Blow-up of solution to Cauchy problem for the generalized damped Boussinesq equation

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Abstract: In this paper we study the local solution, global solution and the Blow-up of the solution for a generalized damped Boussinesq equation. By employing various methods such as the energy method, the Fourier transform method, contraction mapping principle and the concavity method, we proved the existence of local and global solutions under certain conditions. Finally, we gave three the conditions about the Blow-up of the solution.

Key–Words: Cauchy problem, local solution, global solution, Blow-up of solution

1 Introduction

In this paper, we consider the following Cauchy problem of the Boussinesq equation with double damped term:

$$\begin{aligned}&[\Delta^2-\Delta+1]u_{tt}+[a\Delta^2-\Delta+p^2]u\\&+[-b\Delta+c]u_t\\=\ &\beta\Delta f(u),\ \ (x,t)\in\mathbb{R}\times[0,\infty)\end{aligned}\tag{1}$$

with initial datum

$$u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x),\ \ t>0\tag{2}$$

where the coefficients satisfy the following conditions

$$\begin{aligned}&2+2p^2-bc>0,\quad 4p^2-c^2>0; a,b,c>0;\\&\beta,p\in\mathbb{R},\qquad\qquad 4(a+1+p^2)-b^2>0;\end{aligned}$$

and $f(s)$ is a given nonlinear function, and Δ denotes the Laplace operator in $\mathbb{R}$.

It is well known that Boussinesq equation can be written in two basic forms

$$u_{tt}+\alpha u_{xxxx}-u_{xx}=\beta(u^2)_{xx}\tag{3}$$

and

$$u_{tt}-\alpha u_{xxtt}-u_{xx}=\beta(u^2)_{xx}\tag{4}$$

Equation (3) was derived by Boussinesq in 1872 to describe shallow-water waves. And it also arises in a large range of physical phenomena including the propagation of ion-sound waves in a uniform isotropic plasma and nonlinear lattice waves. Moreover, this type of equation governs the dynamics of the inharmonic lattice in Fermi-Pasta-Ulam (FPU) problem.

Eq. (4) is an important model that describes the propagation of long waves on shallow water like the other Boussinesq equations (with u_{xxxx}, instead of u_{xxtt})). In the case $\alpha>0$ Eq. (3) is called the good Boussinesq equation, while this equation with $\alpha<0$ received the name of the bad Boussinesq equation.

There is a considerable mathematical interest in the Boussinesq equations which have been studied from various aspects, see, [2-6],[8-15],[17-18].

Zhijian Yang and Xia Wang [10] studied the blowup of solutions to the initial boundary value problem for the bad Boussinesq-type equation:

$$u_{tt}-u_{xx}-bu_{xxxx}=\sigma(u)_{xx}\tag{5}$$

where $0<x<1, t,b>0$.

Ruying Xue [11] had considered the local and global existence of solutions for a generalized Boussinesq equation:

$$u_{tt}-u_{xx}+u_{xxxx}+(u^{k+1})_{xx}=0\ ,\ k>4\tag{6}$$

with initial data in some homogenous Besov-type space.

Necat Polat, Doğan Kaya [18] considered the existence, both locally and globally in time, and the blow-up of solutions for the Cauchy problem of the generalized damped multidimensional Boussinesq equation:

$$\begin{aligned}&u_{tt}-\Delta u-\Delta u_{tt}+\Delta^2u-k\Delta u_t=\Delta f(u)\\&(x,t)\in\mathbb{R}^n\times(0,\infty),\ k>0\end{aligned}\tag{7}$$

In [12] , R. Z. Xu, Y. C. Liu and B. W. Liu investigated the Cauchy problem for a class of multidimensional Boussinesq equation:

$$u_{tt} - \Delta u + \Delta^2 u + \Delta^2 u_{tt} = \Delta f(u) \quad (8)$$

where $f(u) = \pm\alpha|u|^p$ or $\alpha|u|^{p-1}u,\ \alpha > 0.$

Moreover, some results on the local existence, global existence and finite-time blow-up of the solution were obtained.

Throughout this paper, we use the following notations:

$H^s(\mathbb{R})$ denotes s-order Sobolev space on $\mathbb{R}$ with norm:

$$\|f\|_{H^s} = \|(I - \partial_x^2)^{\frac{s}{2}} f\|_{L^2} = \|(1+\xi^2)^{\frac{s}{2}} \widehat{f}\|_{L^2}$$

where s is a real number. I is unitary operator and $\partial_x = \frac{\partial}{\partial_x}$ denotes the derivative with respect to x.

Let $F[u]$ denote the Fourier transform of u defined by

$$F[u] = \widehat{u}(\xi, t) = \int_{-\infty}^{+\infty} e^{-ix\xi} u(x,t)dx;$$

and we denote its inverse transform by F^{-1}.

2 Existence and uniqueness of local solution

Taking the Fourier transform with respect to x on Eqs. (1) and (2) lead to the following ordinary differential equation with parameter $\xi \in \mathbb{R}$.

$$\begin{aligned}&(1+\xi^2+\xi^4)\widehat{u}_{tt}(\xi,t) + (b\xi^2+c)\widehat{u}_t(\xi,t)\\ &+(a\xi^4+\xi^2+p^2)\widehat{u}(\xi,t) = -\beta\xi^2\widehat{f}(\xi,t) \quad (9)\end{aligned}$$

$$\widehat{u}(\xi,0) = \widehat{u}_0(\xi), \quad \widehat{u}_t(\xi,0) = \widehat{u}_1(\xi). \quad (10)$$

The corresponding characteristic equation is

$$(1+\xi^2+\xi^4)\lambda^2 + (b\xi^2+c)\lambda + (a\xi^4+\xi^2+p^2) = 0 \quad (11)$$

Then the roots $\lambda_{1,2}$ of (11) are

$$\lambda_1 = \frac{-(b\xi^2+c)}{2(1+\xi^2+\xi^4)} + i\delta_\xi,$$
$$\lambda_2 = \frac{-(b\xi^2+c)}{2(1+\xi^2+\xi^4)} - i\delta_\xi,$$

where, $\delta_\xi = \frac{\sqrt{\Gamma}}{2(1+\xi^2+\xi^4)}$ and

$$\begin{aligned}\Gamma &= 4a\xi^8 + 4(a+1)\xi^6 + (4a+4+4p^2-b^2)\xi^4\\ &+(4+4p^2-2bc)\xi^2 + 4p^2 - c^2.\end{aligned}$$

