Periodic solutions for three-species diffusive systems with Beddington-Deangelis and Holling-type III schemes

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Abstract: This paper is concerned with the three-species diffusive systems in a periodic environment, which arises in a one-prey and two-competing-predator population model with Beddington-Deangelis and Holling-type III schemes. By using eigenvalue analysis, bifurcation theories and Schauder estimates, the existence of positive periodic solutions of the single prey species system, two-species predator-prey systems with different functional responses and three-species periodic diffusive systems are investigated. The necessary and sufficient conditions are described by the principal eigenvalue of the periodic parabolic operators. Furthermore, the alternative sufficient conditions characterised by the integral form of the parameters of the systems are more convenient to the biological explanation.

Key–Words: Positive periodic solutions, principal eigenvalue, Schauder estimates, global bifurcation, decoupling

1 Introduction

The study of the dynamic relatives between predators and preys in the muti-species biological systems has been one of the most important themes in population dynamics because of its universal existence in nature (see [1-11] and the references therein). Recently, the three species predator-prey systems with spatial diffusion and some functional responses have been attracted more and more attention [8-11]. As we all know, there are various kinds of functional response models being constructed and investigated in the past half century [12-14]. Among them, three distinct classes have been summed up as following: prey-dependent, predator-dependent and ratio-dependent. Classic “prey-dependent” models assume that the predation rate is a function of only prey density (see [13-15]); “predator-dependent” models estimate the predation rates that depend on densities of both prey and predator (see [16-18],[14]); “Ratio-dependent” models evaluate the predation rates that depend only on the prey to predator ratio (see[19],[10-11]). However, it appears only a few models incorporated two or more different kinds of functional responses [20-27], rarely in the three coupled predator-prey systems. In addition, the investigations on the periodic behaviors of the muti-species models are of theoretical and practical significance, but are more complicated than two species system and appear not much in the literature [6, 24-26].

On the incorporating functional responses models, Aziz-Alaoui and Nindjin et al. in [22, 23] considered earlier a predator-prey model incorporating a modified version of the Leslie-Gower and Holling-type II functional responses. In [20], W. Yang and Y. Li discussed a diffusive predator-prey system with modified Leslie-Gower and Holling-type III schemes. The relevant dynamic behaviors of this two species system were obtained. Some stability analysis were investigated in [21] by Tian and Weng. The latest model were constructed by P. J. Pal and P. K. Mandal in [25]. The paper was concerned with a modified Leslie-Gower delayed predator-prey system where the

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growth of prey population is governed by Allee effect and the predator population consumes the prey according to Beddington-DeAngelis type functional response. To the nonconstant coefficient models, Y. Zhu and K. Wang in [24] assumed that all of the parameters in the system were positive T-periodic functions except for the two parameters in the denominators of the two different functional responses. The well-posed problem of the positive periodic solutions were obtained and the significance of this model was characterized by some theorems and examples. However, one can see that, until now, there are relatively few conclusions on the periodic coefficient, muti-species system models with different functional response schemes.

In this paper, we study a three species periodic predator-prey system with diffusion. We assume in a periodic environment two consumer species compete for the common resource following the Beddington-DeAngelis functional response [16,17] and Holling-type III functional response[14,18], both of them are predator-dependent. We obtain the following periodic reaction-diffusion equations:

$$\begin{array}{l} u_t - \mathcal{D}_1(t)\Delta u = u(r-au) - \dfrac{b_1 u^2 v_1}{m v_1^2 + u^2} \\ \qquad -\dfrac{b_2 u v_2}{\alpha + \beta v_2 + u}, \\ v_{1t} - \mathcal{D}_2(t)\Delta v_1 = v_1(-d_1 - e_1 v_1) \\ \qquad -f_1 v_1 v_2 + \dfrac{c_1 u^2 v_1}{m v_1^2 + u^2}, \\ v_{2t} - \mathcal{D}_3(t)\Delta v_2 = v_2(-d_2 - e_2 v_2) \\ \qquad -f_2 v_1 v_2 + \dfrac{c_2 u v_2}{\alpha + \beta v_2 + u}, \\ \qquad \text{in } \Omega \times (0,+\infty), \\ \left.\begin{array}{l} \frac{\partial u}{\partial \nu} + \gamma u = 0, \\ \frac{\partial v_1}{\partial \nu} + \gamma v_1 = 0, \\ \frac{\partial v_2}{\partial \nu} + \gamma v_2 = 0, \end{array}\right\} \text{on } \partial\Omega \times (0,\infty), \\ (u(x,0), v_1(x,0), v_2(x,0)) \\ = (u(x,T), v_1(x,T), v_2(x,T)) \geq (0,0,0), \\ \qquad \text{in } \Omega, \end{array} \quad (1)$$

where Ω is a bounded region in $\mathbb{R}^n (n \geq 1)$ with sufficiently smooth boundary $\partial\Omega$ and ν is the outward unit vector on $\partial\Omega$. The system is endowed with homogeneous Robin boundary conditions. The three functions $u(x,t)$, $v_1(x,t)$ and $v_2(x,t)$ represent the densities of the prey and the two predators respectively. Two predators share one prey with different functional responses $\frac{b_1 u^2}{m v_1^2 + u^2}$ and $\frac{b_2 u}{\alpha + \beta v_2 + u}$. The first is Holling-type III functional response and the second Beddington-DeAngelis. The predator v_1 consumes the prey with the rate $\frac{b_1 u v_1}{m v_1^2 + u^2}$ and contributes to its growth rate $\frac{c_1 u^2}{m v_1^2 + u^2}$, another predator v_2 consumes the prey with $\frac{b_2 v_2}{\alpha + \beta v_2 + u}$ and contributes to its growth rate $\frac{c_2 u}{\alpha + \beta v_2 + u}$. As it known to all, these two functional responses are more complex than the Lotka-Volterra functional response or Holling-type II functional response, as described in [8, 15, 20, 21] and the references cited therein.

Throughout this paper, we assume that

(H_1) the diffusion coefficients $\mathcal{D}_1(t)$, $\mathcal{D}_2(t)$ and $\mathcal{D}_3(t)$ are smooth strictly positive T-periodic functions;

(H_2) $r = r(x,t)$, $a = a(x,t)$, $b_i = b_i(x,t)$, $c_i = c_i(x,t)$, $d_i = d_i(x,t)$, $e_i = e_i(x,t)$, $f_i = f_i(x,t)$ $(i = 1,2)$ are all smooth positive functions on $\Omega \times \mathbb{R}$ and T-periodic, and $\gamma = \gamma(x,t)$ is smooth positive functions on $\partial\Omega \times \mathbb{R}$ and T-periodic;

(H_3) the parameters $m = m(t)$, $\alpha = \alpha(t)$ and $\beta = \beta(t)$ in the functional response terms are strict positive T-periodic.

In the next section of the present paper we will prove the existence of positive periodic solutions for the single prey species system. In section 3, we introduce two coupled periodic predator-prey systems with diffusion, and acquires the conditions for the existence of positive periodic solutions. Some necessary and sufficient conditions are presented for the three-species coupled system in Section 4 and conclusion are drawn in the last section.

