The Well-posedness of A SARS Epidemic Model

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Abstract: In this paper, a class of age-structured SARS epidemic model described by a set of partial differential equations is discussed. The existence and uniqueness of global solution to the model is obtained by using the fixed point theory. Also, the continuous dependence of the solution on the initial value and the regularity of the solution are discussed.

Key–Words: Partial differential equations, Global solution, Fixed point theory, Regularity of solution

1 Introduction

It is well known that the SARS virus can spread and infect seriously, although its further spread can be terminated by isolation and the curer will not be infected again. Recently, many researchers are interested in the fact that population's age-structure can influence the spread of disease, following with all kinds of age-structured epidemic models, including SIS model, SIR model, SEIR and SEIRS model, etc. (On these models, see [1, 6, 7, 8]). Especially, in [1], the authors proposed a class of SARS epidemic model with paying special attention to the patient who accepting isolation or treatment, then considered the existence and uniqueness of the model solution by applying C_0-semigroup theory and the perturbation theory. In this paper, we present another method to discuss the existence and uniqueness of the global solution to this kind of model by using fixed point theory, moreover, the regularity of model solution is also discussed. In fact, the method shown in this paper can be applied to the study of other similar epidemic models as well as queuing models and repairable system models.

The partial differential equations of the age-structured SARS epidemic model interested in this paper are as follows (see, [1])

$$\frac{\partial S}{\partial t}+\frac{\partial S}{\partial a}=-(\lambda(t)+\mu(a))S(a,t) \tag{1}$$

$$\frac{\partial I}{\partial t}+\frac{\partial I}{\partial a}=-(\mu(a)+\gamma)I(a,t)+\lambda(t)S(a,t) \tag{2}$$

$$\frac{\partial R}{\partial t}+\frac{\partial R}{\partial a}=-\mu(a)R(a,t)+\gamma I(a,t) \tag{3}$$

with the initial conditions

$$S_0(a) = S(a,0) \tag{4}$$

$$I_0(a) = I(a,0) \tag{5}$$

$$R_0(a) = R(a,0) \tag{6}$$

and the boundary conditions

$$\begin{aligned} S(0,t) &= \int_0^A [\beta(a)S(a,t)+\bar{\beta}(a)I(a,t) \\ &\qquad +\beta(a)R(a,t)]\mathrm{d}a \end{aligned} \tag{7}$$

$$I(0,t) = 0 \tag{8}$$

$$R(0,t) = 0 \tag{9}$$

Here $(S(a,t), I(a,t), R(a,t))$ denote the population density function of the susceptible, infected and recovered, respectively, at time t with age a. $\mu(a)$ denotes the relatively natural death rate function with age a, and it is a locally integrable function on $[0, A)$ satisfying $\int_0^A \mu(a)\mathrm{d}a = +\infty$, where A denotes the upper age. $\beta(a)$ is the average birth rate function of the susceptible population and the recovered population at age a, and $\bar{\beta}(a)$ is the average birth rate function of the infected population at age a. Due to the isolation and treatment for the patient, it is generally assumed $\bar{\beta}(a) < \beta(a)$. γ^{-1} denotes the average infectious cycle, and ρ denotes the propagation coefficient. Then the infection rate function at time t is defined as: $\lambda(t) = \rho\int_0^A I(a,t)\mathrm{d}a$.

The rest of the paper is as follows. In Section 2, some assumptions and notations are introduced for convenience, and the partial differential problem is transformed into integral equations by using the characteristic method. In Section 3, the existence and uniqueness of the local positive solution is obtained, following the existence and uniqueness of the global solution discussed by prior estimate of the local solution. And the continuous dependence of the solution on the initial value is also studied. The regularity of the solution is studied in Section 4. And a brief conclusion is presented in Section 5.

2 Assumptions and Problem Transformation

At first, we give some notations that are used throughout this paper.

Let L^1 denotes the Banach space $L^1([0,A])$, with norm:

$$\|u\|_{L^1} = \int_0^A |u(a)|\mathrm{d}a.$$

For $T \geq 0$, let $V = C([0,T], L^1)$, with norm:

$$\|q\|_V = \sup_{0\leq t\leq T} \int_0^A q(a,t)\mathrm{d}a.$$

Further, we assume some conditions on data as follows:

(H_1): $S_0(a), I_0(a), R_0(a), \lambda(t), \beta(a), \bar{\beta}(a), \mu(a)$ are nonnegative and continuous functions satisfying

$$S_0(A) = 0,\ \ I_0(A) = 0,\ \ R_0(A) = 0;$$

(H_2):

$$\lambda_0 = \sup_{0\leq t\leq T} \lambda(t) < \infty;$$

$$\beta_0 = \sup_{0\leq a\leq A} \beta(a) < \infty;$$

(H_3): For any given positive constant δ, there is

$$\lim_{a\to A^-} \int_{a-\delta}^{a} \mu(\xi)\mathrm{d}\xi = +\infty;$$

(H_4): The compatibility conditions hold

$$S_0(0) = \int_0^A [\beta(a)S_0(a) + \bar{\beta}(a)I_0(a) + \beta(a)R_0(a)]\mathrm{d}a;$$

$$I_0(0) = 0,\ \ R_0(0) = 0;$$

(H_5): The first order compatibility conditions hold

$$S_0'(0) = -(\lambda(0) + \mu(0))S_0(0) - \int_0^A [\beta(a)S_0'(a) + \bar{\beta}(a)I_0'(a) + \beta(a)R_0'(a)]\mathrm{d}a$$

$$I_0'(0) = \lambda(0)S_0(0),\ \ R_0'(0) = 0.$$

By direct computation, we can transform the problem (1)-(9) into an integral problem with the characteristic method (see, [2]). In the following, we give it as a lemma and its proof is omitted.

