A Final Bifurcation in a simple duopoly model

NATASCIA ANGELINI
Bologna University
MEDAlics, "Dante Alighieri" University
Via del Torrione, Reggio Calabria
ITALY
natascia.angelini@unibo.it

MASSIMILIANO FERRARA
Mediterranean University
Department of Law and Economics
Palazzo Zani, Reggio Calabria
ITALY
massimiliano.ferrara@unirc.it

Abstract: We study global dynamic behavior of a simple duopoly model already existing in literature, when a supercritical Neimark-Sacker bifurcation occurs. In particular, through the study of the critical lines, we provides numerical examples in order to explain the evolution of a closed invariant curve coexisting with periodic attractors into a "weakly chaotic ring".

Key–Words: Cournot duopoly, critical lines, weakly ring

1 Introduction

Dynamic duopoly theory is still an important strand of research in literature. Relaxing one or more assumptions of the original model by Cournot, the dynamical system representing single-product quantity-setting choices of the firms can exhibit complex dynamics and very peculiar mathematical properties (Rand 1978, Dana and Montrucchio, 1986). In the path-breaking article by Puu (1991), the reaction curves resulting by an 'isoelastic' demand function and constant marginal costs, generates dynamic complexity. Recent literature (Bischi and Naimzada, 2000, Bischi and Kopel, 2001, Puu and Shusko, 2002) has investigated further how heterogeneity among competing firms, in terms of both expectations and output adjustment rules may become a further source of complexity for duopoly models, emphasizing the importance role of global analysis for understanding the mechanisms underlying the dynamic of complex behavior. In a just out book (Bischi, Chiarella, Kopel and Szidarovszky, 2010) many traditional models of oligopolistic competition are restated to include firms' capacity constraints, which naturally lead to the definition of piecewise-smooth maps. Such kind of maps (Lamantia, 2011) may bring about peculiar types of global bifurcations, related to the fact that the Jacobian matrix is typically discontinuous on either side of these borders of the phase space where the definition of the map changes.

However, in this paper, we run through the model of Angelini, Dieci and Nardini (2009) in which the piecewise smooth nature of the system is definitely not necessary for the emergence of complex behavior. By developing and analyzing a very simple model, which however incorporates both 'behavioral' heterogeneity (namely, different approaches to output decisions), and possible differences in production costs, we focus on a simple scheme of interaction between two players/firms: at each time step, the two firms implement different production strategies, conditioned upon their 'beliefs' about next period output of the opponent. One of the firms adopts Best-Reply behavior, thus setting the new output in order to maximize profits, while the second player uses a gradient rule, by adjusting current output in the direction of increasing profit locally. We also assume isoelastic demand and linear cost functions. The model results in a two-dimensional discrete-time nonlinear dynamical system. The dynamics is completely described within the space of only two parameters, one representing the ratio between marginal costs, the other reflecting the speed of gradient adjustment. The new feature of this work is that the analysis of the supercritical Neimark-Saker bifurcation arisen from an increase of the adjustment parameter is conducted by the useful tool of critical lines, introduced by Mira and Gumowsky, 1980 and developed in Mira, Gardini, Barugola and Cathala, 1996. With respect to the attraction basin of non-invertible maps , its shape changes drastically if a critical line crosses its boundary. In our model we look at the qualitative changes of this invariant closed curve born via Neimark-Saker. By numerical simulation it is possible to analyze the basins' position of the periodic attractors with respect to the invariant curve. Moreover the non-linearity of the model leads to the birth of a new attractor, namely "weakly chaotic ring". This attractor, for values very far from the critical Neimark-Saker value, leads to a global bifurcation namely final bifurcation or boundary crisis, due essentially to the "fold and pleats" action of the critical lines.

The paper is organized as follows. Section 2 recalls the static model of Angelini at al. (2009), setting up the dynamical map. The existence and stability of fixed points are reported in Section 3. Section 4.1 describes some of the possible dynamic behaviors that may occur when the fixed point becomes unstable via a supercritical Neimark-Sacker bifurcation. In section 4.2 the role of critical curves in the global bifurcation. Section 5 concludes.

2 The model

We recall the settings of the model proposed in (Angelini et al., 2009). We consider a Cournot duopoly with two firms producing an homogeneous good. The first firm produces a positive quantity $x \geq 0$, and the second firm $y \geq 0$. Production costs are strictly increasing and are assumed linear for both firms,

$$C_1(x) = c_1 x + a_1, \quad C_2(y) = c_2 y + a_2$$

where c_1, $c_2 > 0$, a_1, $a_2 \geq 0$. Given the total supply, $q = x + y$, and p the commodity price, we assume that the demand curve is *isoelastic*, differentiable and with $p\prime(\cdot) < 0$

$$p(q) = \frac{1}{q} = \frac{1}{x+y}. \tag{1}$$

Both firms take production decisions upon a prediction of the opponent's output. Denote by x^e and y^e the expected quantities. If each firm is profit maximizing, the *reaction functions* for firm 1 and 2 are, respectively:

$$x = R_1(y^e) := \arg\max_{x \geq 0} \pi_1(x, y^e) \tag{2}$$

$$y = R_2(x^e) = \arg\max_{y \geq 0} \pi_2(x^e, y) \tag{3}$$

Under Cournot expectations, where $x^e = x$ and $y^e = y$, the solutions to (2) and (3) are then given, respectively, by

$$R_1(y^e) = \begin{cases} \sqrt{\dfrac{y}{c_1}} - y & \text{if } 0 \leq y \leq \dfrac{1}{c_1} \\ 0 & \text{if } y > \dfrac{1}{c_1} \end{cases} \tag{4}$$

and

$$R_2(x^e) = \begin{cases} \sqrt{\dfrac{x}{c_2}} - x & \text{if } 0 \leq x \leq \dfrac{1}{c_2} \\ 0 & \text{if } x > \dfrac{1}{c_2} \end{cases}. \tag{5}$$