2 Solutions for the Single Prey Species System

In this section, we will establish the existence results of positive solutions for single species prey system:

$$\left\{ \begin{array}{ll} u_t - \mathcal{D}_1(t)\Delta u = (r - au)u, & \text{in } \Omega \times \mathbb{R}^+, \\ \frac{\partial u}{\partial \nu} + \gamma(x,t) u = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u(x,T) \geq 0, & x \in \Omega. \end{array} \right. \quad (2)$$

Firstly, we give the strictly positive solution results, which can be get by maximum principle, easily.

Lemma 1. *Suppose that $u = u(x,t)$ is a nonnegative, nontrivial solution for the system (2). Then $u > 0$ for all $x \in \overline{\Omega}$, $t \in \mathbb{R}$.*

Then, We prove the following estimates of u for the system (2) by Schauder estimates.

Lemma 2. *Suppose that $u = u(x,t)$ is a positive solution of the system (2). Then there exists a constant M such that $||u||_{C^{2+\mu,1+\mu/2}(\bar{Q}_T)} \leq M$.*

Proof: We know that, for an arbitrary initial-value u_0, a sufficiently large constant C_1 is a supersolution of the first equation in (2), and so $\|u\|_{C(\bar{Q}_T)} \leq C_1$. By theorem 2.4 in [33], there exists a continuous function f and g such that

$$\begin{aligned}\|u\|_{C^{1+\mu,\frac{1+\mu}{2}}(\bar{Q}_{T,2T})} &\le f(\|u\|_{C(\bar{Q}_T)})\\ &\le g(\|u(r-au)\|_{C(\bar{Q}_{2T})}),\end{aligned} \tag{3}$$

where $\overline{Q}_{T,2T} = \overline{\Omega} \times [T, 2T]$. Thus by embedding theorem $\|u\|_{C^{1+\mu,(1+\mu)/2}(\bar{Q}_{T,2T})}$ and so $\|u\|_{C^{2+\mu,1+\mu/2}(\bar{Q}_{T,2T})}$ is bounded.

Since u is $T-$periodic, $\|u\|_{C^{2+\mu,1+\mu/2}(\bar{Q}_T)} = \|u\|_{C^{2+\mu,1+\mu/2}(\bar{Q}_{T,2T})}$, and so there exists a constant M such that

$$\|u\|_{C^{2+\mu,1+\mu/2}(\bar{Q}_T)} \le M. \tag{4}$$

Let

$$X := \{w \in C^{\mu,\mu/2}(\overline{\Omega}\times\mathbb{R}) : w(x,t) = w(x,t+T)\}$$

$$Y := \left\{ w \;\middle|\; \begin{array}{l} w \in C^{2+\mu,1+\mu/2}(\overline{\Omega}\times\mathbb{R}) \\ Bw = 0, \text{ on } \partial\Omega\times\mathbb{R}, \\ w(x,t) = w(x,t+T) \end{array} \right\},$$

where $0 < \mu < 1$, $B := \partial/\partial\nu + \gamma$ is the Robin boundary operator as given in system (1). We also denote $Q_T := \Omega \times [0, T]$, and the operator

$$L := \partial_t - \mathcal{D}(t)\Delta + q(x,t). \tag{5}$$

where $q \in X$, and $\mathcal{D} \in C^{\mu/2}(\mathbb{R})$ is a positive T-periodic.

Let us introduce the properties of principal eigenvalue to this parabolic operator L. Let $k \in \mathbb{R}$ is sufficiently large and $q(x,t) + k > 0$ for all $(x,t) \in \overline{Q}_T$, then we can get from [29] that $L_k = L + k$ is a closed operator with compact positive inverse and a positive spectral radium. Thus L_k has a positive eigenvalue corresponding to a positive eigenfunction by the Krein Rutman theorem. So, L also has a principal eigenvalue which we denote by $\lambda_1(q)$. By [28], we can state the following results, which will be used later.

Lemma 3. *Let X and Y be defined as before.*

(i) If there is a $u \in Y$, $u > 0$ such that $(L-\lambda)u < 0$, then $\lambda_1(q) < \lambda$.

(ii) If $q_1, q_2 \in X$ with $q_1 < q_2$, then $\lambda_1(q_1) < \lambda_1(q_2)$.

(iii) If $q_n, q \in X$ for all $n \in \mathbf{N}$ and $q_n \to q$ in $C(\overline{Q}_T)$, then $\lim_{n\to\infty} \lambda_1(q_n) = \lambda_1(q)$.

The following theorem gives the existence and uniqueness of positive periodic solutions to system (2).

Theorem 4. *The necessary and sufficient condition for the existence of strictly positive solution of the system (2) is $\lambda_1(-r(x,t)) < 0((x,t) \in \Omega \times \mathbb{R}^+)$. Moreover, if it exists, then it is unique.*

Proof: Let $L_0 := \partial_t - \mathcal{D}(t)\triangle - r(x,t)$. Then the first equation of (2) can be rewritten as

$$L_0 u = -au^2. \tag{6}$$

Suppose the system (2) has a positive solution u. $L_0 u = -au^2 < 0$, so L_0 has negative eigenvalues, therefore, $\lambda_1(-r(x,t)) < 0$.

Suppose $\lambda_1(-r(x,t)) < 0$, where $(x,t) \in \Omega \times R^+$. Let ϕ denote a positive principal eigenfunction of L_0 corresponding to $\lambda_1(-r(x,t))$. By (6), we have

$$\lambda_1(-r(x,t))\phi = L_0\phi = -a\phi^2 \tag{7}$$

Multiplying these equations by a sufficiently small positive number ε, then

$$L_0(\varepsilon\phi) = \lambda_1(-r(x,t))\varepsilon\phi \le -a\varepsilon^2\phi^2. \tag{8}$$

Thus $\varepsilon\phi$ is a subsolution of (2) for arbitrarily small $\varepsilon > 0$. Clearly $u \equiv M$ is a supersolution, where $M > 0$ is an sufficiently big constant so that $(r - aM)M \le 0$ for all $(x,t) \in Q_T$. Hence by the results in Amann [33] there exists a solution between the sub- and supersolutions, that is, $\varepsilon\phi \le u \le M$. Then $u > 0$.

Furthermore, by iterative methods, the solution is unique for the Hölder continuity on x, t and Lipschitz continuity on u of nonhomogeneous term in the right of the equation (2). The Theorem is proved. □

we now give the useful results for a generalized system from (2), which are shown by global bifurcation theory [31].

Theorem 5. $\lambda_1(-r(x,t)) < 0((x,t) \in \overline{\Omega} \times R)$ *is a necessary and sufficient condition for the existence of strictly positive solutions of the equation*

$$\begin{cases} w_t - \mathcal{D}(t)\Delta w = w(r - aw - H(x,t,w)), \\ w \in Y, \end{cases} \tag{9}$$

where $H : X \to X$ is a continuous, increasing function of w and $H(x,t,0) = 0$ on $\bar{\Omega} \times R^+$.

Proof: The necessity can be proved similarly as in Theorem 4. We now prove the sufficiency of the condition by the global bifurcation theories.