The solution of (9) with (10) is

$$\begin{aligned}\widehat{u}(\xi,t) &= \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}\widehat{u}_0(\xi)\\ &+\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\widehat{u}_1(\xi)\\ &-\int_0^t \frac{\beta\xi^2}{2(1+\xi^2+\xi^4)}\widehat{f}(u)(\tau) \times\\ &\frac{e^{\lambda_1(t-\tau)} - e^{\lambda_2(t-\tau)}}{\lambda_1-\lambda_2}d\tau.\end{aligned}$$

Thus the solution of problem (1) with (2) also can be rewritten as

$$\begin{aligned}&u(x,t) = \tilde{g}(t)u_0(x) + g(t)u_1(x) \quad (12)\\ &+ \int_0^t g(t-\tau)(I - \partial_x^2 + \partial_x^4)^{-1}(\beta\partial_x^2)f(x,\tau)d\tau\end{aligned}$$

where

$$\tilde{g}(t)u_0(x) = F^{-1}[J_1(\xi,t)\widehat{u}_0(\xi)],$$

$$g(t)u_1(x) = F^{-1}[J(\xi,t)\widehat{u}_1(\xi)],$$

$$\begin{aligned}&g(t-\tau)(I-\partial_x^2+\partial_x^4)^{-1}(\beta\partial_x^2)f(x,t)\\ &= F^{-1}[J(\xi,t-\tau)(1+\xi^2+\xi^4)^{-1}(\beta\xi^2)\widehat{f}(\xi,t)]\end{aligned}$$

$$J(\xi,t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} = e^{-\frac{b\xi^2+c}{2(1+\xi^2+\xi^4)}t}\frac{\sin\delta_\xi t}{\delta_\xi}$$

$$\begin{aligned}J_1(\xi,t) &= \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1}\\ &= e^{-\frac{b\xi^2+c}{2(1+\xi^2+\xi^4)}t}[\cos\delta_\xi + \frac{-(b\xi^2+c)\sin\delta_\xi t}{2(1+\xi^2+\xi^4)\delta_\xi}]\end{aligned}$$

We can easily get that $J_1(\xi,t)$ and $J(\xi,t)$ are uniformly bounded function on $\mathbb{R}\times[0,+\infty)$.

Lemma 1 *Assume that*

$$f(x) \in H^s \bigcap L^\infty, g(x) \in L^2,$$

then the following inequality holds

$$\|f(x)g(x)\|_{L^2} \le C\|g(x)\|_{L^2} \quad (13)$$

where C is a positive constant.

Proof: Using the Hölder inequality, we can get the result easily. □

Lemma 2 *For any $T > 0$, $s \in \mathbb{R}$, suppose that $u_0(x), u_1(x) \in H^s$ and $f(x,t) \in L^1([0,T];H^s)$, then the Cauchy problem (1) and (2) have a unique solution $u \in C^1([0,T];H^s)$ and the following estimate holds*

$$\begin{aligned}&\|u(x,t)\|_{H^s} + \|u_t(x,t)\|_{H^s} \le C[\|u_0(x)\|_{H^s}\\ &+\|u_1(x)\|_{H^s} + \int_0^t \|f(x,\tau\|_{H^s}d\tau]\end{aligned} \quad (14)$$

Proof: Let $u(x,t) \in C^1([0,T]; H^s)$ be a solution of the Cauchy problem (1) and (2). Employing (12), we have the estimate

$$\begin{aligned}
&\|u(x,t)\|_{H^s} = \|\tilde{g}(t)u_0(x) + g(t)u_1(x) + \\
&\int_0^t g(t-\tau)(I-\partial_x^2+\partial_x^4)^{-1}(\beta\partial_x^2)f(x,t)d\tau\Big\|_{H^s} \\
&\le \|\tilde{g}(t)u_0(x)\|_{H^s} + \|g(t)u_1(x)\|_{H^s} + \\
&\|\int_0^t g(t-\tau)(I-\partial_x^2+\partial_x^4)^{-1}(\beta\partial_x^2)f(x,\tau)d\tau\|_{H^s} \\
&= I_1 + I_2 + I_3 \qquad (15)
\end{aligned}$$

According to the lemma 1, we can estimate three parts in (15) as follows

$$\begin{aligned}
I_1 &= \|\tilde{g}(t)u_0(x)\|_{H^s} \\
&= \|(1+\xi^2)^{\frac{s}{2}} J_1(\xi,t)\widehat{u}_0(\xi)\|_{L^2} \\
&\le C\|(1+\xi^2)^{\frac{s}{2}}\widehat{u}_0(\xi)\|_{L^2} \\
&\le C\|u_0(x)\|_{H^s} \qquad (16)
\end{aligned}$$

$$\begin{aligned}
I_2 &= \|g(t)u_1(x)\|_{H^s} \\
&= \|(1+\xi^2)^{\frac{s}{2}} J(\xi,t)\widehat{u}_1(\xi)\|_{L^2} \\
&\le C\|(1+\xi^2)^{\frac{s}{2}}\widehat{u}_1(\xi)\|_{L^2} \\
&\le C\|u_1(x)\|_{H^s} \qquad (17)
\end{aligned}$$

$$\begin{aligned}
&I_3 = \\
&\Big\|\int_0^t \|g(t-\tau)(I-\partial_x^2+\partial_x^4)^{-1}(\beta\partial_x^2)f(\xi,\tau)d\tau\Big\|_{H^s} \\
&= \|(1+\xi^2)^{\frac{s}{2}}\int_0^t J(\xi,t-\tau)\frac{\beta\xi^2\widehat{f}(\xi,\tau)}{2(1+\xi^2+\xi^4)}d\tau\|_{L^2} \\
&\le C\|(1+\xi^2)^{\frac{s}{2}}\int_0^t \widehat{f}(\xi,\tau)d\tau\|_{L^2} \\
&\le C\int_0^t \|(1+\xi^2)^{\frac{s}{2}}\widehat{f}(\xi,\tau)\|_{L^2}d\tau \\
&\le C\int_0^t \|f(x,\tau)\|_{H^s}d\tau \qquad (18)
\end{aligned}$$

From (16) to (18), we can conclude

$$\begin{aligned}
\|u(x,t)\|_{H^s} \le\ & C[\|u_0(x)\|_{H^s} + \|u_1(x)\|_{H^s} \\
&+ \int_0^t \|f(x,\tau)\|_{H^s}d\tau] \qquad (19)
\end{aligned}$$

Differentiating (12) with respect to t, we get

$$\begin{aligned}
&u_t(x,t) = \tilde{g}_t(t)u_0(x) + g_t(x)u_1(x) \\
&+ \int_0^t \frac{\partial}{\partial t}g(t-\tau)(I-\partial_x^2+\partial_x^4)^{-1}(\beta\partial_x^2)f(x,\tau)d\tau \qquad (20)
\end{aligned}$$