A unique Nash equilibrium exists, namely

$$E^* = (x^*, y^*) = \left(\frac{c_2}{(c_1+c_2)^2}, \frac{c_1}{(c_1+c_2)^2} \right), \tag{6}$$

Now we introduce firms' strategy update rules in time. We assume that agents' behavioral rules are heterogeneous, even if both firms have *naïve* beliefs, i.e. $y^e_{t+1} = y_t$, $x^e_{t+1} = x_t$. Firm 2 adopts the *Best Reply rule* (5), while firm 1 adopts a *gradient rule*, by simply changing the output level in the direction of an increase of its profit function π_1, with a *speed of adjustment* parameter , $k > 0$

The firms' dynamic output strategies are represented by the two-dimensional nonlinear dynamical map $M : \Re^2 \longrightarrow \Re^2$

$$M : \begin{cases} x' = k\dfrac{y}{(x+y)^2} - kc_1 + x \\ y' = \sqrt{\dfrac{x}{c_2}} - x \end{cases}, \tag{7}$$

where the symbol $'$ denotes the unit-time advancement operator.

We restrict our analysis to those orbits for which both x and y remain strictly positive. Denote by $Q = \{(x, y) : x > 0, y > 0\}$ the first quadrant of the phase-plane, and by I^k, $k = 1, 2, ...$, the subset of Q whose points give rise, under repeated applications of T_u, to orbits that remain in Q for at least k iterations, i.e.

$$I^k = \{(x, y) \in Q : \; T^r_u(x, y) \in Q, \; r = 1, 2, ..., k\},$$

where T^r_u is the r-th iterate of map (7). We then denote by $I^\infty := \cap_{k=1}^\infty I^k$ the set of initial points (in the first quadrant) that generate trajectories along which both state variables remain strictly positive forever. Finally, $\mathcal{B}_o := Q \backslash I^\infty$ represents the set of points of the first quadrant that generate trajectories along which one of the state variables becomes negative or zero at some iteration step.

3 Fixed point and local stability

We recall the fixed point and the local properties of the map as already outlined in (Angelini et al., 2009). The map (7) has as unique fixed point the Cournot equilibrium (6). In order to reduce the number of parameters of the model we set the marginal cost of firm 1 equal to one, $c_1 = 1$.

By setting $c_1 = 1$, the map is the following

$$T : \begin{cases} x' = k\dfrac{y}{(x+y)^2} - k + x \\ y' = \sqrt{\dfrac{x}{c_2}} - x \end{cases} , \tag{8}$$

where $x \geq 0$, $y \geq 0$, $(x, y) \neq (0, 0)$.

The unique fixed point of map T, the Cournot equilibrium point E^* becomes

$$E^* = (x^*, y^*) = \left(\frac{c_2}{(1+c_2)^2}, \frac{1}{(1+c_2)^2}\right). \tag{9}$$

where c_2 now represents the ratio between the equilibrium levels of output of firms 1 and 2.

We are now going to study the local asymptotic stability properties of E^*,employing the Jury's stability conditions:

$$P(-1) = 1 - tr(J^*) + \det(J^*) > 0 \tag{10}$$

$$P(1) = 1 + tr(J^*) + \det(J^*) > 0 \tag{11}$$

$$P(0) = 1 - \det(J^*) > 0 \tag{12}$$

where $tr(J^*)$ and $\det(J^*)$ denote the trace and the determinant of J^*, respectively, provides a necessary and sufficient condition for both eigenvalues of J^* to be smaller than unity in modulus (which represents, in its turn, a sufficient condition for local asymptotic stability of E^*). The Jacobian matrix of map T, evaluated at the fixed point E^*, is given by

$$J^* = \begin{bmatrix} 1 - 2k(1+c_2) & k(c_2-1)(c_2+1) \\ \frac{1-c_2}{2c_2} & 0 \end{bmatrix} \tag{13}$$

The three conditions, in terms of c_2 and k, are the following:

$$k(1+c_2)^3 > 0, \tag{14}$$

$$k(1+c_2)(6c_2 - 1 - c_2^2) < 4c_2, \tag{15}$$

$$k(c_2-1)^2(1+c_2) < 2c_2. \tag{16}$$

For c_2, $k > 0$, condition (14) is obviously satisfied. Condition (15) is certainly satisfied for any $k > 0$, if $6c_2 - 1 - c_2^2 \leq 0$, i.e. if $0 < c_2 \leq 3 - 2\sqrt{2}$ or $c_2 \geq 3 + 2\sqrt{2}$, whereas in the opposite case condition (15) holds only for

$$k < \frac{4c_2}{(1+c_2)(6c_2 - 1 - c_2^2)} := F(c_2).$$

Condition (16) becomes (for $c_2 \neq 1$):

$$k < \frac{2c_2}{(c_2-1)^2(1+c_2)} := NS(c_2),$$

whereas it is true for $c_2 = 1$.