Lemma 1 *Assume the conditions* (H_1)-(H_2) *hold. For each given positive constant* $T > 0$, $S(a,t)$, $I(a,t)$, $R(a,t) \in V$ *is the solution to (1)-(9) if and only if* $S(a,t)$, $I(a,t)$, $R(a,t)$ *is the solution to the following integral equations:*

$$S(a,t) = \begin{cases} S_0(a-t)\exp[-\int_0^t (\lambda(\xi) + \mu(a+\xi-t))\mathrm{d}\xi], \\ \text{for} \quad a > t \\ \int_0^A [\beta(\eta)S(\eta,t-a) + \bar{\beta}(\eta)I(\eta,t-a) \\ \quad + \beta(\eta)R(\eta,t-a)]\mathrm{d}\eta \\ \quad \cdot \exp[-\int_{t-a}^t (\lambda(\xi) + \mu(a+\xi-t))\mathrm{d}\xi], \\ \text{for} \quad a \leq t \end{cases} \tag{10}$$

$$I(a,t) = \begin{cases} I_0(a-t)\exp[-\int_0^t (\mu(a+\xi-t)+\gamma)\mathrm{d}\xi] \\ \quad + \int_0^t \lambda(s)S(a+s-t,s) \\ \quad \cdot \exp[-\int_s^t (\mu(a+\xi-t)+\gamma)\mathrm{d}\xi]\mathrm{d}s, \\ \text{for} \quad a > t \\ \int_{t-a}^t \lambda(s)S(a+s-t,s) \\ \quad \cdot \exp[-\int_s^t (\mu(a+\xi-t)+\gamma)\mathrm{d}\xi]\mathrm{d}s, \\ \text{for} \quad a \leq t \end{cases} \tag{11}$$

$$R(a,t) = \begin{cases} R_0(a-t)\exp[-\int_0^t \mu(a+\xi-t)\mathrm{d}\xi] \\ \quad + \int_0^t \gamma I(a+s-t,s)\exp[-\int_s^t \mu(a+\xi-t)\mathrm{d}\xi]\mathrm{d}s, \\ \text{for} \quad a > t \\ \int_{t-a}^t \gamma I(a+s-t,s)\exp[-\int_s^t \mu(a+\xi-t)\mathrm{d}\xi]\mathrm{d}s, \\ \text{for} \quad a \leq t \end{cases} \tag{12}$$

In view of Lemma 1, it is obvious that to get the existence and uniqueness of positive solution to problem (1)-(9), we need to study the existence and uniqueness of positive solution to the above integral equations (10)-(12). In what follows, we will discuss the existence and uniqueness of positive solution to (10)-(12) by using fixed point theory.

To do so, for $\forall q \in V$, based on the integral equations (10)-(12), we define three operators $K_I,\ K_R,\ K_S : V \to V$ as follows:

$$K_I(q)(a,t) = \begin{cases} I_0(a-t)\exp[-\int_0^t(\mu(a+\xi-t)+\gamma)\mathrm{d}\xi] \\ \quad +\int_0^t \lambda(s)q(a+s-t,s) \\ \quad \cdot\exp[-\int_s^t(\mu(a+\xi-t)+\gamma)\mathrm{d}\xi]\mathrm{d}s, \\ \text{for} \quad a>t \\ \int_{t-a}^t \lambda(s)q(a+s-t,s) \\ \quad \cdot\exp[-\int_s^t(\mu(a+\xi-t)+\gamma)\mathrm{d}\xi]\mathrm{d}s, \\ \text{for} \quad a\le t \end{cases} \tag{13}$$

$$K_R(q)(a,t) = \begin{cases} R_0(a-t)\exp[-\int_0^t \mu(a+\xi-t)\mathrm{d}\xi] \\ \quad +\int_0^t \gamma K_I(a+s-t,s) \\ \quad \cdot\exp[-\int_s^t \mu(a+\xi-t)\mathrm{d}\xi]\mathrm{d}s, \\ \text{for} \quad a>t \\ \int_{t-a}^t \gamma K_I(a+s-t,s)\exp[-\int_s^t \mu(a+\xi-t)\mathrm{d}\xi]\mathrm{d}s, \\ \text{for} \quad a\le t \end{cases} \tag{14}$$

$$K_S(q)(a,t) = \begin{cases} S_0(a-t)\exp[-\int_0^t(\lambda(\xi)+\mu(a+\xi-t))\mathrm{d}\xi], \\ \text{for} \quad a>t \\ \int_0^A [\beta(\eta)q(\eta,t-a)+\bar{\beta}(\eta)K_I(\eta,t-a) \\ \quad +\beta(\eta)K_R(\eta,t-a)]\mathrm{d}\eta \\ \quad \cdot\exp[-\int_{t-a}^t(\lambda(\xi)+\mu(a+\xi-t))\mathrm{d}\xi], \\ \text{for} \quad a\le t \end{cases} \tag{15}$$

Remark 2 *We can easily see from (13)-(15) that if the operators K_I, K_R are well defined, then in order to get the unique solution to (1)-(9), we only need to prove that K_S in (15) has a unique fixed point.*

3 The Well-posedness of System Solution

In this section, we will prove that the operator K_S in (15) has unique a fixed point, which leads to the existence and uniqueness of the local solution to problem (1)-(9). And the existence and uniqueness of the global solution to problem (1)-(9) is further obtained by a uniform priori estimate. Meanwhile, we give the continuous dependence of the solution on the initial value.

3.1 The unique existence of local solution

In this subsection, we study the existence and uniqueness of the local solution to (1)-(9). Some notations used in this section are presented as follows

$$L = \max\left\{2\int_0^A S_0(a)\mathrm{d}a, \int_0^A I_0(a)\mathrm{d}a, \int_0^A R_0(a)\mathrm{d}a\right\}$$
$$M = \left\{q \in V \mid q(a,0)=S_0(a), q\ge 0, \|q\|_V \le L\right\}$$

and clearly, M is a closed subspace of V.

The following Lemma is concerned with the operators $K_I(q), K_R(q)$ defined in (13)-(14).

Lemma 3 *Assume conditions* (H_1)-(H_2) *hold. For each given constant $T>0$, $\forall q \in V$, there exist unique $K_I(q), K_R(q) \in V$ satisfying (13)-(14), and if q is nonnegative, $K_I(q), K_R(q)$ are also nonnegative.*

Lemma 3 is obvious from the expressions of $K_I(q)$ and $K_R(q)$ in (13)-(14). Therefore, the operators K_I, K_R are well defined.