Then,

$$\begin{aligned}
&\|u_t(x,t)\|_{H^s} \\
=\ & \|\tilde{g}_t(t)u_0(x) + g_t(t)u_1(x) + \int_0^t \frac{\partial}{\partial_t}g(t-\tau) \\
&(I-\partial_x^2+\partial_x^4)^{-1}(\beta\partial_x^2)f(x,\tau)d\tau\|_{H^s} \\
&\le \|\tilde{g}_t(t)u_0(x)\|_{H^s} + \|g_t(t)u_1(x)\|_{H^s} + \\
&\|\int_0^t \frac{\partial}{\partial t}g(t-\tau) \\
&(I-\partial_x^2+\partial_x^4)^{-1}(\beta\partial_x^2)f(x,\tau)d\tau\|_{H^s} \qquad (21)
\end{aligned}$$

For three parts in (21), using the same method similar to estimate (16), (17) and (18), we can get that

$$\|\tilde{g}_t(t)u_0(x)\|_{H^s} \le C\|u_0(x)\|_{H^s}, \qquad (22)$$

$$\|g_t(t)u_1(x)\|_{H^s} \le C\|u_1(x)\|_{HS}, \qquad (23)$$

$$\begin{aligned}
&\|\int_0^t \frac{\partial}{\partial_t}g(t-\tau)(I-\partial_x^2+\partial_x^4)^{-1} \\
&(\beta\partial_x^2)f(\xi,\tau)d\tau\|_{H^s} \\
&\le C\int_0^t \|f(x,\tau)\|_{H^s}d\tau. \qquad (24)
\end{aligned}$$

Therefore, we have estimate

$$\begin{aligned}
&\|u_t(x,t)\|_{H^s} \le C[\|u_0(x)\|_{HS} + \|u_1(x)\|_{HS} \\
&+ \int_0^t \|f(x,\tau)\|_{HS}d\tau] \qquad (25)
\end{aligned}$$

Combining (19) and (25), we derive

$$\begin{aligned}
&\|u(x,t)\|_{H^s} + \|u_t(x,t)\|_{H^s} \\
\le\ & C[\|u_0(x)\|_{H^s} + \|u_1(x)\|_{H^s} \\
&+ \int_0^t \|f(x,\tau)\|_{H^s}d\tau]. \qquad (26)
\end{aligned}$$

Since H^s and $L^1([0.T]; H^s)$ are complete spaces, taking subsequences

$$\{u_0^j(x)\}_{j=1}^{\infty},\ \{u_1^j(x)\}_{j=1}^{\infty} \subset H^s$$

and

$$\{f^j(x,t)\}_{j=1}^{\infty} \subset L^1([0,T]; H^s)$$

such that when $j \to \infty$ we have

$$\|u_0^j(x) - u_0(x)\|_{H^s} \to 0,$$

$$\|u_1^j(x) - u_1(x)\|_{H^s} \to 0,$$

$$\int_0^t \|f^j(x,t) - f(x,t)\|_{H^s}dt \to 0.$$

Let $\{u^j(x,t)\}_{j=1}^{\infty} \in C^1([0,T];H^s)$ be a solution to the Cauchy problem (1) and (2), determined by the initial value function $\{u_0^j(x)\}_{j=1}^{\infty}$ and $\{u_1^j(x)\}_{j=1}^{\infty}$, and functions $\{f^j(x,t)\}_{j=1}^{\infty}$.

Define the function space as

$$X(T) = \left\{ u \,\middle|\, \begin{array}{l} u \in C^1([0,T];H^s); \text{denote} \\ u(x,0) = u_0(x), \\ u_t(x,0) = u_1(x) \end{array} \right\}$$

with the norm

$$\|u\|_{X(T)} = \max_{0\le t\le T}(\|u(x,t)\|_{H^s} + \|u_t(x,t)\|_{H^s})$$

Clearly, for any $T > 0$, $X(T)$ is Banach space.

For the smooth function $u^j(x,t)$, we have

$$\begin{aligned} &\|u^j(x,t)\|_{H^s} + \|u_t^j(x,t)\|_{H^s} \\ &\le C[\|u_0^j(x)\|_{H^s} + \|u_1^j(x)\|_{H^s} \\ &\quad + \int_0^t \|f^j(x,\tau)\|_{H^s} d\tau] \end{aligned}$$

Since

$$\begin{aligned} &\|u^j(x,t) - u^k(x,t)\|_{X(T)} \\ &= \max_{0\le t\le T}\{\|u^j(x,t) - u^k(x,t)\|_{H^s} \\ &\quad + \|u_t^j(x,t) - u_t^k(x,t)\|_{H^s}\} \\ &\le C[\|u_0^j(x) - u_0^k(x)\|_{H^s} + \|u_1^j(x) - u_1^k(x)\|_{H^s} \\ &+ \int_0^t \|f^j(x,\tau) - f^k(x,\tau)\|_{H^s} d\tau] \\ &(j,k = 1,2\cdots) \end{aligned}$$

$\{u^j(x,t)\}_{j=1}^{\infty}$ is the basic sequence of $X(T)$ and hence it converges to function $u(x,t) \in X(T)$. Therefore, $u(x,t)$ is a solution of Cauchy Problem (1) and (2), and $u(x,t)$ is belong to $C^1([0,T];H^s)$.

The proof of the uniqueness of solution is the same as the first part of the proof of Theorem 6. □

Now, we are going to prove the existence and the uniqueness of the local solution for problem (1) and (2) by the contraction mapping principle.

Let $\|u_0(x)\|_{H^s} + \|u_1(x)\|_{H^s} = M$. Define the set:

$$Y(M,T) = \{u \in X(T) \mid \|u\|_{X(T)} \le 2CM\}$$

Obviously, $Y(M,T)$is a nonempty bounded closed convex subset of $X(T)$ for any $M, T > 0$. For any $w(x,t) \in Y(T)$, let S map w to the unique solution to (1) and (2).

Lemma 3 *(see[8].) Suppose that $s \ge 0$, $f \in C^k(k = [s]+1)$, $f(0) = 0$, $u \in H^s \cap L^\infty$. If $\|u\|_{L^\infty(\mathbb{R})} \le M_1$, then we have*

$$\|f(u)\|_{H^s} \le k_1(M_1)\|u\|_{H^s}$$

where $k_1(M_1)$ is a constant dependent on M_1

Lemma 4 *(see, [8]) Suppose that $s \ge 0$, $f \in C^k(k = [s]+1)$. If $u, v \in H^s \cap L^\infty$, $\|u\|_{L^\infty} \le M_1\|v\|_{L^\infty} \le M_1$, then*

$$\|f(u) - f(v)\|_{H^s} \le k_2(M_1)\|u - v\|_{H^s}$$

where $k_2(M_1)$ is a constant dependent on M_1.