The Cournot equilibrium point E^* is locally asymptotically stable in the area of the space of parameters (c_2, k), $c_2 > 0$, $k > 0$, determined by the union of the following three regions

$$\Omega_1 := \{(c_2, k) : 0 < c_2 \leq 1/3, 0 < k < NS(c_2)\},$$

$$\Omega_2 := \{(c_2, k) : 1/3 < c_2 \leq 3, 0 < k < F(c_2)\},$$

$$\Omega_3 := \{(c_2, k) : c_2 > 3, 0 < k < NS(c_2)\},$$

where

$$F(c_2) = \frac{4c_2}{(c_2+1)\left(6c_2 - 1 - c_2^2\right)}, \tag{17}$$

$$NS(c_2) = \frac{2c_2}{(c_2-1)^2(c_2+1)}. \tag{18}$$

A Flip bifurcation occurs, for $1/3 < c_2 < 3$, on the boundary of equation $k = F(c_2)$ (the eigenvalues of J^* is equal to -1, while the other is smaller than one in modulus); a Neimark-Sacker bifurcation occurs, for $0 < c_2 < 1/3$ (*supercritical type*) or for $c_2 > 3$ (*subcritical type*) , on the boundary of equation $k = NS(c_2)$ (two complex conjugate eigenvalues of modulus equal to one)[1].

4 The supercritical Neimark-Sacker bifurcation and related global bifurcations

This section describes some of the possible dynamic behaviors that may occur when the fixed point becomes unstable via a crossing of the Neimark-Sacker bifurcation curve, in particular we are interested when a *supercritical* NS bifurcation takes place.

[1]For a more detailed analysis of the flip and subcritical NS bifurcations and more numerical investigations of dynamic behavior of the map of this model, even in presence of non-negativity constraints refer to (Angelini el al., 2009).

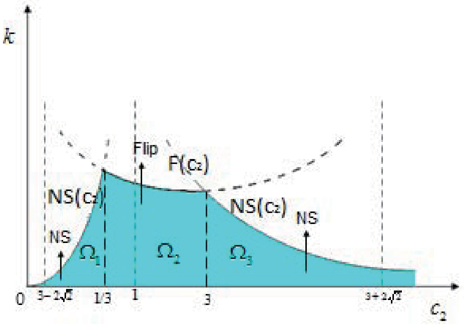

Figure 1: Two dimensional bifurcation diagram in the (c_2, k) parameter plane

4.1 Beyond the supercritical Neimark–Sacker boundary

Recalling (Angelini et al., 2009), starting in the stability region where the parameter of the marginal cost of firm 2 is remarkably smaller than the marginal cost of firm 1, $c_2 < 1/3$, the Cournot equilibrium point (9) is locally asymptotically stable for all values of k lying below the curve $k = NS(c_2)$, defined in (18). It is possible to observe numerically the birth of an attracting closed curve which coexists with cycling attractors. We present here a 'complete' bifurcation sequence that is observed, for instance, by increasing k for fixed $c_2 = 0.19$. In Fig2-a, we can see the attracting curve existing for $k = 0.487$, just beyond the bifurcation boundary, together with its basin of attraction and the region $\mathcal{B}_o$. When k is still very close to its bifurcation value, the closed curve is no longer the unique attractor, but an attracting 4-cycle appears via *saddle-node* bifurcation. This global bifurcation produces a big change of the structure of the basins of attraction (Fig2-b), $k = 0.489$). For increasing values of k, the attracting closed curve increases in size and its basin of attraction gradually reduces (Fig2-c, $k = 0.51$), until it disappears due to a boundary - contact between its basin and that of the 4-cycle. As a consequence, the 4-cycle remains the unique attractor (Fig2-d, $k = 0.52$). Higher values of parameter k bring about a sequence of period-doubling bifurcations and a typical route to chaos for the surviving periodic orbit.

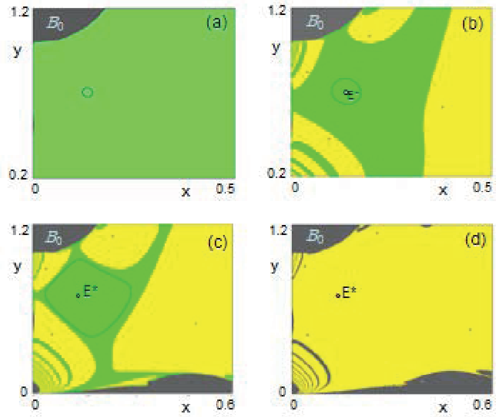

Figure 2: (a-b-c-d): Global dynamics beyond the supercritical Neimark-Sacker bifurcation curve, for $c_2 = 0.19$ and increasing values of k.
(a): the attracting closed curve just after the Neimark-Sacker bifurcation, for $k = 0.487$
(b): coexistence between the attracting curve and a period- 4 orbit, for $k = 0.489$;
(c): the closed curve has become wider and is approaching the basin boundary of the coexisting attractor, for $k = 0.51$;
(d): after the contact, for $k = 0.52$, the period-4 orbit is the orbit is the only surviving attractor.