Further, we give another two lemmas which are useful to prove the main result of this subsection.

Lemma 4 *For each given constant $T>0$, assume the conditions* (H_1)-(H_2) *hold, then for $\forall q \in M$, and $0\le t\le T$, we can get*

$$\begin{aligned} \int_0^A K_I(q)(a,t)\mathrm{d}a &\le (1+\lambda_0 T)L \\ \int_0^A K_R(q)(a,t)\mathrm{d}a &\le (1+\gamma T+\lambda_0\gamma T^2)L \end{aligned}$$

Proof: From the expressions of $K_I(q)$ and $K_R(q)$ in (13)-(14), we have the following estimations

$$\begin{aligned} & \int_0^A K_I(q)(a,t)\mathrm{d}a \\ \le\ & \int_t^A I_0(a-t)\mathrm{d}a+\int_t^A\int_0^t \lambda(s)q(a+s-t,s)\mathrm{d}s\mathrm{d}a \\ & +\int_0^t\int_{t-a}^t \lambda(s)q(a+s-t,s)\mathrm{d}s\mathrm{d}a \\ \le\ & \int_0^A I_0(a)\mathrm{d}a+\int_t^A\int_0^t \lambda(s)q(a+s-t,s)\mathrm{d}s\mathrm{d}a \\ & +\int_0^t\int_0^t \lambda(s)q(a+s-t,s)\mathrm{d}s\mathrm{d}a \\ \le\ & \int_0^A I_0(a)\mathrm{d}a+\int_0^A\int_0^t \lambda(s)q(a+s-t,s)\mathrm{d}s\mathrm{d}a \\ \le\ & L+\lambda_0\int_0^t\int_0^A q(a,s)\mathrm{d}a\mathrm{d}s \le (1+\lambda_0 T)L \end{aligned}$$

and

$$\begin{aligned} & \int_0^A K_R(q)(a,t)\mathrm{d}a \\ \le\ & \int_t^A R_0(a-t)\mathrm{d}a+\int_t^A\int_0^t \gamma K_I(q)(a+s-t,s)\mathrm{d}s\mathrm{d}a \\ & +\int_0^t\int_{t-a}^t \gamma K_I(q)(a+s-t,s)\mathrm{d}s\mathrm{d}a \\ \le\ & \int_0^A R_0(a)\mathrm{d}a+\int_0^A\int_0^t \gamma K_I(q)(a+s-t,s)\mathrm{d}s\mathrm{d}a \\ \le\ & L+\int_0^t\int_0^A \gamma K_I(q)(a+s-t,s)\mathrm{d}a\mathrm{d}s \\ \le\ & L+\int_0^t\int_0^A \gamma K_I(q)(a,s)\mathrm{d}a\mathrm{d}s \\ \le\ & (1+\gamma T+\lambda_0\gamma T^2)L. \end{aligned}$$

The proof of Lemma 4 is completed. □

Lemma 5 *For each given positive constant $T > 0$, assume the conditions* (H_1)-(H_2) *hold, then $\forall q, \tilde{q} \in M$, and $0 \le t \le T$, we have*

$$\begin{aligned} & \int_0^A |K_I(q)(a,t)-K_I(\tilde{q})(a,t)|\mathrm{d}a \\ \le\ & \lambda_0 T\,\|q-\tilde{q}\|_V \\ & \int_0^A |K_R(q)(a,t)-K_R(\tilde{q})(a,t)|\mathrm{d}a \\ \le\ & \lambda_0\gamma T^2\,\|q-\tilde{q}\|_V \end{aligned}$$

Proof: From (13)-(14), the expressions of $K_I(q)$ and $K_R(q)$, we have

$$\begin{aligned} & \int_0^A |K_I(q)(a,t)-K_I(\tilde{q})(a,t)|\mathrm{d}a \\ \le\ & \int_t^A\int_0^t \lambda(s)\,|q(a+s-t,s) \\ & \qquad -\tilde{q}(a+s-t,s)|\mathrm{d}s\mathrm{d}a \\ & +\int_0^t\int_{t-a}^t \lambda(s)\,|q(a+s-t,s) \\ & \qquad -\tilde{q}(a+s-t,s)|\mathrm{d}s\mathrm{d}a \\ \le\ & \int_0^A\int_0^t \lambda(s)\,|q(a+s-t,s) \\ & \qquad -\tilde{q}(a+s-t,s)|\mathrm{d}s\mathrm{d}a \\ \le\ & \lambda_0\int_0^t\int_0^A |q(a,s)-\tilde{q}(a,s)|\mathrm{d}a\mathrm{d}s \\ \le\ & \lambda_0 T\,\|q-\tilde{q}\|_V \end{aligned}$$

and

$$\begin{aligned} & \int_0^A |K_R(q)(a,t)-K_R(\tilde{q})(a,t)|\mathrm{d}a \\ \le\ & \int_t^A\int_0^t \gamma\,|K_I(q)(a+s-t,s) \\ & \qquad -K_I(\tilde{q})(a+s-t,s)|\mathrm{d}s\mathrm{d}a \\ & +\int_0^t\int_{t-a}^t \gamma\,|K_I(q)(a+s-t,s) \\ & \qquad -K_I(\tilde{q})(a+s-t,s)|\mathrm{d}s\mathrm{d}a \\ \le\ & \int_0^A\int_0^t \gamma\,|K_I(q)(a+s-t,s) \\ & \qquad -K_I(\tilde{q})(a+s-t,s)|\mathrm{d}s\mathrm{d}a \\ \le\ & \gamma\int_0^t\int_0^A |K_I(q)(a,s)-K_I(\tilde{q})(a,s)|\mathrm{d}a\mathrm{d}s \\ \le\ & \lambda_0\gamma T^2\,\|q-\tilde{q}\|_V \end{aligned}$$

The proof of Lemma 5 is then completed. □

With the above preparations, we will prove the main result of this subsection.

