi-continuity of F, it is sufficient to testify the closeness of $L(\varepsilon) = \{x \in X : F(x) \leq \varepsilon\}$ for each $\varepsilon \in \mathbb{R}$. As a matter of fact, letting $\{x_n\} \subset L(\varepsilon)$ with $x_n \to \bar{x}$, we have $\xi_{e,C}(x_n, y) \leq \varepsilon,\ \forall\, y \in X$. This implies (3) owing to Lemma 7 (ξ_1). By the similar argument of the proof in Lemma 11, we can obtain $\bar{x} \in L(\varepsilon)$, and so F is lower semi-continuous on X. If $\mathcal{F}^l \neq \emptyset$, then $F(x) = 0,\ \forall\, x \in \mathcal{F}^l$ by Lemma 23. Thus dom$F \neq \emptyset$.

(ii) The analogy argument covers the cases where G is lower semi-continuous and dom$G \neq \emptyset$ by applying Lemma 7 (η_1) and Lemma 23. □

Definition 28. *Suppose that X is a nonempty convex subset of a topological vector space $\mathfrak{X}$. A set-valued mapping $f : X \to 2^Y$ is said*

(i) to be convex-like *(resp.,* concave-like*) on X if for each $x_1, x_2 \in X$ and each $t \in [0, 1]$,*

$$tf(x_1) + (1-t)f(x_2) \subset f(tx_1 + (1-t)x_2)$$

(respectively,

$$f(tx_1 + (1-t)x_2) \subset tf(x_1) + (1-t)f(x_2);$$

*(ii) to be C-*convex *(resp., C-*concave*) if $f + C$ is convex-like (resp., concave-like) on X.*

Remark 29. *(i) If $C(x) = K,\ \forall\, x \in X$, then the notion of $C-$convexity reduces that of $K-$convexity introduced by Fang-Hu-Huang [27].*

(ii) If $X = \mathfrak{X} = \mathbb{R}^m$ and $Y = \mathbb{R}^n$, the convex process from X to Y defined by Rockafellar [28] is just a convex-like mapping.

Lemma 30. *Suppose that $\mathfrak{X}$ is a Hausdorff topological vector space and X is a nonempty closed convex subset of $\mathfrak{X}$. Besides the qualifications (A) and (B), let the following condition hold:*

(C) int$\bar{C} \neq \emptyset$, and $e(x) = \bar{e} \in -\mathrm{int}\bar{C},\ \forall\, x \in X$, where $\bar{C} = \cap\{C(x) :\ x \in X\}$.

(i) If f is C-convex on X, then F defined as (11) is convex on X.

(ii) If f is concave-like on X with convex-values and C is convex-like on X, then G defined as (12) is convex on X

Proof: (i) Apparently, it's enough to testify

$$\begin{aligned} &\xi_{\bar{e},C}(tx_1 + (1-t)x_2, y) \\ &\leq t\xi_{\bar{e},C}(x_1, y) + (1-t)\xi_{\bar{e},C}(x_2, y) \end{aligned} \tag{16}$$

for each $x_1, x_2, y \in X$ and each $t \in [0, 1]$. Indeed, letting $\lambda_i = \xi_{\bar{e},C}(x_i, y),\ i = 1,\ 2$, we have $f(y) \subset \lambda_i\bar{e} + f(x_i) + C(x_i),\ i = 1,\ 2$ by Lemma 7 (ξ_1). In this event,

$$\begin{aligned} f(y) &\subset tf(y) + (1-t)f(y) \\ &\subset t(\lambda_1\bar{e} + f(x_1) + C(x_1)) \\ &\quad +(1-t)(\lambda_2\bar{e} + f(x_2) + C(x_2)) \\ &\subset (t\lambda_1 + (1-t)\lambda_2)\bar{e} \\ &\quad +f(tx_1 + (1-t)x_2) + C(tx_1 + (1-t)x_2) \end{aligned}$$

in view of the C-concavity of f, which yields $\xi_{\bar{e},C}(tx_1 + (1-t)x_2, y) \leq t\lambda_1 + (1-t)\lambda_2$, viz., (16).

(ii) Clearly, $f(y) = tf(y) + (1-t)f(y),\ \forall\, y \in X,\ \forall\, t \in [0, 1]$ by the convex-values of f. For each $x_1,\ x_2,\ y \in X$ and each $t \in [0, 1]$, it follows that

$$\begin{aligned} &f(tx_1 + (1-t)x_2) \subset tf(x_1) + (1-t)f(x_2) \\ &\subset t(f(y) + \eta_{\bar{e},C}(x_1, y)\bar{e} + C(x_1)) \\ &\quad +(1-t)(f(y) + \eta_{\bar{e},C}(x_2, y)\bar{e} + C(x_2)) \\ &\subset f(y) + (t\eta_{\bar{e},C}(x_1, y) + (1-t)\eta_{\bar{e},C}(x_2, y))\bar{e} \\ &\quad +C(tx_1 + (1-t)x_2) \end{aligned}$$

and so $\eta_{\bar{e},C}(tx_1 + (1-t)x_2, y) \leq t\eta_{\bar{e},C}(x_1, y) + (1-t)\eta_{\bar{e},C}(x_2, y)$ by Lemma 7 (η_1). Therefore, G is convex.

It is easy to see that the convexity-like of f implies that f is convex-valued. However, the concavity-like of f cannot guarantee this property. See the following example.