Suppose $\lambda_1(-r(x,t)) < 0$. Choose a sufficiently large fixed constant $K_0 > 0$ such that $-r(x,t)+K_0 > 0$ in Q_T. Consider the problem

$$\begin{cases} w_t - \mathcal{D}(t)\Delta w + (-r(x,t) + K_0)w \\ = \lambda w - (aw + H(x,t,w))w, \\ w \in Y, \end{cases} \tag{10}$$

which may be expressed in operator form in the space Y as $w = \lambda\mathcal{S}w - \mathcal{S}\mathcal{N}(w)$. We let

$$f(\lambda, w) := w - \lambda\mathcal{S}w + \mathcal{S}\mathcal{N}(w) = 0, \tag{11}$$

where $\mathcal{S}: X \to Y$ is the inverse of the linear operator $L_{K_0} := L_0 + K_0 = \partial_t - \mathcal{D}(t)\Delta + (-r(x,t) + K_0)$ and $\mathcal{N}: X \to X$ is given by $\mathcal{N}(w) = (aw + H(x,t,w))w$.

Let

$$L_{01}(\bar{\lambda}, 0)w = D_2 f(\bar{\lambda}, 0)w = (1 - \bar{\lambda}\mathcal{S})w, \quad (12)$$

$$L_{11}(\bar{\lambda}, 0)w = D_1 D_2 f(\bar{\lambda}, 0)w = (-\mathcal{S})w. \quad (13)$$

Then $L_{01}(\bar{\lambda}, 0)w = 0$ implies $w_t - \mathcal{D}(t)\Delta w + (-r(x,t) + K_0)w = \overline{\lambda} w$. So there exists $w = w_0 > 0$ as the principal eigenfunction of $\overline{\lambda}$. Hence, the null space of $L_{01}(\overline{\lambda}, 0)$,

$$\mathcal{N}(L_{01}(\overline{\lambda}, 0)) = \text{spans}\{w_0\}$$

Since the linearized operator of $(L_{01}$ is self-adjoint, it follows by Fredholm alternative axiom that

$$\mathcal{R}(L_{01}(\overline{\lambda}, 0)) = (\mathcal{N}(L^*_{01}(\overline{\lambda}, 0)))^{\perp} = \left\{ w \mid \int w \cdot w_0 = 0 \right\}. \quad (14)$$

where,the operator $L^*_{01}(\overline{\lambda}, 0)$ is the adjoint operator of $L_{01}(\overline{\lambda}, 0)$, and $\mathcal{R}(L_{01}(\overline{\lambda}, 0))$ has codimension one. Obviously, $L_{11}(\bar{\lambda}, 0)w_0 \notin \mathcal{R}(L_{01}(\bar{\lambda}, 0))$. Hence, according to theorem 13.5 of [30], there exists a $\delta > 0$ and a C^1-curve

$$(\lambda, \phi) : (-\delta, \delta) \to \mathbb{R} \times Z \ (Z = \{w_0\}^{\perp})$$

such that (i) $\lambda(0) = \bar{\lambda}$; (ii) $\phi(0) = 0$; and (iii) $f(\lambda(s), w(s)) = 0$ for $|s| < \delta$. Therefore, $C = \{(\lambda(s), w(s)) | w(s) = s(w_0 + \phi(s)), |s| < \delta\}$ is the solution branch of equation (11), and

$$C_+ := C - \{(\lambda(s), s(w_0 + \phi(s))) : -\delta < s < 0\} \quad (15)$$

contains a positive solution branch of (11).

By the theories of global bifurcation (see [31-30]), the continuum $C_+ - \{(\overline{\lambda}, 0)\}$ must satisfy one of the two alternatives (i) joining up with $(\hat{\lambda}, 0)$, where$\hat{\lambda} \neq \overline{\lambda}$ and $1/\hat{\lambda}$ is an eigenvalue of $\mathcal{S}$; (ii) joining up with ∞.

Firstly, we suppose (i) holds. Let

$$P = \{w \in X | w(x,t) > 0, (x,t) \in \bar{Q}_T\}. \quad (16)$$

Then $C_+ - \{(\overline{\lambda}, 0)\} \not\subseteq P$. Thanks to standard Schauder estimates and Sobolev embedding theorem, we find that $\|w\|$ is uniformly bounded in $C^{2,1}(\overline{Q}_T)$, and so there exists $(\hat{\lambda}, 0) \in (C_+ - \{(\overline{\lambda}, 0)\}) \bigcap \partial P$ which is the limit of a sequence

$$\{(\lambda_m, w_m)\} \subset C_+ \bigcap P, \ w_m > 0 \ \text{on} \ \overline{Q}_T. \quad (17)$$

Since $\mathcal{S}$ is compact on X, $\mathcal{S}$ has a discrete spectrum and 0 is unique a limit point of the spectrum. Thus ∞ is unique a limit point of the spectrum of $L_{K_0}(= \mathcal{S}^{-1})$ and so $\lambda_m = \hat{\lambda}$ for sufficiently large m, say $m \geq m_0$. Thus, by Schauder estimates and Sobolev embedding theorem, the systems

$$\begin{cases} w_{mt} - \mathcal{D}(t)\Delta w_m + (K_0 - r(x,t))w_m \\ = \hat{\lambda} w_m, \qquad \text{in } Q_T, \\ \frac{\partial w_m}{\partial \nu} + \gamma(x,t) w_m = 0, \quad \text{on } \partial\Omega \times [0,T], \end{cases} \quad (18)$$

exists a convergent subsequence of $\{w_m\}$, which still denote by $\{w_m\}$ for the sake of convenience, such that $w_m \to w^*$ as $m \to \infty$, and $w^* \geq 0, \not\equiv 0, (x,t) \in \bar{Q}_T$. So taking the limit in (18) as $m \to \infty$, we get

$$\begin{cases} {w^*}_t - \mathcal{D}(t)\Delta w^* + (K_0 - r(x,t))w^* \\ = \hat{\lambda} w^*, \qquad \text{in } Q_T, \\ \frac{\partial w^*}{\partial \nu} + \gamma(x,t) w^* = 0, \qquad \text{on } \partial\Omega \times [0,T]. \end{cases} \quad (19)$$

It follows from the maximum principle that $w^* > 0$ on $\bar{Q}_T$ which implies $\hat{\lambda} = \bar{\lambda}$, a contradiction. Therefore (ii) is right, that is, the continuum $C_+ - \{(\overline{\lambda}, 0)\}$ joins up with ∞.

By lemma 2, $\|w\|_{C(\overline{Q}_T)}$ is bounded, and there exists a continuous function g' such that

$$\|w\|_{C^{1+\mu, \frac{1+\mu}{2}}(\bar{Q}_{T,2T})} \leq g'(\|(\lambda - K_0 + r - aw - H(x,t,w))w\|_{C(\bar{Q}_{2T})}). \quad (20)$$

Since $H(x,t,w)$ is a continuous, increasing function of w, $\|H(x,t,w)\|_{C(\bar{Q}_{2T})}$ can be bounded by a function depending on $\|w\|_{C(\bar{Q}_T)}$. Hence $\|w\|_{C^{1+\mu,(1+\mu)/2}(\bar{Q}_{T,2T})}$ can be bounded solely by a constant depending on $\lambda, K_0, \max r(x,t)$. By similar analysis of lemma 2, we get that the solution of (10) has standard Schauder estimates

$$\|w\|_{c^{2+\mu,1+\mu/2}(\bar{Q}_T)} \leq M_1, \quad (21)$$

where M_1 is a constant depending only on $\lambda, K_0, \max r(x,t)$.