Lemma 5 *Suppose $s \ge 0$, $u_0(x), u_1(x) \in H^s$, $f \in C^{[s]+1}$ and $f(0) = 0$, then S is strictly contractive mapping from $Y(M,T)$ into itself for T sufficiently small relative to M.*

Proof: Let $w(x,t) \in Y(M,T)$. Using of (14) and Lemma 4, we have

$$\begin{aligned} &\|u(x,t)\|_{H^s} + \|u_t(x,t)\|_{H^s} \\ &\le C(\|u_0(x)\|_{H^s} + \|u_1(x)\|_{H^s}) \\ &+C\int_0^t \|f(w(x,\tau))\|_{H^s} d\tau \\ &\le CM + C\max_{0\le t\le T} k_1(M_1)\|w(x,t)\|_{H^s} T \\ &\le CM + Ck_1(M_1)\|w(x,t)\|_{X(T)} T \\ &\le CM + Ck_1(M_1)2CMT. \end{aligned}$$

Taking

$$T \le \frac{1}{2k_1(M_1)CM} \tag{27}$$

then $\|u(x,t)\|_{H^s} + \|u_t(x,t)\|_{H^s} \le 2CM$, S maps $Y(M,T)$ into $Y(M,T)$.

We are going to prove that S:$Y(T) \to Y(T)$ is strictly contractive.

For any $T > 0$, $w_1(x,t), w_2(x,t) \in Y(T)$, i.e.,

$$u_1(x,t) = Sw_1(x,t), u_2(x,t) = Sw_2(x,t).$$

Set

$$u(x,t) = u_1(x,t) - u_2(x,t),$$

$$w(x,t) = w_1(x,t) - w_2(x,t).$$

Then

$$\begin{aligned}&\|u(x,t)\|_{H^s}+\|u_t(x,t)\|_{H^s}\\&\le C\int_0^t\|f(Sw_1(x,\tau)-Sw_2(x,\tau))\|_{H^s}d\tau\\&\le C\int_0^t\|f(w_1(x,t)-w_2(x,t))\|_{H^s}d\tau\\&\le C\int_0^t\|f(w(x,\tau))\|_{H^s}d\tau\\&\le Ck_1(M_1)\max_{0\le t\le T}\|w(x,t)\|_{H^s}T\\&\le Ck_1(M_1)\|w(x,t)\|_{X(T)}T.\end{aligned}$$

Taking

$$T\le\frac{1}{2Ck_1(M_1)}, \tag{28}$$

we have

$$\|u\|_{X(T)}\le\frac{1}{2}\|w(x,t)\|_{X(T)}.$$

Hence, $S:Y(T)\to Y(T)$ is strictly contractive. □

Theorem 6 *Suppose $s>\frac{1}{2}$, $u_0,u_1\in H^s$, $f\in C^{[s]+1}$ and $f(0)=0$. Then problem (1) and (2) admit a unique local solution $u(x,t)$ defined on a maximal time interval $[0,T^0)$ with $u(x,t)\in C^1([0,T^0);H^s)$. Moreover, if*

$$\sup_{0\le t\le T^0}(\|u(x,t)\|_{H^s}+\|u_t(x,t)\|_{H^s}<\infty \tag{29}$$

then $T^0=\infty$.

Proof: By the lemma 5, we can chose a $\grave{T}>0$ appropriately such that S has a unique fixed point $u(x,t)\in Y(M,T)$, which is a strong solution of problem(1) and (2). And it is easy to prove the uniqueness of solution which belongs to $X(\grave{T})$ for each $\grave{T}$.

Now, let $[0,T^0)$ be the maximal time interval of existence for $u\in X(T^0)$. We want to show that, if

$$\sup_{0\le t\le T^0}(\|u(x,t)\|_{H^s}+\|u_t(x,t)\|_{H^s})<\infty$$

is satisfied, then $T^0=\infty$.

Suppose that

$$\sup_{0\le t\le T^0}(\|u(x,t)\|_{H^s}+\|u_t(x,t)\|_{H^s})<\infty$$

holds and $T^0<\infty$. For each $T\in[0,T^0)$.

we consider the follow Cauchy problem

$$\begin{aligned}&\upsilon_{tt}-\Delta\upsilon-\Delta\upsilon_{tt}+\Delta^2\upsilon_{tt}+a\Delta^2\upsilon\\&-b\Delta\upsilon_t+c\upsilon_t+p^2\upsilon=\beta\Delta f(\upsilon)\end{aligned} \tag{30}$$

$$\upsilon(x,0)=u(x,\grave{T}),\upsilon_t(x,0)=u_t(x,\grave{T}) \tag{31}$$

By

$$\sup_{0\le t\le T^0}(\|u(x,t)\|_{H^s}+\|u_t(x,t)\|_{H^s})<\infty$$

we have

$$\sup_{0\le t\le T^0}(\|u(x,t)\|_{H^s}+\|u_t(x,t)\|_{H^s})<K$$

where K is a positive constant independent of $\grave{T}\in[0,T^0)$. From Lemma 3 and the contraction mapping principle we see that there exists a constant $T_1\in(0,T^0)$ such that for each $\grave{T}\in[0,T^0)$, the problem (30) and (31) has a unique solution $\upsilon(x,t)\in X(T_1)$. In particular, (1) and (2) reveal T_1 that can be selected independently of $\grave{T}\in[0,T^0)$. Take $\grave{T}=T^0-\frac{T_1}{2}$ and define

$$\widetilde{u}=\begin{cases}u(x,t), & t\in[0,\grave{T}]\\ \upsilon(x,t-\grave{T}), & t\in[\grave{T},T^0+\frac{T_1}{2}]\end{cases}$$

then $\tilde{u}(x,t)$ is a solution to equations (1) and (2) on interval $[0,T^0+\frac{T_1}{2}]$, and by the uniqueness, $\widetilde{u}(x,t)$ extends $u(x,t)$, which violates the maximalist of $[0,T^0)$. Therefore, if (29) holds, then

$$T^0=\infty$$

Theorem 6 is proved. □

3 Existence and uniqueness of global solution

Lemma 7 *Suppose that $f(u)\in C$, $F(u)=\int_0^u f(s)ds$, $u_0\in H^1$, $u_1,(-\Delta)^{-\frac{1}{2}}u_1\in L^2$ and $F(u_0)\in L^1$. Then for the solution $u(x,t)$ to problem (1) and (2), we have the energy identity*

$$\begin{aligned}&E=\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2+\|u\|_{L^2}^2+\|u_t\|_{L^2}^2+\\&\|u_{tx}\|_{L^2}^2+a\|u_x\|_{L^2}^2+p^2\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2\\&+2\beta\int_{-\infty}^{+\infty}F(u)dx+2b\int_0^t\|u_\tau\|_{L^2}^2d\tau+\\&2c\int_0^t\|(-\Delta)^{-\frac{1}{2}}u_\tau\|_{L^2}^2d\tau=E(0)\end{aligned} \tag{32}$$