Theorem 6 *Assume the conditions* (H_1)-(H_2) *hold, then there exists a positive constant $T_0 > 0$, which only depends on the L^1 norm of the initial value, such that for $t \in [0, T_0]$, the problem (1)-(9) has a unique local positive solution.*

Proof: Let T_0 satisfy $0 < T_0 < 1$ and

$$T_0\beta_0(3+\lambda_0+\gamma+\lambda_0\gamma) < \frac{1}{2}.$$

Firstly, we prove that operator K_S is a mapping from M to M. From the expression of K_S in (15), it is easy to know that $\forall q \in M, K_S(q) \in V$, and $K_S(q)(a,0) = S_0(a)$. Then by (15) and Lemma 4, we have the following estimation:

$$\begin{aligned}
&\int_0^A K_S(q)(a,t)\mathrm{d}a \\
\leq\ & \int_t^A S_0(a-t)\mathrm{d}a + \int_0^t\int_0^A [\beta(\eta)q(\eta,t-a)+\bar{\beta}(\eta) \\
&\cdot K_I(q)(\eta,t-a) + \beta(\eta)K_R(q)(\eta,t-a)]\mathrm{d}\eta\mathrm{d}a \\
\leq\ & \int_0^A S_0(a)\mathrm{d}a + \beta_0\int_0^t\int_0^A [q(\eta,t-a) \\
&\quad +K_I(q)(\eta,t-a) + K_R(q)(\eta,t-a)]\mathrm{d}\eta\mathrm{d}a \\
\leq\ & \frac{L}{2} + \beta_0\int_0^t\int_0^A [q(\eta,s) + K_I(q)(\eta,s) \\
&\quad +K_R(q)(\eta,s)]\mathrm{d}\eta\mathrm{d}s \\
\leq\ & \frac{L}{2} + \beta_0 T_0(3+\lambda_0+\gamma+\lambda_0\gamma)L \leq L
\end{aligned}$$

which implies that $K_S(q) \in M$.

Secondly, we prove that K_S is a strictly compressed mapping on M. For $\forall q, \tilde{q} \in M$, we can see from (15) and Lemma 5 that

$$\begin{aligned}
&\int_0^A |K_S(q)(a,t) - K_S(\tilde{q})(a,t)|\mathrm{d}a \\
\leq\ & \int_0^t\int_0^A [\beta(\eta)\,|q(\eta,t-a) - \tilde{q}(\eta,t-a)| \\
&+\bar{\beta}(\eta)\,|K_I(q)(\eta,t-a)-K_I(\tilde{q})(\eta,t-a)| \\
&+\beta(\eta)\,|K_R(q)(\eta,t-a)-K_R(\tilde{q})(\eta,t-a)|]\mathrm{d}\eta\mathrm{d}a \\
\leq\ & \beta_0\int_0^t\int_0^A [|q(\eta,t-a) - \tilde{q}(\eta,t-a)| \\
&+|K_I(q)(\eta,t-a) - K_I(\tilde{q})(\eta,t-a)| \\
&+|K_R(q)(\eta,t-a) - K_R(\tilde{q})(\eta,t-a)|]\mathrm{d}\eta\mathrm{d}a \\
\leq\ & \beta_0\int_0^t\int_0^A [|q(\eta,s) - \tilde{q}(\eta,s)| \\
&\quad +|K_I(q)(\eta,s) - K_I(\tilde{q})(\eta,s)| \\
&\quad +|K_R(q)(\eta,s) - K_R(\tilde{q})(\eta,s)|]\mathrm{d}\eta\mathrm{d}s \\
\leq\ & \beta_0 T_0(1+\lambda_0+\lambda_0\gamma)\,\|q-\tilde{q}\|_V \\
\leq\ & \frac{1}{2}\,\|q-\tilde{q}\|_V
\end{aligned}$$

it follows that

$$\|K_S(q) - K_S(\tilde{q})\|_V \leq \frac{1}{2}\,\|q-\tilde{q}\|_V$$

Hence, by the Banach contraction principle, K_S has unique a fixed point on M, which follows that the integral equations (10)-(12) has a unique positive solution. And by Lemma 1 and Lemma 3, the problem (1)-(9) has a unique local positive solution. Thus, the proof of Theorem 6 is completed. □

3.2 The unique existence of global solution

In this subsection, we will prove the existence and uniqueness of the global solution to problem (1)-(9). For simplicity, set

$$W = \left\{\phi \in L^1 \;\middle|\; \int_0^A |\phi(x)|\mathrm{d}x \leq L, \phi(x) \geq 0. a.e.\right\}$$

Lemma 7 *For each given positive constant $T > 0$, assume the conditions (H_1)-(H_2) hold, and let $(S,I,R) \in V \times V \times V$ is the positive solution to problem (1)-(9), its initial value $(S_0, I_0, R_0) \in W \times W \times W$, then for each given $t \in [0,T]$, we have*

$$\begin{aligned}
E(t) \;=\; & \int_0^A S(a,t)\mathrm{d}a + \int_0^A I(a,t)\mathrm{d}a \\
& + \int_0^A R(a,t)\mathrm{d}a \leq \frac{5}{2}Le^{QT}
\end{aligned}$$

where $Q = \max\{\beta_0+\lambda_0, \beta_0+\gamma\}$.

Proof: From the definitions of $S(a,t)$, $I(a,t)$ and $R(a,t)$ in (10)-(12), we have the following estimations