Suppose $(w, \lambda) \in C_+$ and $w \not\equiv 0$. Then $w > 0$. Since $w_t - \mathcal{D}(t)\Delta w + (K_0 - r(x,t) - \lambda)w = -(-aw + H(x,t,w))w < 0$, so $\lambda_1(K_0 - r(x,t) - \lambda) < 0$, i.e., $\lambda > \bar{\lambda}$. Hence $C_+ \subset Y \times \mathbb{R}$ lies in the half plane $\lambda \geq \bar{\lambda}$ and can approach ∞ only as $\lambda \to +\infty$. Therefore for all $\lambda > \bar{\lambda} = \lambda_1(-r(x,t)) + K_0$, in particular for $\lambda = K_0$, there exists $(w, \lambda) \in C_+$, i.e., there exists a positive solution w of the system (9), the proof is then complete. □

Remark 6. *In Theorem 5 we suppose the function $H(x,t,w)$ satisfy condition $H(x,t,0) = 0$ on $\bar{\Omega} \times$*

$\mathbb{R}^+$. *If H is bounded by some increasing function of u, or itself a continuous, nonnegative bounded function, then the results of Theorem 5 are true. In fact, from the in equation (20), $\|w\|_{C^{1+\mu,(1+\mu)/2}(\bar{Q}_{T,2T})}$ can be bounded by a constant only depending on $\lambda, K_0, \max r(x,t)$. The proof can be carried out similarly.*

3 Two coupled species equations

Denote by $\lambda_{1,i}(q)(i = 1, 2, 3)$ the principal eigenvalue of the operator $\mathcal{L} : \partial_t - \mathcal{D}_i(t)\Delta + q(i = 1, 2, 3)$ for the convenience. Then the Theorem 4 can be described by another form as following.

Theorem 4*. *The single species system (2) has a strict positive periodic solution if and only if $\lambda_{1,1}(-r(x,t)) < 0$. Moreover, this solution is unique.*

We denote by $u_+(x,t)$ this unique strict positive periodic solution. So $(u_+, 0, 0)$ is one kind of the solutions of three species system (1), which is usually called a semi-trival solution.

In this section, we introduce the another kinds of semi-trival solutions, that is, the existence of the solution $(\tilde{u}_+, \tilde{v}_{1+}, 0)$ and $(\tilde{u}_+, 0, \tilde{v}_{2+})$ of the system (1).

Consider the following coupled species systems, two kinds of predator-prey models degenerated from system (1):

$$\begin{cases} u_t - \mathcal{D}_1(t)\Delta u = u(r - au) - \frac{b_1 u^2 v_1}{m v_1^2 + u^2}, \\ v_{1t} - \mathcal{D}_2(t)\Delta v_1 = v_1(-d_1 - e_1 v_1) + \frac{c_1 u^2 v_1}{m v_1^2 + u^2}, \\ \frac{\partial u}{\partial \nu} + \gamma u = 0,\ \frac{\partial v_1}{\partial \nu} + \gamma v_1 = 0, \\ u(x,0) = u(x,T), v_1(x,0) = v_1(x,T), \end{cases} \tag{22}$$

$$\begin{cases} u_t - \mathcal{D}_1(t)\Delta u = u(r - au) - \frac{b_2 u v_2}{\alpha + \beta v_2 + u}, \\ v_{2t} - \mathcal{D}_3(t)\Delta v_2 = v_2(-d_2 - e_2 v_2) + \frac{c_2 u v_2}{\alpha + \beta v_2 + u}, \\ \frac{\partial u}{\partial \nu} + \gamma u = 0,\ \frac{\partial v_2}{\partial \nu} + \gamma v_2 = 0, \\ u(x,0) = u(x,T), v_2(x,0) = v_2(x,T). \end{cases} \tag{23}$$

where, all of the parameters satisfy the assumptions (H_1)-(H_3).

We also denote by $(\tilde{u}_+, \tilde{v}_{1+})$ and $(\tilde{u}_+, \tilde{v}_{2+})$ the strict positive solutions of the above two species systems (22) and (23), where $\tilde{u}_+ > 0$, $\tilde{v}_{1+} > 0$ and $\tilde{v}_{2+} > 0$. We will discuss the coexistence of the systems (22) and (23). That is, the necessary and sufficient conditions of the strict positive periodic solutions of these coupled species systems are to be considered.

Theorem 7. *The two principal eigenvalues on $\overline{\Omega} \times \mathbb{R}$ satisfying:*

(i) $\lambda_{1,1}(-r(x,t)) < 0$*;*

(ii) $\lambda_{1,2}(d_1(x,t) - c_1(x,t)) < 0$*,*

are the necessary and sufficient conditions for coexistence of the strictly positive periodic solutions $(\tilde{u}_+, \tilde{v}_{1+})$ of the system (22).

Proof: Since $(\tilde{u}_+, \tilde{v}_{1+})$ is the positive periodic solution of (22), that is, there are the coupled $u > 0$ and $v_1 > 0(\overline{\Omega} \times \mathbb{R})$ satisfying the equations (22), and the parameters as $\mathcal{D}_1, \mathcal{D}_2, r, a, b_1, c_1, d_1$ and m satisfy the assumptions (H_1)-(H_3). We get

$$\begin{aligned} u_t - \mathcal{D}_1(t)\Delta u + (-r(x,t))u \\ = u(-au - \tfrac{b_1 u v_1}{m v_1^2 + u^2}) < 0, \end{aligned} \tag{24}$$

$$\begin{aligned} v_{1t} - \mathcal{D}_2(t)\Delta v_1 + (d_1(x,t) - c_1(x,t))v_1 \\ = v_1(-e_1 v_1 + \tfrac{c_1 u^2}{m v_1^2 + u^2} - c_1) < 0. \end{aligned} \tag{25}$$

So we have $\lambda_{1,1}(-r(x,t)) < 0$ and $\lambda_{1,2}(d_1(x,t) - c_1(x,t)) < 0$. We finished the necessity.