Proof: Multiplying the equation (1) by $(-\Delta)^{-1}u_t$ and integrating the product with respect to x, we get

$$\begin{aligned}&(u_{tt}-\Delta u-\Delta u_{tt}+\Delta^2u_{tt}+a\Delta^2u\\&-b\Delta u_t+cu_t+p^2u-\beta\Delta f(u),(-\Delta)^{-1}u_t)=0\end{aligned}$$

$$((-\Delta)^{-1}u_{tt} + u + u_{tt} - \Delta u_{tt} - a\Delta u + bu_t$$
$$+c(-\Delta)^{-1}u_t + p^2((-\Delta)^{-1})u + \beta f(u), u_t) = 0$$

$$((-\Delta)^{-1}u_{tt}, u_t) + (u, u_t) + (u_{tt}, u_t)$$
$$-(\Delta u_{tt}, u_t) - a(\Delta u, u_t) + b(u_t, u_t)$$
$$+c((-\Delta)^{-1}u_t, u_t) + p^2((-\Delta)^{-1}u, u_t)$$
$$+\beta(f(u), u_t) = 0$$

and

$$\frac{1}{2}\frac{d}{dt}[\|(\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + \|u\|_{L^2}^2 + \|u_t\|_{L^2}^2$$
$$+\|u_{tx}\|_{L^2}^2 + a\|u_x\|_{L^2}^2 + p^2\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2$$
$$+2\beta\int_{-\infty}^{+\infty} F(u)dx] + b\|u_t\|_{L^2}^2$$
$$+c\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 = 0$$

So

$$E = \|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + \|u\|_{L^2}^2 + \|u_t\|_{L^2}^2$$
$$+\|u_{tx}\|_{L^2}^2 + a\|u_x\|_{L^2}^2 + p^2\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2$$
$$+2\beta\int_{-\infty}^{+\infty} F(u)dx + 2b\int_0^t \|u_\tau\|_{L^2}^2 d\tau$$
$$+2c\int_0^t \|(-\Delta)^{-\frac{1}{2}}u_\tau\|_{L^2}^2 d\tau = E(0)$$

where

$$E(0) = \|(-\Delta)^{-\frac{1}{2}}u_1\|_{L^2}^2 + \|u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2$$
$$+a\|(-\Delta)^{-\frac{1}{2}}u_0\|_{L^2}^2 + p^2\|(-\Delta)^{-\frac{1}{2}}u_0\|_{L^2}^2$$
$$+2\beta\int_{-\infty}^{+\infty} F(u_0)dx$$

and $(.,.)$ denotes the inner product of L^2 space.

Theorem 8 *Suppose $s \geq \frac{1}{2}$, $u_0, u_1 \in H^s$, $f \in C^{[s]+1}$ and $T^0 > 0$ is a maximal time of the solution of the problem (1) and (2). Then $T^0 < \infty$ if and only if*

$$\limsup_{t\to T^0} \|u(t)\|_{L^\infty} = \infty \tag{33}$$

Proof: If (33) set up, by the Sobolev imbedding Theorem, we have,

$$\limsup_{t\to T^0} [\|u(\cdot,t)\|_{H^s} + \|u_t(\cdot,t)\|_{H^s}] = \infty$$

then $T^0 < \infty$.

On the other hand, we prove that $T^0 = \infty$ when

$$\sup_{t\in[0,T^0)} \|u(t)\|_{L^\infty} < \infty.$$

Since

$$\|u(\cdot,t)\|_{H^s} + \|u_t(\cdot,t)\|_{H^s}$$
$$\leq C[\|u_0\|_{H^s} + \|u_1\|_{H^s}$$
$$+\int_0^t \|f(\cdot,\tau)\|_{H^s} d\tau],$$

clearly,

$$\|u(\cdot,t) + u_t(\cdot,t)\|_{H^s}$$
$$\leq C[\|u_0\|_{H^s} + \|u_1\|_{H^s}$$
$$+\int_0^t \|f(\cdot,\tau)\|_{H^s} d\tau] < \infty$$

So $\limsup_{t\to T^0} \|u(\cdot,t) + u_t(\cdot,t)\|_{H^s} < \infty$.

By the Sobolev imbedding Theorem, we get

$$\lim_{t\to T^0} \|u(\cdot,t)\|_{L^\infty} < \infty$$

Therefore, $T^0 = \infty$. □

Theorem 9 *Suppose $s \geq \frac{1}{2}, u_0, u_1 \in H^s, F(u_0) \in L^1$, $f(u) \in C^{[s]+1}$, $F(u) \geq 0$ or $f'(u)$ is bounded below, then there is a unique global solution $u \in C^1([0,T]; H^s)$ to problem (1) and (2).*

Proof: If $F(u) \geq 0$, then

$$E_1 = \|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + \|u\|_{L^2}^2 + \|u_t\|_{L^2}^2$$
$$+a\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2 + p^2\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2 \leq E(0)$$

If $f'(u)$ is bounded below, i.e., there is a constant A_0 such that $f'(u) \geq A_0$ for any $u \in \mathbb{R}$.

set $f_0(u) = f(u) - k_0 u$, $k_0 = \min\{A_0, 0\}$, then

$$f_0(0) = 0,\ f_0'(u) = f'(u) - k_0 \geq 0$$

So $f_0(u)$ is a monotonically increasing function. And we have $F_0(u) = \int_0^u f_0(s)ds > 0$.