4.2 Role of the critical curves in the global bifurcation

For certain ranges of the parameters the map

$$T : \begin{cases} x' = k\dfrac{y}{(x+y)^2} - k + x \\ y' = \sqrt{\dfrac{x}{c_2}} - x \qquad \text{with } 0 \le x \le \dfrac{1}{c_2} \end{cases} \tag{19}$$

is a non-invertible map. This means that starting from the initial condition (x_0, y_0) the forward iteration of (19) uniquely defines the trajectory $\{x_t, y_t\} = M_2^t(x_0, y_0)$, with $t = 0, 1, 2, ...$, while the backward iteration of (19) is not uniquely defined. So each point of the phase-space may have several rank-1 preimages. The rank-1 preimages of a point (u, v) are the solutions of the system

$$\begin{cases} u = k\dfrac{y}{(x+y)^2} - k + x \\ v = \sqrt{\dfrac{x}{c_2}} - x \end{cases} \tag{20}$$

where the unknown variables are x and y. It is easy to deduce that given a point $(u, v) \in \Re^2$, its rank-1 preimages $M_2^{-1}(u,v)$, obtained by solving the fourth degree algebraic system (20) with respect to the variables x and y, may be up to four. In fact following the notation used in Mira *et al.*, this is a $Z_0 - Z_2 - Z_4$ non-invertible map. The inverse maps are not easy to write in an elementary analytical way, but it easy to draw *critical lines,* i.e. the locus of points having two merging rank-1 preimages, in order to identify the different regions Z_k $(k = 0, 2, 4)$, each point of which has k distinct rank-1 preimages. We recall that the critical lines LC can be obtained by

$$LC = M_2\left(LC_{-1}\right)$$

where LC_{-1} [2] is defined letting the determinant of the Jacobian matrix equal to zero. So from the Jacobian matrix

$$J = \begin{bmatrix} \dfrac{-2ky}{(x+y)^3}+1 & k\dfrac{(x+y)(x-y)}{(x+y)^4} \\ \dfrac{1}{2c_2}\sqrt{\dfrac{c_2}{x}}-1 & 0 \end{bmatrix}$$

the equation of LC_{-1} is given by

$$\left[\frac{1}{2c_2}\sqrt{\frac{c_2}{x}}-1\right]\left[k\frac{(x+y)(x-y)}{(x+y)^4}\right] = 0$$

So LC_{-1} is formed by the two branches $LC_{-1}^{(a)}$ and $LC_{-1}^{(b)}$:

$$LC_{-1} = LC_{-1}^{(a)} \cup LC_{-1}^{(b)} = \left\{x = \frac{1}{4c_2}\right\} \cup \{y = x\}$$

and also LC is formed by two different branches: $LC = LC^{(a)} \cup LC^{(b)}$, where $LC^{(a)} = M_2\left(LC_{-1}^{(a)}\right)$ and $LC^{(b)} = M_2\left(LC_{-1}^{(b)}\right)$. $LC^{(a)}$ separates the regions Z_0 and Z_2, while $LC^{(b)}$ separates the regions Z_2 and Z_4 (Fig3). Analytically the equation of the $LC^{(a)}$ is given by $y = \dfrac{1}{4c_2}$. So if we have $y < \dfrac{1}{4c_2}$, we get two preimages for the x, otherwise we have none. In this latter case, we are in region Z_0. For the same logic, $LC^{(b)}$ determines the numbers of preimages of the y, that combined with the ones of the x, give the total numbers of rank-1 preimages of a point (u,v). Instead, the areas R_1 and R_2 respectively denote the regions of definition of the inverses of the map: if we consider $x < \dfrac{1}{4c_2}$,which is the region on the left of $LC_{-1}^{(a)}$, the one we are interested in, the x*−preimage* of the inverses is defined by

$$x = \frac{1-2vc_2-\sqrt{1-4vc_2}}{2c_2}$$

while the y*−preimage*is obtained solving the system with respect to y, given u, v and the expression for x (two solutions for y, two inverses for M_2, one belongs to R_1 and the other to R_2)

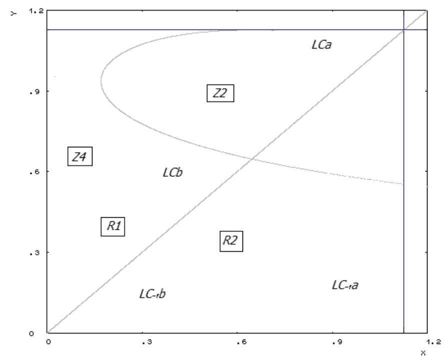

Figure 3: Critical lines

If we fix the marginal cost $c_2 \approx 0.22$, as k increases through the bifurcation value $k = \dfrac{2c_2}{(c_2-1)^2(c_2+1)} \approx 0.55$ a Neimark- Saker bifurcation of supercritical type occurs, and an invariant closed curve Γ arises around the unstable focus E^*.