In what follows, we prove the sufficiency. Firstly, we consider a system:

$$\begin{cases} v_{1t} - \mathcal{D}_2(t)\Delta v_1 = v_1(-d_1 - e_1 v_1 + \frac{c_1 u^2}{m v_1^2 + u^2}), \\ v_1 \in Y. \end{cases} \tag{26}$$

which is the predator equation with Holling Type-III functional response with the arbitrary given $u \in X$. We rewrite it as

$$\begin{aligned} v_{1t} - \mathcal{D}_2(t)\Delta v_1 + (d_1 - c_1)v_1 \\ = v_1(-e_1 v_1 - (c_1 - \tfrac{c_1 u^2}{m v_1^2 + u^2})). \end{aligned} \tag{27}$$

Let $L_{d_1 - c_1} := \partial_t - \mathcal{D}_2 + (d_1 - c_1)$. We know that $L_{d_1 - c_1}$ is a linear operator, and the second factor in the right of the equation (27), $-e_1 v_1 - (c_1 - \frac{c_1 u^2}{m v_1^2 + u^2})$, is a decreasing function of v_1. So, if $\lambda_{1,2}(d_1(x,t) - c_1(x,t)) < 0$, by the similar discussion as in Theorem 5, we show the existence of the unique strictly positive solution, noted by $\tilde{v}_1(u)_+$ or $\tilde{v}_{1+}$.

Consider the another system:

$$\begin{cases} u_t - \mathcal{D}_1(t)\Delta u = u(r - au - \frac{b_1 u \tilde{v}_1(u)_+}{m \tilde{v}_1(u)_+^2 + u^2}), \\ u \in Y. \end{cases} \tag{28}$$

which is the prey equation only with the Holling Type-III functional response, and the predator v_1 is given as the unique strictly positive solution $\tilde{v}_1(u)_+$.

Let

$$H(x,t,u) = \frac{b_1 u \tilde{v}_1(u)_+}{m \tilde{v}_1(u)_+^2 + u^2}.$$

Clearly,

$$H(x,t,u) = \frac{b_1}{\sqrt{m}} \frac{(\sqrt{m}\tilde{v}_1(u)_+)u}{(\sqrt{m}\tilde{v}_1(u)_+)^2 + u^2} \le \frac{b_1}{2\sqrt{m}} \text{ in } \bar{Q}_T$$

so H is bounded on $\bar{\Omega} \times R^+$ for the continuous and $T-$ periodical hypothesis H_1 and H_2. By the Remark 6 after Theorem 5, we know that if and only if $\lambda_{1,1}(-r(x,t)) < 0$, the system (28) has a unique strictly positive solution $\tilde{u}_+$ for the unique strictly positive solution $\tilde{v}_{1+}$. That is, we get the coupled positive periodic solution $(\tilde{u}_+, \tilde{v}_{1+})$ of the system (22). This completes the proof. □

Theorem 8. *The necessary and sufficient conditions for the existence of strictly positive periodic solutions $(\tilde{u}_+, \tilde{v}_{2+})$ of the system (23) is the two principal eigenvalues on $\overline{\Omega} \times \mathbb{R}$ satisfying:*

(i) $\lambda_{1,1}(-r(x,t)) < 0$;

(ii) $\lambda_{1,3}(d_2(x,t) - \frac{c_2(x,t)u_+}{\alpha(t)+u_+}) < 0$, where $u_+ = u_+(x,t)$ is the unique positive solution of (2) in Theorem 4.

Proof: Suppose $u > 0$ and $v_2 > 0$. Then

$$\begin{aligned} &u_t - \mathcal{D}_1(t)\Delta u + (-r(x,t))u \\ &\quad = u(-au - \tfrac{b_2 v_2}{\alpha+\beta v_2+u}) < 0, \end{aligned} \tag{29}$$

so we can claim that $\lambda_{1,1}(-r(x,t)) < 0$.

Since $u_t - \mathcal{D}_1(t)\Delta u = u(r - au - \frac{b_2 u v_2}{\alpha+\beta v_2+u}) \le u(r-au)$, it is clear that u is a subsolution for the system (2). The unique solution u_+ of (2) in Theorem 4 must lie between this subsolution and a supersolution corresponding to any sufficiently large positive constant. Thus $0 < u \le u_+$. By the following equation

$$\begin{aligned} &v_{2t} - \mathcal{D}_3(t)\Delta v_2 + (d_2 - \tfrac{c_2 u_+}{\alpha+u_+})v_2 \\ &\le v_{2t} - \mathcal{D}_3(t)\Delta v_2 + (d_2 - \tfrac{c_2 u}{\alpha+u})v_2 \\ &= v_2(-e_2 v_2 + \tfrac{c_2 u}{\alpha+\beta v_2+u} - \tfrac{c_2 u}{\alpha+u}) \\ &< 0, \end{aligned} \tag{30}$$

we get $\lambda_{1,3}(d_2(x,t) - \frac{c_2(x,t)u_+}{\alpha(t)+u_+}) < 0$.

We now consider the sufficient proof. Suppose that $\lambda_{1,1}(-r(x,t)) < 0$ and $\lambda_{1,3}(d_2(x,t) - \frac{c_2(x,t)u_+}{\alpha(t)+u_+}) < 0$. Rewrite the second equation of (23) as

$$\begin{aligned} &v_{2t} - \mathcal{D}_3(t)\Delta v_2 + (d_2 - \tfrac{c_2 u_+}{\alpha+u_+})v_2 \\ &\quad = v_2(-e_2 v_2 + \tfrac{c_2 u}{\alpha+\beta v_2+u} - \tfrac{c_2 u_+}{\alpha+u_+}). \end{aligned} \tag{31}$$

By the similar argument as in the Theorem 4, or the upper and lower solutions methods in Amann [33], when $\lambda_{1,3}(d_2(x,t) - \frac{c_2(x,t)u_+}{\alpha(t)+u_+}) < 0$, there exists unique positive solution $v_2(u)$ for any given u in X.

Here, we use the decoupling method as in [29]. We can get, easily, if $\lambda_{1,3}(d_2 - \frac{c_2 u}{\alpha+u}) < 0$, the equation

$$\begin{aligned} &v_{2t} - \mathcal{D}_3(t)\Delta v_2 \\ &\quad = v_2(-d_2 - e_2 v_2 + \tfrac{c_2 u}{\alpha+\beta v_2+u} - \tfrac{c_2 u}{\alpha+u}) \end{aligned} \tag{32}$$

has a unique positive solution. Define

$$v_2(u) = \begin{cases} 0, & \text{if } \lambda_{1,3}(d_2 - \frac{c_2 u}{\alpha+u}) \ge 0 \\ \text{unique positive solution of (31),} & \text{if } \lambda_{1,3}(d_2 - \frac{c_2 u}{\alpha+u}) < 0. \end{cases} \tag{33}$$

This function $v_2(u)$ is continuous from X to X shown as in [29]. Here, we get its increasing when $u > -\alpha$ on $\overline{\Omega} \times R$. In fact, suppose $-\alpha < u_1 \le u_2$. if $\lambda_{1,3}(d_2 - \frac{c_2 u_1}{\alpha+u_1}) \ge 0$, then $v_2(u_1) = 0$ and so $v_2(u_1) \le v_2(u_2)$. If$\lambda_{1,3}(d_2 - \frac{c_2 u_1}{\alpha+u_1}) < 0$, then $\lambda_{1,3}(d_2 - \frac{c_2 u_2}{\alpha+u_2}) \le \lambda_{1,3}(d_2 - \frac{c_2 u_1}{\alpha+u_1}) < 0$, the following equation

$$v_{2t} - \mathcal{D}_3(t)\Delta v_2 = v_2(-d_2 - e_2 v_2 + \frac{c_2 u_2}{\alpha+\beta v_2+u_2}) \tag{34}$$

has the unique positive solution $v_2(u_2)$. And $v_2(u_1)$ is a subsolution of (34) for $\frac{c_2 u_1}{\alpha+\beta v_2+u_1} \le \frac{c_2 u_2}{\alpha+\beta v_2+u_2}$, so $v_2(u_1) \le v_2(u_2)$. In addition, we can get $v_2(0) = 0$ in Y by maximum principle.