Exploiting (32) and we notice that

$$\begin{aligned} F(u) &= \int_0^u f(s)ds = \int_0^t [f_0(s) + k_0 s]ds \\ &= F_0(u) + \frac{k_0}{2}u^2 \end{aligned}$$

Then

$$\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + \|u\|_{L^2}^2 + \|u_t\|_{L^2}^2$$
$$+a\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2 + p^2\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2$$
$$+2\beta\int_{-\infty}^{+\infty} F_0(u)dx$$
$$= E(0) - 2b\int_0^t \|u_\tau\|_{L^2}^2 d\tau$$
$$-2c\int_0^t \|(-\Delta)^{-\frac{1}{2}}u_\tau\|_{L^2}^2 d\tau - \beta k_0\|u\|_{L^2}^2$$
$$\leq E(0) - 2b\int_0^t \|u_\tau\|_{L^2}^2 d\tau - \beta k_0\|u\|_{L^2}^2.$$

We have

$$E_1 + 2\beta\int_{-\infty}^{+\infty} F_0(u)dx$$
$$\le E(0) - 2b\int_0^t \|u_\tau\|_{L^2}^2 d\tau - \beta k_0\|u\|_{L^2}^2$$
$$\le E(0) - 2b\int_0^t \|u_\tau\|_{L^2}^2 d\tau - \beta k_0\|u_0\|_{L^2}^2$$
$$+\int_0^t (\beta^2 k_0^2\|u\|_{L^2}^2 + \|u_\tau\|_{L^2}^2)d\tau$$
$$\le E(0) - \beta k_0\|u_0\|_{L^2}^2 + (2b+\beta^2 k_0^2)\cdot$$
$$\int_0^t (\|u\|_{L^2}^2 + \|u_\tau\|_{L^2}^2)d\tau.$$

By the Gronwall inequality

$$\|u\|_{L^2}^2 + \|u_t\|_{L^2}^2$$
$$\le (E(0) - \beta k_0\|u_0\|_{L^2}^2)\exp[(2b+\beta^2 k_0^2)T]$$

Thus, for any $t \in [0, T^0)$, $\|u(x,t)\|_{H^1}$ is Uniformly bounded, by the Sobolev imbedding Theorem, we have

$$\|u(x,t)\|_{L^\infty} < \infty, \forall t \in [0, T^0).$$

Therefore, the problem (1),(2)have a unique global solution $u(x,t)$,which is belong to $C^1([0,T]; H^s)$. □

4 Blow-up of solution

Lemma 10 *Suppose that a positive twice differentiable function $H(t)$ satisfies on $t \ge 0$ the inequality*

$$H(t)H''(t) - (1+e)(H'(t))^2$$
$$\ge -2M_1 H(t)H'(t) - M_2(H(t))^2$$

where $e > 0$ and $M_1, M_2 \ge 0$ are constants.

If $H(0) > 0$ and $H'(0) > -\gamma_2 e^{-1}H(0)$,and $M_1 + M_2 > 0$,then $H(t)$ tends to infinity, as $t \to t_1 \le t_2$,

$$t_2 = \frac{1}{2\sqrt{M_1^2 + eM_2}}\ln\frac{\gamma_1 H(0) + eH'(0)}{\gamma_2 H(0) + eH'(0)}$$

where $\gamma_{1,2} = -M_1 \mp \sqrt{M_1^2 + eM_2}$.

If $H(0) > 0$ and $H'(0) > 0$,and $M_1 = M_2 = 0$, then $H(t) \to \infty$ as $t \to t_1 \le t_2 = \frac{H(0)}{eH'(0)}$.

Theorem 11 *Assume that $k \ge 0, f(u) \in C, u_0 \in H^1$, $u_1, (-\Delta)^{-\frac{1}{2}}u_0, (-\Delta)^{-\frac{1}{2}}u_1 \in L^2, F(u) = \int_0^u f(s)ds$, $F(u_0) \in L^1$ and there exists a constant $\alpha > 0$ such that*

$$uf(u) \le (\alpha + k + 2)F(u) + \frac{\alpha}{2\beta}u^2, \forall u \in \mathbb{R}$$

then the solution $u(x,t)$ of problem (1) and (2) blow up in finite time if one of the following conditions is valid:

(i): $E(0) < 0$

(ii): $E(0) = 0$ and

$$((-\Delta)^{-\frac{1}{2}}u_0, (-\Delta)^{-\frac{1}{2}}u_1) + (u_0, u_1) + (u_{0x}, u_{1x}) > 0$$

(iii): $E(0) > 0$ and

$$((-\Delta)^{-\frac{1}{2}}u_0, (-\Delta)^{-\frac{1}{2}}u_1) + (u_0, u_1) + (u_{0x}, u_{1x})$$
$$> \sqrt{\frac{2+\alpha+k}{2+\alpha}E(0)}\cdot$$
$$\sqrt{\|(-\Delta)^{-\frac{1}{2}}u_0\|_{L^2}^2 + \|u_0\|_{L^2(R)}^2 + \|u_{0x}\|_{L^2}^2}$$

Proof: Let

$$H(t) = \|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2 + \|u\|_{L^2}^2 + \|u_x\|_{L^2}^2$$
$$+\gamma(t+t_0)^2$$

where γ and t_0 are nonnegative constants to be specified later.

$$H'(t) = 2((-\Delta)^{-\frac{1}{2}}u, (-\Delta)^{-\frac{1}{2}}u_t)$$
$$+2(u, u_t) + 2(u_x, u_{xt}) + 2\gamma(t+t_0)$$

Using the Schwartz inequality,

$$(H'(t))^2 = 4H(t)[\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2$$
$$+\|u_t\|_{L^2}^2 + \|u_{xt}\|_{L^2}^2 + \gamma]$$

$$\begin{aligned} H''(t) &= 2\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + 2\|u_t\|_{L^2}^2 \\ &\quad +2\|u_{tx}\|_{L^2}^2 + 2(u, (-\Delta)^{-1})u_{tt} \\ &\quad +u_{tt} - \Delta u_{tt}) + 2\gamma \end{aligned}$$

We get from (1) that

$$\begin{aligned} &(-\Delta)^{-1}u_{tt} + u_{tt} - \Delta u_{tt} \\ = \ &-\beta f(u) - p^2(-\Delta)^{-1}u \\ &-c(-\Delta)^{-1}u_t - bu_t + a\Delta u - u. \end{aligned}$$

Then, we have

$$H''(t) = 2\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + 2\|u_t\|_{L^2}^2 +$$
$$2\|u_{tx}\|_{L^2}^2 - 2p^2\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2 - 2\|u\|_{L^2}^2$$
$$-a\|u_x\|_{L^2}^2 - 2c(u, (-\Delta)^{-1}u_t) - 2b(u, u_t)$$
$$-2\beta\int_{-\infty}^{\infty} uf(u)dx.$$