$$\begin{aligned}
&\int_0^A S(a,t)\mathrm{d}a \\
\leq\ & \int_t^A S_0(a-t)\mathrm{d}a + \int_0^t\int_0^A [\beta(\eta)S(\eta,t-a) \\
&+\bar{\beta}(\eta)I(\eta,t-a) + \beta(\eta)R(\eta,t-a)]\mathrm{d}\eta\mathrm{d}a \\
\leq\ & \int_0^A S_0(a)\mathrm{d}a + \beta_0\int_0^t\int_0^A [S(\eta,t-a) \\
&+I(\eta,t-a) + R(\eta,t-a)]\mathrm{d}\eta\mathrm{d}a \\
\leq\ & \frac{L}{2} + \beta_0\int_0^t\int_0^A [S(\eta,s)+I(\eta,s)+R(\eta,s)]\mathrm{d}\eta\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&\int_0^A I(a,t)\mathrm{d}a \\
\leq\ & \int_t^A I_0(a-t)\mathrm{d}a + \int_t^A\int_0^t \lambda(s)S(a+s-t,s)\mathrm{d}s\mathrm{d}a \\
&\quad + \int_0^t\int_{t-a}^t \lambda(s)S(a+s-t,s)\mathrm{d}s\mathrm{d}a \\
\leq\ & \int_0^A I_0(a)\mathrm{d}a + \int_0^A\int_0^t \lambda(s)S(a+s-t,s)\mathrm{d}s\mathrm{d}a \\
\leq\ & L + \lambda_0\int_0^t\int_0^A S(a,s)\mathrm{d}a\mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}&\int_0^A R(a,t)\mathrm{d}a\\ \leq\ &\int_t^A R_0(a-t)\mathrm{d}a+\int_t^A\int_0^t \gamma I(a+s-t,s)\mathrm{d}s\mathrm{d}a\\ &+\int_0^t\int_{t-a}^t \gamma I(a+s-t,s)\mathrm{d}s\mathrm{d}a\\ \leq\ &\int_0^A R_0(a)\mathrm{d}a+\int_0^A\int_0^t \gamma I(a+s-t,s)\mathrm{d}s\mathrm{d}a\\ \leq\ & L+\gamma\int_0^t\int_0^A I(a,s)\mathrm{d}a\mathrm{d}s\end{aligned}$$

For each $0\leq t\leq T$, let

$$E(t)=\int_0^A S(a,t)\mathrm{d}a+\int_0^A I(a,t)\mathrm{d}a+\int_0^A R(a,t)\mathrm{d}a$$

Then it follows that

$$\begin{aligned}E(t)\ \leq\ &\frac{5}{2}L+(\beta_0+\lambda_0)\int_0^t\int_0^A S(a,s)\mathrm{d}a\mathrm{d}s\\ &+(\beta_0+\gamma)\int_0^t\int_0^A I(a,s)\mathrm{d}a\mathrm{d}s\\ &+\beta_0\int_0^t\int_0^A R(a,s)\mathrm{d}a\mathrm{d}s\\ \leq\ &\frac{5}{2}L+Q\int_0^t E(s)\mathrm{d}s\end{aligned}$$

where $Q=\max\{\beta_0+\lambda_0,\beta_0+\gamma\}$. The Gronwall Inequality gives the estimation

$$E(t)\leq\frac{5}{2}Le^{QT}.$$

The proof of Lemma 7 is then completed. □

By Theorem 6 ($T_0>0$ is only depend on the L^1 norm of the initial value) and Lemma 7, the existence and uniqueness of the global solution to problem (1)-(9) is obvious with the Continuation Theorem. (This theory was introduced in [3, 4, 5]). Hence we have

Theorem 8 *Assume the conditions* (H_1)-(H_2) *hold. For each given positive constant* $T>0$*, the problem (1)-(9) has a unique positive solution.*

Here we present the continuous dependence of the solution to problem (1)-(9) on its initial value to end this section.

Theorem 9 *For* (S_0,I_0,R_0), $(\tilde{S}_0,\tilde{I}_0,\tilde{R}_0)\in W\times W\times W$*, let* (S,I,R) *and* $(\tilde{S},\tilde{I},\tilde{R})$ *are the solutions to problem (1)-(9) respectively on the initial values* (S_0,I_0,R_0) *and* $(\tilde{S}_0,\tilde{I}_0,\tilde{R}_0)$*, then for each* $0\leq t\leq T$*, we can get:*

$$\begin{aligned}G(t)\ =\ &\|S(t)-\tilde{S}(t)\|_{L^1}+\|I(t)-\tilde{I}(t)\|_{L^1}\\ &+\|R(t)-\tilde{R}(t)\|_{L^1}\\ \leq\ & e^{QT}[\|S_0-\tilde{S}_0\|_{L^1}\\ &+\|I_0-\tilde{I}_0\|_{L^1}+\|R_0-\tilde{R}_0\|_{L^1}]\end{aligned}$$

where $Q=\max\{\beta_0+\lambda_0,\beta_0+\gamma\}$.

Proof: From (10)-(12), the definitions of $S(a,t)$, $I(a,t)$ and $R(a,t)$, we have

$$\begin{aligned}&\int_0^A |S(a,t)-\tilde{S}(a,t)|\mathrm{d}a\\ \leq\ &\int_t^A |S_0(a-t)-\tilde{S}_0(a-t)|\mathrm{d}a\\ &+\int_0^t\int_0^A [\beta(\eta)|S(\eta,t-a)-\tilde{S}(\eta,t-a)|\\ &+\bar{\beta}(\eta)|I(\eta,t-a)-\tilde{I}(\eta,t-a)|\\ &+\beta(\eta)|R(\eta,t-a)-\tilde{R}(\eta,t-a)|]\mathrm{d}\eta\mathrm{d}a\\ \leq\ &\int_0^A |S_0(a)-\tilde{S}_0(a)|\mathrm{d}a\\ &+\beta_0\int_0^t\int_0^A [|S(\eta,t-a)-\tilde{S}(\eta,t-a)|\\ &+|I(\eta,t-a)-\tilde{I}(\eta,t-a)|\\ &+|R(\eta,t-a)-\tilde{R}(\eta,t-a)|]\mathrm{d}\eta\mathrm{d}a\\ \leq\ &\|S_0-\tilde{S}_0\|_{L^1}+\beta_0\int_0^t\int_0^A [|S(\eta,s)-\tilde{S}(\eta,s)|\\ &+|I(\eta,s)-\tilde{I}(\eta,s)|+|R(\eta,s)-\tilde{R}(\eta,s)|]\mathrm{d}\eta\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}&\int_0^A |I(a,t)-\tilde{I}(a,t)|\mathrm{d}a\\ \leq\ &\int_t^A |I_0(a-t)-\tilde{I}_0(a-t)|\mathrm{d}a\\ &+\int_t^A\int_0^t \lambda(s)|S(a+s-t,s)-\tilde{S}(a+s-t,s)|\mathrm{d}s\mathrm{d}a\\ &+\int_0^t\int_{t-a}^t \lambda(s)|S(a+s-t,s)-\tilde{S}(a+s-t,s)|\mathrm{d}s\mathrm{d}a\\ \leq\ &\int_0^A |I_0(a)-\tilde{I}_0(a)|\mathrm{d}a\\ &+\lambda(s)\int_0^A\int_0^t |S(a+s-t,s)-\tilde{S}(a+s-t,s)|\mathrm{d}s\mathrm{d}a\\ \leq\ &\|I_0-\tilde{I}_0\|_{L^1}+\lambda_0\int_0^t\int_0^A |S(a,s)-\tilde{S}(a,s)|\mathrm{d}a\mathrm{d}s\end{aligned}$$