Clearly $(u, v_2(u))$ will be a coexistence solution of the system (23) if and only if u is a positive solution of

$$u_t - \mathcal{D}_1(t)\Delta u = u(r - au - \frac{b_2 v_2(u)}{\alpha+\beta v_2(u)+u}), \ u \in Y. \tag{35}$$

That is to say, if $u > 0$, then there must be $v_2(u) > 0$. Otherwise,

$$\begin{aligned} &u_t - \mathcal{D}_1(t)\Delta u \\ &\quad = u(r - au - \tfrac{b_2 v_2(u)}{\alpha+\beta v_2(u)+u}) \ge u(r - au). \end{aligned} \tag{36}$$

For $v_2(u)$ increasing on u and $v_2(0) = 0$, we get that u is a upper solution of (22), a contradiction.

Let $H = \frac{b_2 v_2(u)}{\alpha+\beta v_2(u)+u}$. Then H is continuous, increasing function of u in (35). By Theorem 5, we have $u > 0$ if and only if $\lambda_{1,1}(-r(x,t)) < 0$ on $\overline{\Omega} \times \mathbb{R}$. Thus the sufficient proof is complete. □

We can also obtain the alternative sufficient conditions for the existence of the system (22) and (23) by integral form of some parameters, which are more convenient to the biological explanation.

Corollary 9. *Suppose the system (22) satisfy the following two inequations:*

(i) $\int_0^T \int_\Omega r(x,t)\mathrm{d}x\mathrm{d}t > 0$, and

(ii)$\int_0^T \int_\Omega [d_1(x,t) - c_1(x,t)]\,\mathrm{d}x\mathrm{d}t < 0$.

Then, there exists positive $T-$ periodic solutions of system (22).

Proof: Let ϕ be a eigenfunction of $\lambda_{1,1}(-r(x,t))$, which is the principal eigenvalue of the operator $\partial_t -$

$\mathcal{D}_1(t)\Delta + (-r(x,t))$. Then $\phi(x,t) > 0$ on $\overline{Q}_T$ by maximum principle. Since

$$\phi_t - \mathcal{D}_1(t)\Delta\phi + (-r(x,t))\phi = \lambda_{1,1}(-r(x,t))\phi, \tag{37}$$

dividing by ϕ and integrating over Q_T, we can obtain that

$$\begin{aligned}&\int_\Omega\int_0^T \frac{\phi_t}{\phi}\mathrm{d}t\mathrm{d}x - \int_0^T \mathcal{D}_1(t)\int_\Omega \frac{\Delta\phi}{\phi}\mathrm{d}x\mathrm{d}t \\ &- \int_0^T\int_\Omega r(x,t)\mathrm{d}x\mathrm{d}t \\ &= \lambda_{1,1}(-r(x,t))|Q_T|,\end{aligned} \tag{38}$$

where $|Q_T|$ is the Lebesgue measure of Q_T. Since ϕ is $T-$ periodic, $\int_0^T \frac{\phi_t}{\phi}\mathrm{d}t = 0$. By the periodic properties of $\gamma(x,t)$ and $\mathcal{D}_1(t)$, we have

$$\begin{aligned}&\int_0^T \mathcal{D}_1(t)\int_\Omega \frac{\Delta\phi}{\phi}\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \mathcal{D}_1(t)\left(\int_{\partial\Omega}\frac{1}{\phi}\nabla\phi\cdot\mathrm{d}S + \int_\Omega \frac{|\nabla\phi|^2}{\phi^2}\mathrm{d}x\right)\mathrm{d}t \\ &= \int_0^T \mathcal{D}_1(t)\left(-\int_{\partial\Omega}\gamma(x,t)\mathrm{d}x + \int_\Omega \frac{|\nabla\phi|^2}{\phi^2}\mathrm{d}x\right)\mathrm{d}t \\ &= -\int_{\partial\Omega}\left(\int_0^T \mathcal{D}_1(t)\gamma(x,t)\mathrm{d}t\right)\mathrm{d}x \\ &+ \int_0^T\int_\Omega \mathcal{D}_1(t)\frac{|\nabla\phi|^2}{\phi^2}\mathrm{d}x\mathrm{d}t \\ &\geq 0.\end{aligned} \tag{39}$$

It follows from (38) and the condition (i), i.e., $\int_0^T\int_\Omega r(x,t)\mathrm{d}x\mathrm{d}t > 0$ that $\lambda_{1,1}(-r(x,t)) < 0$.

By the analogous discussion to the operator $\partial_t - \mathcal{D}_1(t)\Delta + (d_1(x,t) - c_1(x,t))$, we can get the principle eigenvalue $\lambda_{1,2}(d_1(x,t) - c_1(x,t)) < 0$ easily. So, by Theorem 7, the system (22) has the positive $T-$ periodic solutions. The proof is completed. □

We also can get the following results for the system (23), the proof is omitted.

Corollary 10. *Suppose the system (23) satisfy the following two inequations:*

(i) $\int_0^T\int_\Omega r(x,t)\mathrm{d}x\mathrm{d}t > 0$, *and*

(ii) $\int_0^T\int_\Omega \left[d_2(x,t) - \frac{c_2(x,t)u_+}{\alpha(t)+u_+}\right]\mathrm{d}x\mathrm{d}t < 0$.

Then, there exists positive $T-$ periodic solutions of system (23).

4 Three coupled species systems

In this section, we investigate the coexistence of the three coupled species periodic diffusive system (1). By the properties of period parabolic operator, principal eigenvalue, decoupled methods and the similar argument of the former sections, the necessary and sufficiency conditions for the existence of positive solutions can be given. Note these positive solutions of the coupled system by $(u_+^*, v_{1+}^*, v_{2+}^*)$, that is, $u_+^* > 0, v_{1+}^* > 0, v_{2+}^* > 0$ on $\overline{\Omega}\times\mathbb{R}$.