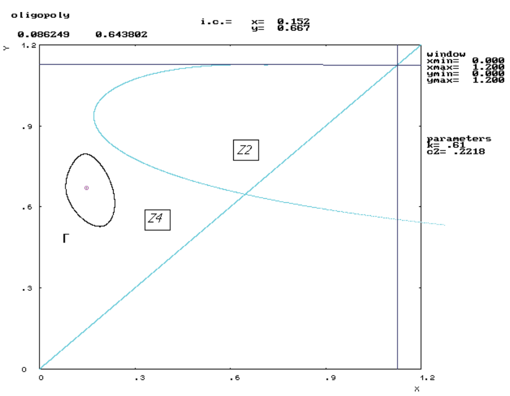

Figure 4: Soon after the supercritical N-S bifurcation of the fixed point E^* an ICC appears.

In the above figure (Fig4) the invariant closed curve seems to be smooth and of approximately circular shape. Moreover Γ has no intersections with

[2]For a continuously differentiable maps the critical curve LC_{-1} is a subset of the locus of points (u,v) for which the Jacobian determinant vanishes.

LC_{-1}.It can be noticed that the invariant curve cannot be too close to LC_{-1}[3], because when a Neimark-Saker bifurcation occurs the eigenvalues are complex conjugate and belong to the unit circle, while along LC_{-1} one eigenvalue must be zero being the determinant of the Jacobian matrix equal to zero along LC_{-1}. So it is possible to have intersections between Γ and LC_{-1} only when the value of the parameter k is quite far from the bifurcation value.

Now it is interesting to analyze the qualitative changes of the invariant closed curve Γ as the parameter k is increased. We can recall the properties (see Mira *at al.*) that the ICC (invariant closed curve) Γ satisfies when it has no intersection with LC_{-1}:

When $\Gamma \cap LC_{-1} = \varnothing$, then:

(a) The area enclosed by Γ, say $a(\Gamma)$ $E^* \in a(\Gamma)$ is invariant, $M_2 a(\Gamma) = a(\Gamma)$;

(b) When an "external pair"[4] P_n exists, either all the P_n points are inside $a(\Gamma)$ or they are outside $a(\Gamma)$.

The properties (b) could be observed in the following picture (Fig5), when a stable 4-cycle coexists with the ICC, and all the points of the cycle are outside $a(\Gamma)$ .Increasing the parameter of the adjustment speed k of the gradient rule, we notice that the stable 4-cycle becomes a 4-pieces chaotic attractor, still outside $a(\Gamma)$ .As the chaotic 4-pieces attractor hurts the boundary of its basin it disappears (Fig6 and Fig7).

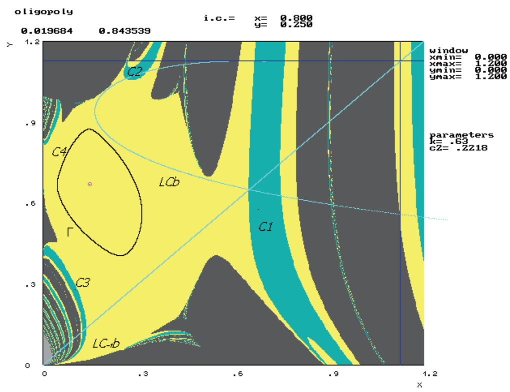

Figure 5: All the points of 4-cycle are outside $a(\Gamma)$

This scenario changes when the ICC has a contact with LC_{-1} as shown in Fig9, with $k = 0.66$.

Let A and B the points of intersection between the ICC Γ and LC_{-1}. A_1 and B_1 are the respective

[3]A steady- state which bifurcates via Neimark-Saker cannot lie on LC_{-1}.

[4]With "external pair" we call an external periodic k-cycle with respect to the invariant closed curve (ICC).

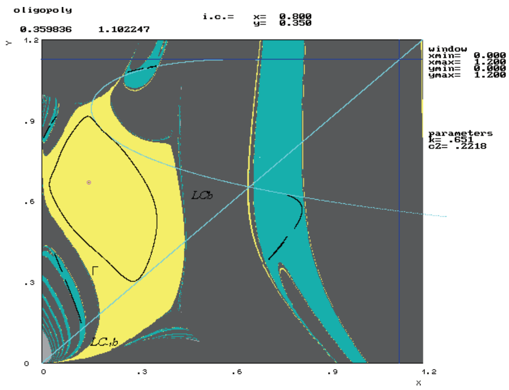

Figure 6: 4-pieces chaotic attractor

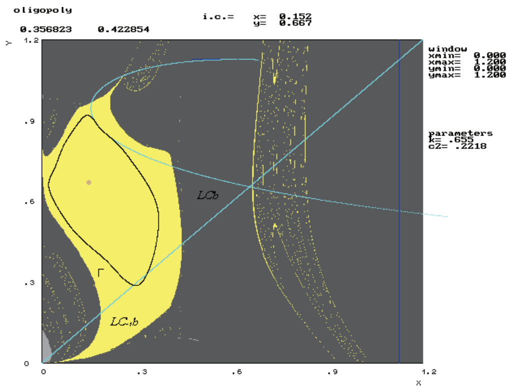

Figure 7: extinction of the attractor

images through M_2.By simple geometrical considerations it can be noticed that all the points belonging to the areas $s_1$and s_2 are mapped in the unique area formed by the arch A_1B_1 (image of the arch AB, through M_2 and and the piece of the critical line $LC^{(b)}$ from A_1 to B_1 (see Fig9). Referring to the properties in Mira *at al.* when $\Gamma \cap LC_{-1} \neq \varnothing$, the area $a(\Gamma)$ bounded by the invariant curve is no longer forward invariant: the two portions of the area $a(\Gamma)$, s_1 and s_2, are both "folded" by M_2 along LC *outside* $a(\Gamma)$, i.e. a point in s_1 which is inside Γ is mapped in a area which does not belong to $a(\Gamma)$.