also

$$\begin{aligned}
&\int_0^A |R(a,t)-\tilde{R}(a,t)|da \\
\le\ & \int_t^A |R_0(a-t)-\tilde{R}_0(a-t)|\mathrm{d}a \\
&+\int_t^A\int_0^t \gamma|I(a+s-t,s)-\tilde{I}(a+s-t,s)|\mathrm{d}s\mathrm{d}a \\
&+\int_0^t\int_{t-a}^t \gamma|I(a+s-t,s)-\tilde{I}(a+s-t,s)|\mathrm{d}s\mathrm{d}a \\
\le\ & \int_0^A |R_0(a)-\tilde{R}_0(a)|\mathrm{d}a \\
&+\int_0^A\int_0^t \gamma|I(a+s-t,s)-\tilde{I}(a+s-t,s)|\mathrm{d}s\mathrm{d}a \\
\le\ & \|R_0-\tilde{R}_0\|_{L^1}+\gamma\int_0^t\int_0^A |I(a,s)-\tilde{I}(a,s)|\mathrm{d}a\mathrm{d}s
\end{aligned}$$

For each $0 \le t \le T$, let

$$\begin{aligned}
G(t) = &\|S(t)-\tilde{S}(t)\|_{L^1} + \|I(t)-\tilde{I}(t)\|_{L^1} \\
&+ \|R(t)-\tilde{R}(t)\|_{L^1}
\end{aligned}$$

Then it follows that

$$\begin{aligned}
G(t) \le\ & \|S_0-\tilde{S}_0\|_{L^1} + \|I_0-\tilde{I}_0\|_{L^1} \\
&+\|R_0-\tilde{R}_0\|_{L^1} + Q\int_0^t G(s)\mathrm{d}s
\end{aligned}$$

where $Q = \max\{\beta_0+\lambda_0, \beta_0+\gamma\}$. The Gronwall Inequality gives the estimation

$$\begin{aligned}
G(t) \le e^{QT}[&\|S_0-\tilde{S}_0\|_{L^1} + \|I_0-\tilde{I}_0\|_{L^1} \\
&+\|R_0-\tilde{R}_0\|_{L^1}]
\end{aligned}$$

□

4 The Regularity of Solution

In this section, we will prove the regularity (i.e., C^1 continuity) of the solution to problem (1)-(9).

Theorem 10 *Assume the conditions* (H_1)-(H_4) *hold, and* $(S, I, R) \in V \times V \times V$ *is the positive solution to problem (1)-(9), then* $S, I, R \in C([0, A] \times [0, T])$.

Proof: From the condition (H_1), we observe

$$\lambda(t) + \mu(a) \in C([0, A] \times [0, T])$$

Taking account of (H_1) and S, I, $R \in V$, we have

$$\begin{aligned}
&\int_0^A [\beta(\eta)S(\eta, t-a) + \bar{\beta}(\eta)I(\eta, t-a) \\
&\quad + \beta(\eta)R(\eta, t-a)]\mathrm{d}\eta \in C[0, T]
\end{aligned}$$

Then, consider the definition of $S(a, t)$ in (10) and the compatibility condition (H_4), we easily know that

$$S(a, t) \in C([0, A) \times [0, T]) \tag{16}$$

Now, we consider the case that $a = A$

(i) When $t < A$, we can obtain from (H_2), (H_3), $S_0(A) = 0$ and (10) that

$$\begin{aligned}
&\lim_{a\to A} S(a,t) \\
&= S_0(A-t) \\
&\quad \cdot \lim_{a\to A} \exp[-\int_0^t (\lambda(\xi)+\mu(a+\xi-t))\mathrm{d}\xi] \\
&\le S_0(A-t)\lim_{a\to A}\exp[-\int_{a-t}^a \mu(\xi)\mathrm{d}\xi] = 0
\end{aligned} \tag{17}$$

(ii) When $t > A$, we observe that

$$\begin{aligned}
&\lim_{a\to A} S(a,t) \\
&= \int_0^A [\beta(\eta)S(\eta,t-A)+\bar{\beta}(\eta)I(\eta,t-A) \\
&\quad + \beta(\eta)R(\eta,t-A)]\mathrm{d}\eta \\
&\quad \cdot \lim_{a\to A}\exp[-\int_{t-a}^t (\lambda(\xi)+\mu(a+\xi-t))\mathrm{d}\xi] \\
&\le \int_0^A [\beta(\eta)S(\eta,t-A)+\bar{\beta}(\eta)I(\eta,t-A) \\
&\quad + \beta(\eta)R(\eta,t-A)]\mathrm{d}\eta \\
&\quad \cdot \lim_{a\to A}\exp[-\int_0^a \mu(\xi)\mathrm{d}\xi] \\
&= 0
\end{aligned} \tag{18}$$

Hence, we see from (16)-(18) that

$$S(a, t) \in C([0, A] \times [0, T]) \tag{19}$$

By the similar procedure, we can know from (H_1) and (19) that

$$\begin{aligned}
&\mu(a)+\gamma \in C[0,A], \\
&\int_{t-a}^t \lambda(s)S(a+s-t,s) \\
&\quad \cdot \exp[-\int_s^t (\mu(a+\xi-t)+\gamma)\mathrm{d}\xi]\mathrm{d}s \\
&\quad \in C([0,A]\times[0,T]).
\end{aligned}$$