We decoupled the equations of system (1) in order to show the main results of this section. Let $u \in X$, $v_1 \in X$ and consider the following equation for v_2 derived from the system (1)

$$\begin{cases} v_{2t} - \mathcal{D}_3(t)\Delta v_2 \\ \qquad = v_2(-d_2 - e_2v_2 - f_2v_1 + \frac{c_2u}{\alpha+\beta v_2+u}), \text{ in } X, \\ v_2 \in Y. \end{cases} \tag{40}$$

If $\lambda_{1,3}(d_2 - \frac{c_2u}{\alpha+u} + f_2v_1) < 0$, then (40) has a unique positive solution by Theorem 4. We define a map from $X\times X$ to X by

$$v_2(u) = \begin{cases} 0, & \text{if } \lambda_{1,3}(d_2 - \frac{c_2u}{\alpha+u} + f_2v_1) \geq 0 \\ \text{unique positive solution of (40),} & \\ & \text{if } \lambda_{1,3}(d_2 - \frac{c_2u}{\alpha+u} + f_2v_1) < 0. \end{cases} \tag{41}$$

Lemma 11. *$v_2(u, v_1)$ is a monotone increasing function of u when $u > -\alpha$ on $\overline{\Omega}\times\mathbb{R}$, and monotone decreasing function of v_1 on $\overline{\Omega}\times\mathbb{R}$.*

Proof: Firstly, By the similar arguments of lemma 4.2 of [29], we can get the continuity of $v_2(u, v_1)$ on u and v_1, respectively. Suppose $-\alpha < u_1 \leq u_2$ to fixed v_1 in X, $v_2(u,\cdot)$ is an increasing function of u shown in the discussion of theorem 8.

Now, suppose $\underline{v}_1 \leq \overline{v}_1$ to any given u in X. If $\lambda_{1,3}(d_2 - \frac{c_2u}{\alpha+u} + f_2\overline{v}_1) \geq 0$, then $v_2(u,\overline{v}_1) = 0$ and so $v_2(u,\underline{v}_1) \geq v_2(u,\overline{v}_1)$. If $\lambda_{1,3}(d_2 - \frac{c_2u}{\alpha+u} + f_2\overline{v}_1) < 0$, then

$$\begin{aligned} v_{2t} &- \mathcal{D}_3(t)\Delta v_2 \\ &= v_2(-d_2 - e_2v_2 - f_2\overline{v}_1 + \tfrac{c_2u}{\alpha+\beta v_2+u}) \\ &\leq v_2(-d_2 - e_2v_2 - f_2\underline{v}_1 + \tfrac{c_2u}{\alpha+\beta v_2+u}),\end{aligned} \tag{42}$$

it follows that $v_2(u,\overline{v}_1)$ is a subsolution of

$$\begin{aligned} v_{2t} &- \mathcal{D}_3(t)\Delta v_2 \\ &= v_2(-d_2 - e_2v_2 - f_2\underline{v}_1 + \tfrac{c_2u}{\alpha+\beta v_2+u}).\end{aligned} \tag{43}$$

Clearly (43) has arbitrarily large constant supersolutions. Since $v_2(u,\underline{v}_1)$ is the unique positive solution of (43), it must lie between the subsolution $v_2(u,\overline{v}_1)$ and a sufficiently large constant supersolution, i.e. $v_2(u,\underline{v}_1) \geq v_2(u,\overline{v}_1)$. This completes the proof. □

To any nonnegative, nontrivial solution u of system (1), we know that u is smaller than the solution u of system (2),which is the single prey species system without predators. So $0 < u \leq u_+$, where u_+ is the unique positive solution in Theorem 4. If $v_2(u, v_1)$ is one solution of system (1) and $u \geq 0, v_1 \geq 0$, by lemma 11, we get, easily, that

$$v_2(u, v_1) \leq v_2(u_+, 0) \leq \tilde{v}_2(u_+)_+. \quad (44)$$

Here, $v_2(u_+, 0)$ is acted as a degenerated solution of (1) and so a solution of the coupled system (23).

By the similar discussion, $v_1(u, v_2)$ is a monotone increasing function of u when $u \geq 0$ on $\overline{\Omega} \times R$, monotone decreasing function of v_2 on $\overline{\Omega} \times R$, and we have

$$v_1(u, v_2) \leq v_1(u_+, 0) \leq \tilde{v}_1(u_+)_+. \quad (45)$$

Lemma 12. *Suppose the system (1) have a positive solution $(u_+^*, v_{1+}^*, v_{2+}^*)$. Then the three principal eigenvalues on $\overline{\Omega} \times R$ satisfy*

(i) $\lambda_{1,1}(-r(x,t)) < 0$;

(ii) $\lambda_{1,2}(d_1(x,t) - c_1(x,t) + f_1\tilde{v}_2(u_+)_+) < 0$;

(iii) $\lambda_{1,3}(d_2(x,t) - \frac{c_2(x,t)u_+}{\alpha(t)+u_+} + f_2\tilde{v}_1(u_+)_+) < 0$.

Proof: Since $u > 0$, $v_1 > 0$ and $v_2 > 0$, then

$$\begin{aligned} &u_t - \mathcal{D}_1(t)\Delta u + (-r)u \\ &\quad = u(-au - \tfrac{b_1uv_1}{mv_1^2+u^2} - \tfrac{b_2v_2}{\alpha+\beta v_2+u}) < 0, \end{aligned} \quad (46)$$

it follows that $\lambda_{1,1}(-r(x,t)) < 0$. To the other two species v_1 and v_2, we have

$$\begin{aligned} &v_{1t} - \mathcal{D}_2(t)\Delta v_1 + (d_1 - c_1 + f_1\tilde{v}_2(u_+)_+)v_1 \\ &= v_1(-e_1v_1 - f_1(v_2 - \tilde{v}_2(u_+)_+) - \tfrac{c_1mv_1^2}{mv_1^2+u^2}) < 0, \end{aligned} \quad (47)$$

$$\begin{aligned} &v_{2t} - \mathcal{D}_3(t)\Delta v_2 \\ &\quad +(d_2 - \tfrac{c_2u_+}{\alpha+u_+} + f_2\tilde{v}_1(u_+)_+)v_2 \\ &= v_2\Big(- e_2v_2 - f_2(v_1 - \tilde{v}_1(u_+)_+) \\ &\qquad -(\tfrac{c_2uv_2}{\alpha+\beta v_2+u} - \tfrac{c_2u_+}{\alpha+u_+})\Big) \\ &< 0. \end{aligned} \quad (48)$$

Then the principal eigenvalues on $\overline{\Omega} \times R$ satisfy $\lambda_{1,2}(d_1(x,t) - c_1(x,t) + f_1\tilde{v}_2(u_+)_+) < 0$ and $\lambda_{1,3}(d_2(x,t) - \frac{c_2(x,t)u_+}{\alpha(t)+u_+} + f_2\tilde{v}_1(u_+)_+) < 0$ and so the proof is complete. □

Theorem 13. *The following three principal eigenvalues on $\overline{\Omega} \times R$ satisfying:*

(i)$\lambda_{1,1}(-r(x,t)) < 0$;

(ii)$\lambda_{1,2}(d_1(x,t) - c_1(x,t) + f_1\tilde{v}_2(u_+)_+) < 0$,

(iii)$\lambda_{1,3}(d_2(x,t) - \frac{c_2(x,t)u_+}{\alpha(t)+u_+} + f_2\tilde{v}_1(u_+)_+) < 0$,

are the necessary and sufficient conditions for coexistence of the strictly positive periodic solutions $(u_+^*, v_{1+}^*, v_{2+}^*)$ of the system (1).

Proof: The necessity of the conditions is proved in lemma 12.