Example 31. *Let $X = \mathfrak{X} = Y = \mathbb{R}$ and $f : X \to 2^Y$ defined as*

$$f(x) = \begin{cases} (-|x|, |x|) \setminus \{0\}, & x \neq 0, \\ \{0\}, & x = 0. \end{cases}$$

It is easy to see that f is concave-like, but not convex-valued.

In addition, neither C-convexity nor C-concavity of f can guarantee that f is convex-valued. For example, let $X = \mathfrak{X} = Y = \mathbb{R}$ and $C(x) = \mathbb{R}_+,\ \forall\, x \in X$, and let $f : X \to 2^Y$ be defined by

$$f(x) = \begin{cases} \{0, x\}, & x \geq 0, \\ \{0\}, & x < 0. \end{cases}$$

By simply calculating, f is both C-convex and C-concave, while not convex-valued.

On the other hand, the property of convex-values of f cannot guarantee any one of convexity-like, concavity-like, C-convexity and C-concavity. In terms of single-valued mapping f, both convexity-like and concavity-like of f are required to agree with

$$f(tx_1 + (1-t)x_2) = tf(x_1) + (1-t)f(x_2), \\ \forall\, x_1,\ x_2 \in X,\ \forall\, t \in [0, 1],$$

while this equation fails in general.

Now let $X = \mathfrak{X} = Y = \mathbb{R}$ and $f, g : X \to 2^Y$ be defined as

$$f(x) = \begin{cases} \{0\}, & x = 0, \\ \{1\}, & x \neq 0, \end{cases} \quad g(x) = \begin{cases} \{1\}, & x = 0, \\ \{0\}, & x \neq 0. \end{cases}$$

Obviously, both f and g are convex-valued, but f fails C-convex and g fails C-concave, since

$$\tfrac{1}{2}f(0) + \tfrac{1}{2}f(1) + \mathbb{R}_+ \not\subset f(\tfrac{1}{2}) + \mathbb{R}_+$$

and

$$g(\tfrac{1}{2}) + \mathbb{R}_+ \not\subset \tfrac{1}{2}g(0) + \tfrac{1}{2}g(1) + \mathbb{R}_+.$$

Incidentally, the mapping defined by (2) satisfying int$\bar{C} \neq \emptyset$.

Theorem 32. *Assume that $(\mathfrak{X}, d)$ is a Hausdorff linear, locally compact metric space, X is a nonempty closed convex subset of $\mathfrak{X}$. If (A)-(C) and (a1)-(a3) hold, f is C-convex on X, and $\mathcal{F}^l$ is a singleton, then (lP) is well-posed.*

Proof: Consider (S) with $\phi = F$ defined by (11). It follows from Lemmas 27 and 30 that ϕ is convex and lower semi-continuous on the convex set X. Since F defined by (11) is a gap function of (lP), argmin$\phi = \mathcal{F}^l$ is a singleton. Thus (S) with $\phi = F$ is well-posedness according as Theorem 2.1 in [26] and so (lP) is well-posed by Theorem 24 (i).

Also using Theorem 2.1 in [26], we obtain the following similarly by applying Theorem 24 (ii) instead of Theorem 24 (i).

Theorem 33. *Let $(\mathfrak{X}, d)$ be a Hausdorff linear, locally compact metric space and X a nonempty closed convex subset of $\mathfrak{X}$. If (A)-(C) and (a2)-(a4) hold, f is concave-like on X with convex-values, C is convex-like on X and $\mathcal{G}^l$ is a singleton, then (lQ) is well-posed.*

By the way, based on the assumptions of (A) and (B), the conclusion in Theorem 15 still holds if (b1) is replaced by one of the following:

(b3) F is level-compact on $\mathcal{F}^l(\varepsilon_0)$ for some $\varepsilon_0 > 0$;

(b4) $\mathfrak{X}$ is a finite dimension normed linear space and F is level-bounded on X.

In fact, (b3) implies that $\mathcal{F}^l(\varepsilon_0) = \{x \in \mathcal{F}^l(\varepsilon_0) : F(x) \leq \varepsilon\}$ for each $\varepsilon \geq \varepsilon_0$ and so it is compact, that is, (b1) is satisfied. In addition, (b4) deduces that $A(\varepsilon) = \{x \in X : F(x) \leq \varepsilon\}$ is bounded for each $\varepsilon \in \mathbb{R}$. Otherwise, $\|x_n\| \to +\infty$ and $F(x_n) \leq \varepsilon_0$ for some $\{x_n\} \subset A(\varepsilon_0)$ and some $\varepsilon_0 \in \mathbb{R}$, which is absurd by (b4). In addition, $A(\varepsilon_0)$ is closed by Lemma 27 (i). Then $A(\varepsilon_0)$ is compact by its boundedness and closeness and so (b1) is true by $\mathcal{F}^l(\varepsilon_0) \subset A(\varepsilon_0)$.

Likewise, the conclusion in Theorem 16 is true as well if one of the criteria below is substituted for (b2) in case of (A) and (B).

(b5) G is level-compact on $\mathcal{G}^l(\varepsilon_0)$ for some $\varepsilon_0 > 0$;

(b6) $\mathfrak{X}$ is a finite dimension normed linear space and G is level-bounded on X.

Acknowledgements: This research was supported by the Project of Siping Science and Technology Development Program (2013040)

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