Also, with the definition of $I(a, t)$ in (11) and (H_4), we can easily get that

$$I(a, t) \in C([0, A) \times [0, T]) \tag{20}$$

Also, we consider the case that $a = A$

(i) When $t < A$, we obtain from (H_3), $I_0(A) = 0$ and (11) that

$$\begin{aligned}
&\lim_{a\to A} I(a,t)\\
&= I_0(A-t)\\
&\quad \cdot \lim_{a\to A}\exp[-\int_0^t(\mu(a+\xi-t)+\gamma)\mathrm{d}\xi]\\
&\quad + \lim_{a\to A}\int_0^t \lambda(s)S(a+s-t,s)\\
&\quad\quad \cdot \exp[-\int_s^t(\mu(a+\xi-t)+\gamma)\mathrm{d}\xi]\mathrm{d}s \\
&\le I_0(A-t)\lim_{a\to A}\exp[-\int_{a-t}^a \mu(\xi)\mathrm{d}\xi]\\
&\quad + \lim_{a\to A}\int_{a-t}^a \lambda(s+t-a)S(s,s+t-a)\\
&\quad\quad \cdot \exp[-\int_s^a \mu(\xi)\mathrm{d}\xi]\mathrm{d}s\\
&= 0
\end{aligned} \tag{21}$$

(ii) When $t > A$, we have

$$\begin{aligned}
&\lim_{a\to A} I(a,t)\\
&= \lim_{a\to A}\int_{t-a}^t \lambda(s)S(a+s-t,s)\\
&\quad\quad \cdot \exp[-\int_s^t(\mu(a+\xi-t)+\gamma)\mathrm{d}\xi]\mathrm{d}s\\
&\le \lim_{a\to A}\int_0^a \lambda(s+t-a)S(s,s+t-a)\\
&\quad\quad \cdot \exp[-\int_s^a \mu(\xi)\mathrm{d}\xi]\mathrm{d}s\\
&= 0
\end{aligned} \tag{22}$$

Hence from (20)-(22), we have

$$I(a,t) \in C([0,A]\times[0,T]) \tag{23}$$

Last, for $R(a,t)$, we have from (H_1) and (23) that $\mu(a) \in C[0,A]$ and

$$\begin{aligned}
&\int_{t-a}^t \gamma I(a+s-t,s)\exp[-\int_s^t \mu(a+\xi-t)\mathrm{d}\xi]\mathrm{d}s\\
&\in C([0,A]\times[0,T]).
\end{aligned}$$

By the arguments above, the definition of $R(a,t)$ in (12) and (H_4), we can observe that

$$R(a,t) \in C([0,A)\times[0,T]) \tag{24}$$

Now, we consider the case that $a = A$:

(i) When $t < A$, we obtain from (H_3), $R_0(A) = 0$ and (12) that

$$\begin{aligned}
&\lim_{a\to A} R(a,t)\\
&= R_0(A-t)\lim_{a\to A}\exp[-\int_0^t \mu(a+\xi-t)\mathrm{d}\xi]\\
&\quad + \lim_{a\to A}\int_0^t \gamma I(a+s-t,s)\\
&\quad\quad \cdot \exp[-\int_s^t \mu(a+\xi-t)\mathrm{d}\xi]\mathrm{d}s\\
&= R_0(A-t)\lim_{a\to A}\exp[-\int_{a-t}^a \mu(\xi)\mathrm{d}\xi]\\
&\quad + \lim_{a\to A}\int_{a-t}^a \gamma I(s,s+t-a)\\
&\quad\quad \cdot \exp[-\int_s^a \mu(\xi)\mathrm{d}\xi]\mathrm{d}s\\
&= 0
\end{aligned} \tag{25}$$

(ii) When $t > A$, we observe that

$$\begin{aligned}
&\lim_{a\to A} R(a,t)\\
&= \lim_{a\to A}\int_{t-a}^t \gamma I(a+s-t,s)\\
&\quad\quad \cdot \exp[-\int_s^t \mu(a+\xi-t)\mathrm{d}\xi]\mathrm{d}s\\
&= \lim_{a\to A}\int_0^a \gamma I(s,s+t-a)\exp[-\int_s^a \mu(\xi)\mathrm{d}\xi]\mathrm{d}s\\
&= 0
\end{aligned} \tag{26}$$

Hence, we see from (24)-(26) that

$$R(a,t) \in C([0,A]\times[0,T]).$$

The proof of Theorem 10 is completed. □

Then, similar to the proof of Theorem 10, we can get the following theorem concerned with the regularity of the solution to problem (1)-(9). To do so, we need another condition:

(H_6): $S_0(a), I_0(a), R_0(a), \mu(a)$ are continuously differentiable functions.

Theorem 11 *Under the assumptions of Theorem 10 and* (H_5), (H_6), *then the solution* $S, I, R \in C^1([0,A)\times[0,T])$.

Proof: For simplicity, set

$$\begin{aligned}
\frac{\partial S(x,y)}{\partial x} &\triangleq S'_1(x,y), & \frac{\partial S(x,y)}{\partial y} &\triangleq S'_2(x,y)\\
\frac{\partial I(x,y)}{\partial x} &\triangleq I'_1(x,y), & \frac{\partial I(x,y)}{\partial y} &\triangleq I'_2(x,y)\\
\frac{\partial R(x,y)}{\partial x} &\triangleq R'_1(x,y), & \frac{\partial R(x,y)}{\partial y} &\triangleq R'_2(x,y)
\end{aligned}$$

Now, we will show the continuity of the partial derivative of S, I, R respectively.