Suppose that $\lambda_{1,1}(-r) < 0$, $\lambda_{1,2}(d_1 - c_1 + f_1\tilde{v}_2(u_+)_+) < 0$ and $\lambda_{1,3}(d_2 - \frac{c_2u_+}{\alpha+u_+} + f_2\tilde{v}_1(u_+)_+) < 0$ on $\overline{\Omega} \times \mathbb{R}$.

We affirm that $(u, v_1(u, v_2), v_2)$will be a coexistence solution of the system (1) if and only if (u, v_2) is a positive solution of

$$\begin{cases} u_t - \mathcal{D}_1(t)\Delta u \\ \quad = u(r - au - \frac{b_1uv_1(u,v_2)}{mv_1^2(u,v_2)+u^2} - \frac{b_2v_2}{\alpha+\beta v_2+u}), \\ v_{2t} - \mathcal{D}_3(t)\Delta v_2 \\ \quad = v_2(-d_2 - e_2v_2 - f_2v_1(u, v_2) + \frac{c_2u}{\alpha+\beta v_2+u}). \end{cases} \quad (49)$$

That is, if $u > 0$, and $v_2 > 0$, then we must have $v_1(u, v_2) > 0$, or else we get a contradiction.

In fact, if $v_1(u, v_2) \leq 0$, we have

$$\begin{cases} u_t - \mathcal{D}_1(t)\Delta u \\ \quad = u(r - au - \frac{b_1uv_1(u,v_2)}{mv_1^2(u,v_2)+u^2} - \frac{b_2v_2}{\alpha+\beta v_2+u}) \\ \quad \leq u(r - au - \frac{b_2v_2}{\alpha+\beta v_2+u}), \\ v_{2t} - \mathcal{D}_3(t)\Delta v_2 \\ \quad = v_2(-d_2 - e_2v_2 - f_2v_1(u, v_2) + \frac{c_2u}{\alpha+\beta v_2+u}) \\ \quad \leq v_2(-d_2 - e_2v_2 + \frac{c_2u}{\alpha+\beta v_2+u}), \end{cases} \quad (50)$$

and (u, v_2) is one subsolution of the two coupled system (23) by comparison theorem. On the other hand, by the inequation results of (44) and (45), if $\lambda_{1,3}(d_2 - \frac{c_2u_+}{\alpha+u_+} + f_2\tilde{v}_1(u_+)_+) < 0$ on $\overline{\Omega} \times R$ then $\lambda_{1,3}(d_2 - \frac{c_2u_+}{\alpha+u_+}) < 0$ and so the system (23) has the strictly positive solution $(\tilde{u}_+, \tilde{v}_{2+})$ by theorem 8. We get $0 < (u, v_2) \leq (\tilde{u}_+, \tilde{v}_{2+})$. Furthermore, by the hypothesis (H_1)-(H_3), the nonhomogeneous terms of system (23) satisfy Hölder continuity on x, t and Lipschitz continuity on u, v_2, the positive solution $(\tilde{u}_+, \tilde{v}_{2+})$ is unique. So we have $(u, v_2) \equiv (\tilde{u}_+, \tilde{v}_{2+})$, the problem is degenerated to the system (23).

So we need only prove the system (49) has a positive solution (u, v_2). Using the inequation results of (44) and (45), if $\lambda_{1,3}(d_2 - \frac{c_2u_+}{\alpha+u_+} + f_2\tilde{v}_1(u_+)_+) < 0$ then $\lambda_{1,3}(d_2 - \frac{c_2u}{\alpha+u} + f_2v_1) < 0$ and so there exists a unique positive solution of the equation $v_{2t} - \mathcal{D}_3(t)\Delta v_2 = v_2(-d_2 - e_2v_2 - f_2v_1(u, v_2) + \frac{c_2u}{\alpha+\beta v_2+u})$ for any u in X by Theorem 4. We get that if $u > 0$ then $v_2(u) > 0$, otherwise a contradiction shown by analogue method in theorem 8. For $\lambda_{1,1}(-r) < 0$, $v_2(u)$ is increasing on u, the equation

$$\begin{cases} u_t - \mathcal{D}_1(t)\Delta u \\ \quad = u(r - au - \frac{b_1uv_1(u,v_2(u))}{mv_1^2(u,v_2(u))+u^2} - \frac{b_2v_2(u)}{\alpha+\beta v_2(u)+u}), \\ u \in Y, \end{cases} \quad (51)$$

has a unique solution $u > 0$ by the Remark 2.6 after the Theorem 5, where $H(x,t,0) = 0$ for $v_2(0) = 0$ as in the discussion of theorem 8, and $H(x,t,u) = \frac{b_1 u v_1(u,v_2(u))}{m v_1^2(u,v_2(u))+u^2} + \frac{b_2 v_2(u)}{\alpha+\beta v_2(u)+u} \le \frac{b_1}{2\sqrt{m}} + \frac{b_2 v_2(u)}{\alpha+\beta v_2(u)+u}$ bounded by some increasing function of u in X. So there is a positive solution (u, v_1, v_2) of the system (1) and the sufficiency proof is complete. □

We can also obtain the following alternative sufficient conditions for the existence of positive periodic solutions of the three coupled species system (1) by the analogous discussion as Corollary 9 and Corollary 10. Here we omitted the proof.

Corollary 14. *Suppose the hypothesis conditions (H_1)-(H_2) be satisfied. The system (1) has positive $T-$periodic solutions provided*

(i) $\iint_{Q_T} r(x,t)dxdt > 0$,

(ii) $\iint_{Q_T}[d_1(x,t) - c_1(x,t) + f_1\tilde{v}_2(u_+)_+]dxdt < 0$,

(iii) $\iint_{Q_T}[d_2(x,t) - \frac{c_2(x,t)u_+}{\alpha(t)+u_+} + f_2\tilde{v}_1(u_+)_+]dxdt < 0$.

5 Conclusion

In this paper, we have investigated the existence of positive periodic solutions for a one-prey and two-competing-predator diffusive model with Beddington-Deangelis and Holling-type III schemes. In the system (1), one knows that there are two types of competition between the two predators: the first type is direct interference where individuals of each predator species act with aggression against individuals of the other predator species, which is described by the coefficients $f_1(x,t)$ and $f_2(x,t)$. The second type of competition is interference competition that occurs during hunting prey with different functional responses. By use of the properties of the periodic parabolic operators, theories of global bifurcation and Schauder estimates, the existence and uniqueness of positive periodic solutions to single prey system are given firstly. By decoupling and calculous technique, the existence of time-periodic solutions to the two-coupled, especially, the three-species systems are investigated. The necessary and sufficient conditions for the positive periodic solutions are obtained in Section 3 and Section 4.

Furthermore, the three Corollaries 9, 10 and 14 can give us reasonable biological explanations. In the case of Robin boundary conditions, the sufficient conditions require that the birth-rate of the prey u is on average positive and the average death-rates of two predators are not too large and controlled by the competition coefficients and other factors derived by coupled.

For our model, we only consider the existence of positive periodic solutions for the coupled systems. The asymptotic behaviors, the properties on steady state system and the optimal control problems are also important topics for us, which will be pursued in future works.

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