By the Schwartz inequality

$$2c(u,(-\Delta)^{-1}u_t) \le c[\|u\|_{L^2}^2 + \|(-\Delta)^{-1}u_t\|_{L^2}^2] \quad (34)$$

$$2b(u,u_t) \le b[\|u\|_{L^2}^2 + \|u_t\|_{L^2}^2] \quad (35)$$

From that, we get

$$\begin{aligned} &H(t)H''(t) - (1+\frac{\alpha}{4})(H'(t))^2 \\ &\ge H(t)H''(t) - (4+\alpha)H(t)[\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|u_{tx}\|_{L^2}^2 + \gamma] \\ &\ge H(t)\{H''(t) - (4+\alpha)[\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|u_{tx}\|_{L^2}^2 + \gamma]\} \end{aligned}$$

Taking $k = \max\{a,b,c,1\}$, since

$$\begin{aligned} E &= \|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + \|u\|_{L^2}^2 + \|u_t\|_{L^2}^2 \\ &+\|u_{tx}\|_{L^2}^2 + a\|u_x\|_{L^2}^2 + p^2\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2 \\ &+2\beta\int_{-\infty}^{\infty}F(u)dx + b\int_0^t \|u_\tau\|_{L^2}^2 d\tau + c\int_0^t \|(-\Delta)^{-\frac{1}{2}}u_\tau\|_{L^2}^2 d\tau = E(0) \end{aligned}$$

we have

$$\begin{aligned} kE(0) &\le k[\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + \|u\|_{L^2}^2 \\ &+\|u_t\|_{L^2}^2] + k[\|u_{tx}\|_{L^2}^2 + \|u_x\|_{L^2}^2 \\ &+p^2\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2 + 2\beta\int_{-\infty}^{+\infty}F(u)dx \\ &+b\int_0^t \|u_\tau\|_{L^2}^2 d\tau + c\int_0^t \|(-\Delta)^{-\frac{1}{2}}u_\tau\|_{L^2}^2 d\tau] \end{aligned} \quad (36)$$

We get from (34) and (35) that

$$\begin{aligned} &2c(u,(-\Delta)^{-1}u_t) + 2b(u,u_t) \\ &\le k[2\|u\|_{L^2}^2 + \|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2] \end{aligned}$$

From the above inequality and (36), we get

$$\begin{aligned} H''(t) &\ge 2\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + 2\|u_t\|_{L^2}^2 \\ &+2\|u_{tx}\|_{L^2}^2 - 2p^2\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2 - 2\|u\|_{L^2}^2 \\ &-a\|u_x\|_{L^2}^2 - 2\beta\int_{-\infty}^{+\infty}uf(u)dx \\ &-k[2\|u\|_{L^2}^2 + \|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2] \\ &\ge 2\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + 2\|u_t\|_{L^2}^2 + 2\|u_{tx}\|_{L^2}^2 \\ &-2p^2\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2 - 2\|u\|_{L^2}^2 - \|u_x\|_{L^2}^2 \\ &-2\beta\int_{-\infty}^{\infty}uf(u)dx - kE(0) + k[\|u_{tx}\|_{L^2}^2 \\ &+\|u_x\|_{L^2}^2 + p^2\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2 \\ &+b\int_0^t \|u_\tau\|_{L^2}^2 d\tau + 2\beta\int_{-\infty}^{+\infty}F(u)dx \\ &+c\int_0^t \|(-\Delta)^{-\frac{1}{2}}u_\tau\|_{L^2}^2 d\tau] \end{aligned}$$

Therefore, we have

$$\begin{aligned} &H''(t) - (4+\alpha)[\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 \\ &+\|u_{tx}\|_{L^2}^2 + \gamma] \\ &\ge 2\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + 2\|u_t\|_{L^2}^2 + 2\|u_{tx}\|_{L^2}^2 \\ &-2p^2\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2 - 2\|u\|_{L^2}^2 - a\|u_x\|_{L^2}^2 \\ &-2\beta\int_{-\infty}^{+\infty}uf(u)dx - kE(0) + k[\|u_{tx}\|_{L^2}^2 \\ &+\|u_x\|_{L^2}^2 + p^2\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2 \\ &+2\beta\int_{-\infty}^{+\infty}F(u)dx + b\int_0^t \|u_\tau\|_{L^2}^2 d\tau \\ &+c\int_0^t \|(-\Delta)^{-\frac{1}{2}}u_\tau\|_{L^2}^2 d\tau] \\ &-(4+\alpha)[\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 \\ &+\|u_{tx}\|_{L^2}^2 + \gamma] \\ &\ge (-2-\alpha)\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + (-2-\alpha)\|u_t\|_{L^2}^2 \\ &+(k-2-\alpha)\|u_{tx}\|_{L^2}^2 + (-2p^2+kp^2)\cdot \\ &\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2 - 2\|u\|_{L^2}^2 + (k-a)\|u_x\|_{L^2}^2 \\ &-(4+\alpha)\gamma - 2\beta\int_{-\infty}^{+\infty}uf(u)dx - kE(0) \\ &+k[2\beta\int_{-\infty}^{+\infty}F(u)dx + b\int_0^t \|u_\tau\|_{L^2}^2 d\tau \\ &+c\int_0^t \|(-\Delta)^{-\frac{1}{2}}u_\tau\|_{L^2}^2 d\tau] \end{aligned} \quad (37)$$

We also know that

$$\begin{aligned} &(-2-\alpha)\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + (-2-\alpha)\|u_t\|_{L^2}^2 \\ &+(k-2-\alpha)\|u_{tx}\|_{L^2}^2 + (-2p^2+kp^2)\cdot \\ &\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2 - 2\|u\|_{L^2}^2 + (k-\alpha)\|u_x\|_{L^2}^2 \\ &\ge (-2-\alpha)[\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 \end{aligned}$$

$$
\begin{aligned}
&+\|u_{tx}\|_{L^2}^2 + p^2\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2}^2 + \|u\|_{L^2}^2 \\
&+\alpha\|u_x\|_{L^2}^2] + \alpha\|u\|_{L^2}^2 \\
&\geq (-2-\alpha)[E(0) - 2\beta\int_{-\infty}^{\infty} F(u)dx \\
&-b\int_0^t \|u_\tau\|_{L^2}^2 d\tau - c\int_0^t \|(-\Delta)^{-\frac{1}{2}}u_\tau\|_{L^2}^2 d\tau] \\
&+\alpha\|u\|_{L^2}^2
\end{aligned}
$$