Another characteristic due to the folding of the plane along first rank critical curve LC is that if there is a periodic cycle that coexists with the ICC, some of the periodic point of the cycle could be outside the invariant curve Γ and some inside, as it can be noted in Fig10. For example for $k = 0.67363$ a stable period 11 coexists with Γ, but some of its points are inside the area $a(\Gamma)$, as C_1, C_6 and C_9, while the others are outside.

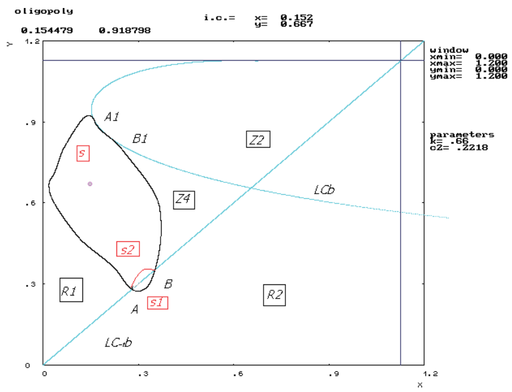

Figure 8: k is far from the bifurcation value and the area $a(\Gamma)$ is no longer invariant.

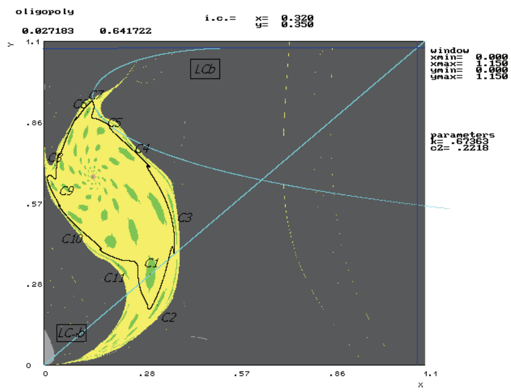

Figure 10: Properties related to the position of the points of a periodic cycle coexsisting with Γ.

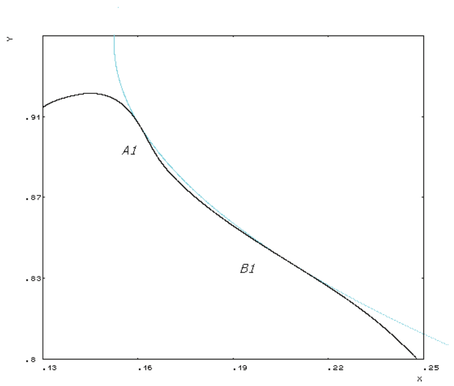

Figure 9: Enlargement

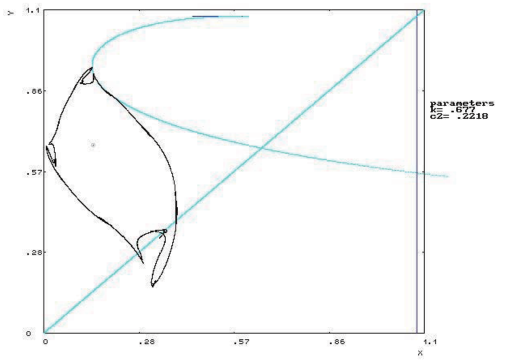

Figure 11: weakly chaotic ring

The stable 11-cycle presents a basin like a spiral sequence of islands converging towards the unstable focus E^*. Notice that the lake containing C_5 crosses through LC_b, and its first rank preimage produces a lake crossing through $LC_{-1}^{(b)}$. As it can be seen in the previous figure (Fig10), the convolutions of Γ become more pronounced due to the nonlinearities of the map and it can be noticed in the next pictures (Fig11–Fig12) the presence of loops and self-intersections for a value of $k = 0.677$. The closed invariant curve is no longer invariant, and the attractive curve turns into a chaotic attractor, which is called in literature "weakly chaotic ring" (Mira *at al.*) for its shape.

When the parameter k reaches the value 0.6772, the chaotic ring is completely developed and, despite a strong cyclic pattern, trajectories appear to be quite irregular, as shown in the time series related to the quantities x and y (Fig14–Fig15). Another thing interesting to notice is that the chaotic attractor is near to the boundary of its basin: as k increases further, up to 0.6777, there will be a contact between the chaotic ring and its boundary, leading to the destruction of the attractor itself. This kind of bifurcation is called *final bifurcation*, or *boundary crisis*, and it is a global bifurcation that occurs at a value of the parameter k really far from the Neimark-Saker bifurcation value, and after a significant transformation of the invariant closed curve Γ, due to the folding action of the critical curves.