(i) We can know from the expression of $S(a,t)$ in (10) that when $a > t$

$$\begin{aligned}\frac{\partial S}{\partial t} = & -\exp[-\int_0^t (\lambda(\xi)+\mu(a+\xi-t))\mathrm{d}\xi] \\ & \cdot\{S'_0(a-t)+S_0(a-t) \\ & \cdot[\int_0^t -\mu'(a+\xi-t)\mathrm{d}\xi+\lambda(t)+\mu(a)]\}\end{aligned} \quad (27)$$

and when $a \le t$

$$\begin{aligned}\frac{\partial S}{\partial t} = & \{\int_0^A [\beta(\eta)S'_2(\eta,t-a)+\bar{\beta}(\eta)I'_2(\eta,t-a) \\ & +\beta(\eta)R'_2(\eta,t-a)]\mathrm{d}\eta \\ & -\int_0^A [\beta(\eta)S(\eta,t-a)+\bar{\beta}(\eta)I(\eta,t-a) \\ & +\beta(\eta)R(\eta,t-a)]\mathrm{d}\eta \\ & \cdot[\int_{t-a}^t -\mu'(a+\xi-t)\mathrm{d}\xi \\ & +\lambda(t)+\mu(a)-\lambda(t-a)-\mu(0)]\} \\ & \cdot\exp[-\int_{t-a}^t (\lambda(\xi)+\mu(a+\xi-t))\mathrm{d}\xi].\end{aligned} \quad (28)$$

Then from (27)-(28), (H_6) and the first order compatibility condition (H_5), we have

$$\frac{\partial S}{\partial t} \in C([0,A)\times[0,T]) \quad (29)$$

Similarly, we can prove that

$$\frac{\partial S}{\partial a} \in C([0,A)\times[0,T]) \quad (30)$$

Hence, taking account of (29)-(30), we can observe that

$$S(a,t) \in C^1([0,A)\times[0,T]) \quad (31)$$

(ii) From the expression of $I(a,t)$ in (11), we can obtain that when $a > t$

$$\begin{aligned}\frac{\partial I}{\partial t} = & \lambda(t)S(a,t)-\exp[-\int_0^t (\mu(a+\xi-t)+\gamma)\mathrm{d}\xi] \\ & \cdot\{I'_0(a-t)+I_0(a-t) \\ & \cdot[\int_0^t -\mu'(a+\xi-t)\mathrm{d}\xi+\mu(a)+\gamma]\} \\ & -\int_0^t \{[\lambda(s)S'_1(a+s-t,s) \\ & +\lambda(s)S(a+s-t,s) \\ & \cdot(\int_s^t -\mu'(a+\xi-t)\mathrm{d}\xi+\mu(a)+\gamma)] \\ & \cdot\exp[-\int_s^t (\mu(a+\xi-t)+\gamma)\mathrm{d}\xi]\}\mathrm{d}s\end{aligned} \quad (32)$$

and when $a \le t$

$$\begin{aligned}\frac{\partial I}{\partial t} = & -\int_{t-a}^t \{\exp[-\int_s^t (\mu(a+\xi-t)+\gamma)\mathrm{d}\xi] \\ & \cdot[\lambda(s)S(a+s-t,s) \\ & \cdot(\int_s^t -\mu'(a+\xi-t)\mathrm{d}\xi+\mu(a)+\gamma) \\ & +\lambda(s)S'_1(a+s-t,s)]\}\mathrm{d}s \\ & +\lambda(t)S(a,t)-\lambda(t-a)S(0,t-a) \\ & \cdot\exp[-\int_{t-a}^t (\mu(a+\xi-t)+\gamma)\mathrm{d}\xi]\end{aligned} \quad (33)$$

Then from (31)-(33), (H_6) and (H_5), we can obtain that

$$\frac{\partial I}{\partial t} \in C([0,A)\times[0,T]) \quad (34)$$

Similarly, we can prove that

$$\frac{\partial I}{\partial a} \in C([0,A)\times[0,T]) \quad (35)$$

Hence, taking account of (34)-(35), we have

$$I(a,t) \in C^1([0,A)\times[0,T]) \quad (36)$$

(iii) By the expression of $R(a,t)$ in (12), we can get that when $a > t$

$$\begin{aligned}\frac{\partial R}{\partial t} = & \gamma I(a,t)-\exp[-\int_0^t \mu(a+\xi-t)\mathrm{d}\xi] \\ & \cdot\{R_0(a-t)[\int_0^t -\mu'(a+\xi-t)\mathrm{d}\xi \\ & +\mu(a)]+R'_0(a-t)\} \\ & -\int_0^t \{[\gamma I'_1(a+s-t,s)+\gamma I(a+s-t,s) \\ & \cdot(\int_s^t -\mu'(a+\xi-t)\mathrm{d}\xi+\mu(a))] \\ & \cdot\exp[-\int_s^t \mu(a+\xi-t)\mathrm{d}\xi]\}\mathrm{d}s\end{aligned} \quad (37)$$

and when $a \le t$

$$\begin{aligned}\frac{\partial R}{\partial t} = & -\int_{t-a}^t \{\exp[-\int_s^t \mu(a+\xi-t)\mathrm{d}\xi] \\ & \cdot[\gamma I'_1(a+s-t,s)+\gamma I(a+s-t,s) \\ & \cdot(\int_s^t -\mu'(a+\xi-t)\mathrm{d}\xi+\mu(a))]\}\mathrm{d}s \\ & +\gamma I(a,t)-\gamma I(0,t-a) \\ & \cdot\exp[-\int_{t-a}^t \mu(a+\xi-t)\mathrm{d}\xi]\end{aligned} \quad (38)$$

Also, from (36)-(38), (H_6) and (H_5), we have

$$\frac{\partial R}{\partial t} \in C([0,A)\times[0,T]) \quad (39)$$

Similarly, we can prove that

$$\frac{\partial R}{\partial a} \in C([0, A) \times [0, T]) \tag{40}$$

Finally, taking account of (39)-(40), we have

$$R(a, t) \in C^1([0, A) \times [0, T])$$

thus the proof of the Theorem 11 is completed. □

Remark 12 *It follows from Theorem 6, Theorem 8 and Theorem 11 that there exists unique solution* $S, I, R \in C^1([0, A) \times [0, T])$ *to problem (1)-(9).*

5 Conclusion

In this paper, we mainly consider the existence and uniqueness of the classical solution to a dynamic model of a class of SARS infectious diseases proposed in [1], because it is important both in theory and in practice. Firstly, we transform the model into integral equations with the characteristic method. Secondly, the existence and uniqueness of the local solution of the model is obtained by using the fixed point theory. Then the existence and uniqueness of the global solution of the model is further studied by a uniform priori estimate, and the continuous dependence of the model solution on its initial value is also presented. Lastly, the regularity or the C^1 continuity of the solution is studied at the end of the paper. The most important thing is that this paper provides a practical method for the study of the existence and uniqueness of solution to systems with more than four variables in partial differential equations.

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