From (37),we have

$$
\begin{aligned}
&H^{''}(t) - (4+\alpha)[\|(-\Delta)^{-\frac{1}{2}}u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \\
&\|u_{tx}\|_{L^2}^2 + \gamma] \\
&\geq (-2-\alpha)[E(0) - 2\beta\int_{-\infty}^{+\infty} F(u)dx \\
&-b\int_0^t \|u_\tau\|_{L^2}^2 d\tau - c\int_0^t \|(-\Delta)^{-\frac{1}{2}}u_\tau\|_{L^2}^2 d\tau] \\
&+\alpha\|u\|_{L^2}^2 - (4+\alpha)\gamma \\
&-2\beta\int_{-\infty}^{+\infty} uf(u)dx - kE(0) \\
&+k[2\beta\int_{-\infty}^{+\infty} F(u)dx + b\int_0^t \|u_\tau\|_{L^2}^2 d\tau \\
&+c\int_0^t \|(-\Delta)^{-\frac{1}{2}}u_\tau\|_{L^2}^2 d\tau] \\
&\geq (-2-\alpha-k)E(0) - (4+\alpha)\gamma + \\
&\int_{-\infty}^{+\infty} \{[2\beta(2+\alpha) + 2\beta k]F(u) - 2\beta uf(u)\}dx \\
&+\alpha\|u\|_{L^2}^2 + \int_0^t \{[b(2+\alpha) + kb]\|u_\tau\|_{L^2}^2 \\
&+[kc + c(2+\alpha)]\|(-\Delta)^{-\frac{1}{2}}u_\tau\|_{L^2}^2\}d\tau \\
&\geq (-2-\alpha-k)E(0) - (4+\alpha)\gamma + \int_{-\infty}^{+\infty}\{[2\beta \\
&(2+\alpha) + 2\beta k]F(u) - 2\beta uf(u) + \alpha u^2\}dx.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&H(t)H^{''}(t) - (1+\frac{\alpha}{4})(H^{'}(t))^2 \\
&\geq -[(2+\alpha+k)E(0) \\
&+(4+\alpha)\gamma]H(t). \qquad (38)
\end{aligned}
$$

If $E(0) < 0$, taking $\gamma = -\frac{2+\alpha+k}{4+\alpha}E(0) \geq 0$, then

$$H(t)H^{''}(t) - (1+\frac{\alpha}{4})(H^{'}(t))^2 \geq 0$$

We may now choose t_0 so large that $H^{'}(0) > 0$.

From Lemma 10 we know

$$T_2 = \frac{4H(0)}{\alpha H'(0)} < \infty$$

If $E(0) < 0$, taking $\gamma = 0$, then we get from (38)

$$H(t)H^{''}(t) - (1+\frac{\alpha}{4})(H^{'}(t))^2 \geq 0$$

Also $H^{'}(0) > 0$ by assumption (ii). Thus, we obtain from Lemma 10 that $H(t)$ becomes infinite at a time T_1 at most equal to $T_2 = \frac{4H(0)}{\alpha H'(0)} < \infty$.

If $E(0) > 0$, then taking $\gamma = 0$, inequality (38) becomes

$$
\begin{aligned}
&H(t)H^{''}(t) - (1+\frac{\alpha}{4})(H^{'}(t))^2 \\
&\geq -(2+\alpha+k)E(0)H(t) \qquad (39)
\end{aligned}
$$

Define $\mu(t) = (H(t))^{-\theta}$,where $\theta = \frac{\alpha}{4}$, then

$$\mu^{'}(t) = -\theta(H(t))^{-\theta-1}H^{'}(t) \qquad (40)$$

and

$$
\begin{aligned}
&\mu^{''}(t) = -\theta(H(t))^{-\theta-2} \times \\
&[H(t)H^{''}(t) - (\theta-1)(H^{'}(t))^2] \\
&\leq (2+k+4\theta)E(0)(H(t))^{-\theta-1} \qquad (41)
\end{aligned}
$$

where inequality (39) is used. And (40) implies $\mu^{'}(0) < 0$.

Let $t^* = \sup\{t \mid \mu^{'}(\tau) < 0, \tau \in (0,t)\}$. By the continuity of $\mu^{'}(t)$, t^* is positive. Multiplying (41) by $2\mu^{'}(t)$ yields

$$
\begin{aligned}
&[(\mu^{'}(t))^2]^{'} \\
&\geq -2\theta^2(2+k+4\theta)E(0)(H(t))^{-2\theta-2}H^{'}(t) \\
&= 2\theta^2\frac{2+k+4\theta}{2\theta+1}E(0)[(H(t))^{-2\theta-1}]^{'} \qquad (42)
\end{aligned}
$$

For any $t \in [0, t^*)$, integrating (42) with respect to t over $[0,t)$ to get

$$
\begin{aligned}
&(\mu^{'}(t))^2 \\
&\geq 2\theta^2\frac{2+k+4\theta}{2\theta+1}E(0)(H(t))^{-2\theta-1} + (\mu^{'}(0))^2 \\
&-2\theta^2\frac{2+k+4\theta}{2\theta+1}E(0)(H(0))^{-2\theta-1} \\
&\geq (\mu^{'}(0))^2 - 2\theta^2\frac{2+k+4\theta}{2\theta+1}E(0)(H(0))^{-2\theta-1}
\end{aligned}
$$

By assumption (iii)

$$(\mu^{'}(0))^2 - 2\theta^2\frac{2+k+4\theta}{2\theta+1}E(0)(H(0))^{-2\theta-1} > 0$$

Hence by continuity of $\mu^{'}(t)$, we obtain

$$\mu^{'}(t) \leq -[(\mu^{'}(0))^2 - 2\theta^2 \frac{2+k+4\theta}{2\theta+1} E(0) \cdot (H(0))^{-2\theta-1}]^{\frac{1}{2}}, \quad \forall t \in [0, t^*] \tag{43}$$

By the continuity of t^*, it follows that inequality (43) holds for all $t \geq 0$. Therefore,

$$\mu(t) \leq (0) - [(\mu^{'}(0))^2 - 2\theta^2 \frac{2+k+4\theta}{2\theta+1} E(0) (H(0))^{-2\theta-1}]^{\frac{1}{2}} t, \quad \forall t > 0$$

So $\mu(T_1) = 0$ for some T_1 and

$$0 < T_1 \leq T_2 = \mu(0)/[(\mu^{'})^2 - \alpha^2 \frac{2+k+\alpha}{4\alpha+8} E(0) \cdot (H(0))^{-\frac{\alpha}{2}-1}]^{\frac{1}{2}}$$

Thus, $H(t)$ becomes infinite at a time T_1.

Therefore, $H(t)$ becomes infinite at a time T_1 under either assumptions (i),(ii) or (iii). We have a contradiction with the fact that the maximal time of existence is infinite. Hence the maximal time of existence is finite. □

5 Conclusion

In this paper we considered the generalized Boussinesq equation with double damped term. We discussed the existence and uniqueness of local solution and the global solution under certain conditions. Finally we gave three conditions about the Blow-up of the solution.

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