5 Conclusions

This simple dynamic duopoly model is able to generate very complex behaviors. We observe that the new attractors born after the local bifurcations become themselves more and more complex objects when the adjustment parameter is increased further. Another relevant topic is multistability. In such a case, the phase space is shared among different basins of at-

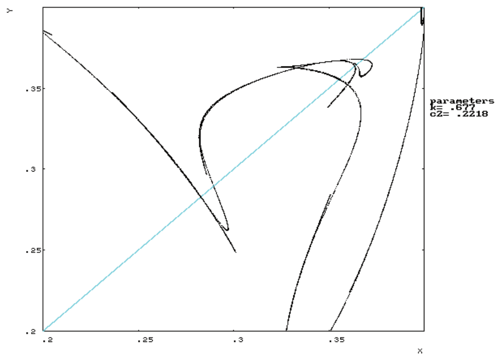

Figure 12: enlargement

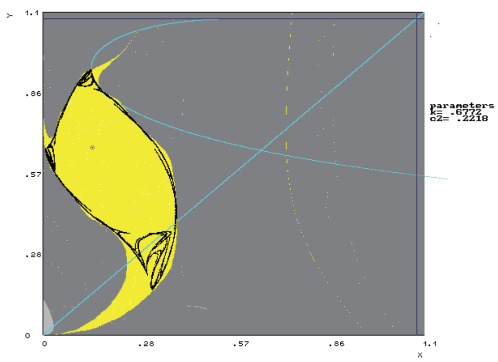

Figure 13: The fully developed weakly chaotic ring

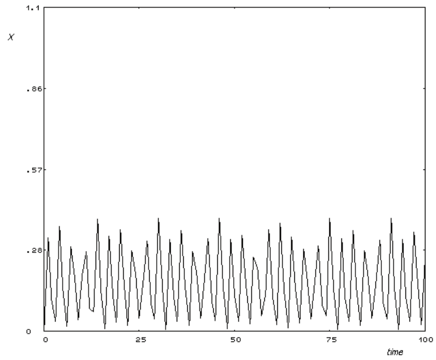

Figure 14: Time series for the quantity x

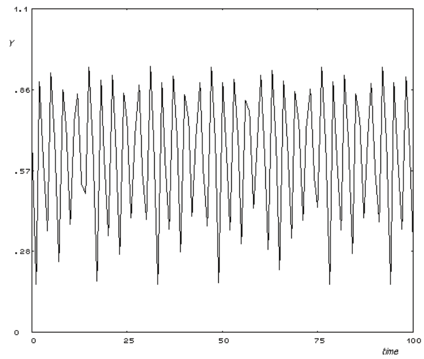

Figure 15: Time series for the quantity y

traction, which may be themselves characterized by complex structures, as numerical simulation suggests. A characteristic due to the folding of the plane along first rank critical curve LC is the coexistence of periodic cycles with the invariant closed curve of supercritical nature. The instrument of analysis of the critical curves allows us to explore how these bifurcations arise from contacts between the boundary of attracting basins and critical lines.

Further research for instance, may introduce production adjustment costs and capacity limitations (not only non-negativity constraints as in the original work of Angelini, Dieci Nardini, 2009). Moreover, we want to carry out a model with a generalized demand function with only local properties, in order to identify general conditions under which particular bifurcations may occur.

Acknowledgements: The research was supported by the research centre MEDAlics, "Dante Alighieri" University (Reggio Calabria).

References

[1] H. N. Agiza, A. A. Elsadany, Nonlinear dynamics in the Cournot duopoly game with heterogeneous players, *Physica A*, 320, (2003), pp.512–524.

[2] A. Agliari, L. Gardini and T. Puu, Some global bifurcations related to the appearance of closed invariant curves,*Mathematics and Computers in Simulation*, 68, (2005), pp.201–219.

[3] A. Agliari, L. Gardini and T. Puu, Global bifurcation in duopoly when the fixed point is destabilized via a subcritical Neimark bifurcation, *International Game Theory Review*, 8, (2006), pp.1–20.

[4] N. Angelini, R. Dieci, F. Nardini, Bifurcation analysis of a dynamic duopoly model with heterogeneous costs and behavioral rules, *Mathematics and Computers in Simulation* , 79, (2009), pp.3179–3196

[5] S. Banerjee, M. S. Karthik, G. Yuan and J. A. Yorke, Bifurcations in one-dimensional piecewise smooth maps – theory and applications in switching circuits, *IEEE Transactions on Circuits and Systems - I: Fundamental Theory and Applications*, 47, (2000a), pp.389–394.

[6] S. Banerjee, P.Ranjan, C. Grebogi, Bifurcations in two-dimensional piecewise smooth maps – theory and applications in switching circuits, *IEEE Transactions on Circuits and Systems - I: Fundamental Theory and Applications*, 47, (2000b), pp.633–643.

[7] G. I. Bischi, C. Chiarella, M. Kopel and F. Szidarovszky, *Nonlinear oligopolies: Stability and bifurcations*, Springer, Berlin, 2010.

[8] G. I. Bischi and M. Kopel, Equilibrium selection in a nonlinear duopoly game with adaptive expectations, *Journal of Economic Behavior & Organization*, 46,(2001), pp.73-100.

[9] G. I. Bischi and A. Naimzada, Global analysis of a dynamic duopoly game with bounded rationality, In: Filar, J. A., Gaitsgory, V. and Mizukami, K. (Eds.), *Advanced in Dynamics Games and Application*, Annals of the International Society of Dynamic Games, Vol. 5, pp.361–385. Birkhauser, Basel, 2000.

[10] G. I. Bischi, F. Lamatia and I. Sushko, Border collision bifurcation in a simply oligopoly model with constraints, *International Journal of Applied Mathematics and Statistics,* 26, (2012), 2: pp.121–136.

[11] W. A. Brock and C. H. Hommes, A Rational Route to Randomness, *Econometrica*, 65, (1997), pp.1059–1095.

[12] L. C. Corchon and A. Mas-Colell, On the stability of best reply and gradient systems with applications to imperfectly competitive models, *Economic Letters*, 51, (1996), pp.59–65.

[13] A.-A. Cournot, *Récherches sur les Principes Mathématiques de la Théorie des Richesses*, Hachette, Paris , 1838 .

[14] M. Di Bernardo, C. J. Budd, A. R. Champneys and P. Kowalczyk, *Piecewise-smooth dynamical systems: theory and applications*, Springer, Berlin, 2007.

[15] E. Droste, C. H. Hommes and J. Tuinstra, Endogenous fluctuations under evolutionary pressure in Cournot competition, *Games and Economic Behavior,* 40, (2002), pp.32–269.

[16] M. Ferrara and C. Udriste, Multitime models of optimal growth, *WSEAS Transactions on Mathematics*, 7, (2008), 1: pp.51–55

[17] L. Gardini, I. Sushko and A. K. Naimzada, Growing through chaotic intervals, *Journal of Economic Theory*. 143,(2008), pp.541–557.

[18] C. H. Hommes, *Chaotic dynamics in economic models: some simple case studies*, Thesis, University of Groningen, Wolters-Noordhoff, Groningen, 1991.

[19] W. Huang, The long-run benefits of chaos to oligopolistic firms, *Journal of Economic Dynamics & Control*, 32, (2008), pp.1332–1355.

[20] F. A. Lamantia, Nonlinear Duopoly with Efficient Production-Capacity Levels, *Computational Economics* , 38, (2011), pp. 295–309

[21] Y. L. Maistrenko, V. L. Maistrenko and S. I. Vikul, On period-adding sequences of attracting cycles in piecewise linear maps, *Chaos, Solitons & Fractals*, 9, (1998), pp.67–75.

[22] A. Medio and M. Lines,*Nonlinear dynamics - A primer,* Cambridge University Press, 2001.

[23] H. E. Nusse and J. A. Yorke, Border-collision bifurcations including period two to period three for piecewise smooth systems, *Physica D*, 57, (1992), pp.39–57.

[24] H. E. Nusse and J. A. Yorke, Border-collision bifurcation for piecewise smooth one-dimensional maps. *International Journal of Bifurcation and Chaos***,** 5, (1995), pp.189–207.

[25] T. Puu, Chaos in duopoly pricing, *Chaos, Solitons & Fractals,* 1, (1991) pp.573-581.

[26] T. Puu, *Attractors, bifurcations, and chaos - Nonlinear phenomena in Economics* (2nd. ed.). Springer, Berlin, 2003.

[27] T. Puu and A. Norin, Cournot duopoly when the competitors operate under capacity constraints, *Chaos Solitons & Fractals*, 18, (2003), pp.577-592.

[28] T. Puu and I. Sushko, *Oligopoly Dynamics, Models and Tools*, Springer, New York, 2002.

[29] T. Puu and I. Sushko, *Business Cycle Dynamics, Models and Tools*, Springer, New York, 2006.

[30] T. Puu, (*Oligopoly old ends—New means*, Springer, Berlin: 2011 .

[31] I. Sushko,T. Puu and L. Gardini, The Hicksian floor-roof model for two regions linked by interregional trade, *Chaos, Solitons & Fractals*, 18, (2003), pp.593–612.

[32] F. Tramontana, L. Gardini, and T. Puu, Cournot duopoly when the competitors operate multiple production plants, *Journal of Economic Dynamics and Control*, 33, (2009), 1:pp.250–265

[33] F. Tramontana, L. Gardini, T. Puu, Global bifurcations in a piecewise-smooth Cournot Duopoly game. *Chaos, Solitons & Fractals,* 43, (2010), pp.15–24.

[34] F. Tramontana, L. Gardini, T. Puu, Mathematical properties of a discontinuous Cournot-Stackelberg model. *Chaos, Solitons & Fractals*, 44, (2011), pp.58–70.

[35] H. R. Varian, *Microeconomic analysis* (3rd ed.). W. W. Norton & Company, Inc., New York, 1992.

[36] Z. T.Zhusubaliyev, and E. Mosekilde, *Bifurcations and chaos in piecewise-smooth dynamical systems*. World Scientific, Singapore, 2003.

[37] J. Zhang, Q. Da, and Y. Wang, Analysis of a nonlinear duopoly game with heterogeneous players. *Economic Modeling*, 24, (2007), pp.138-148.