Robust Stability of Uncertain Discrete-time State-delayed Systems with State Saturation Nonlinearities

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Abstract: This paper is concerned with the robust stability of a class of uncertain discrete-time systems with time-varying delays and state saturation nonlinearities. The saturation nonlinearities are assumed to be the standard saturation function, and the delay is allowed to be time-varying with known bounds. By applying the Lyapunov stability theorem and the delay-fractioning approach, a suitable Lyapunov-Krasovskii functional and a non-negative scalar are constructed respectively. A delay-dependent criterion of the robust stability is given for the addressed systems. Finally, a numerical example is given to illustrate the effectiveness of the presented criterion.

Key–Words: Robust stability, Time-varying delay, State saturation nonlinearities, Lyapunov-Krasovskii functional, Delay-fractioning approach

1 Introduction

Due to its important significance, the stability of dynamical systems has been widely studied over the past decades. On the other hand, in various engineering systems, time delays are introduced into the model of physical systems due to the inevitable reason such as measurement, transmission and transport lags, computational delays or un-modeled inertia of system components. The time delay is an important source of instability of the systems. So far, a large amount of results have been published for the stability and control of time-delay systems, see e.g. [1, 2]. Moreover, it is also well known that parameter uncertainties are inherent features of many physical systems. The uncertainties exist due to the variations in system parameters, modeling errors or some other ignored factors. Accordingly, the problems of analysis and synthesis have been addressed for uncertain systems [3–5].

Dynamical systems with state saturation nonlinearities exist commonly in neural networks, analogue circuits and control systems, and hence the stability analysis of such systems is highly nontrivial [6, 7]. Recently, many important results have been reported on this issue, see e.g. [8–14]. To be specific, the problems of the global asymptotic stability of the equilibrium have been investigated in [8, 9] for n-order discrete-time systems with state saturations and partial state saturation. An improved version of Ritzerfeld-Werters criterion for the nonexistence of overflow oscillations in second-order state-space digital filters has been presented. A new zero-input limit cycle-free realizability condition has been given in [10] for a generalized overflow characteristic and a complete stability analysis has been proposed in [11] for a planar discrete-time linear system with saturation. In [12], a criterion of the global asymptotic stability has been presented for discrete-time systems with partial state saturation nonlinearities. Subsequently, the extension of this approach has been performed in [13] to a situation involving partial state saturation nonlinearities.

Recently, the problem of global asymptotic stability has been studied in [15] for uncertain discrete-time state-delayed systems with saturation nonlinearities. By using the linear matrix inequality (LMI) technique, a new criterion has been presented to guarantee the global asymptotic stability for the related systems. It has been shown that the presented result has improved the results in [8, 13]. However, it is worth pointing out that the delays have been assumed to be time-invariant in most relevant literature concerning the robust stability problem for the uncertain time-delay systems with state saturation nonlinearity. To the best of the authors knowledge, there has been little work undertaken on the robust stability of uncertain discrete-time systems with time-varying delay and state saturation nonlinearity. Note that the delay-fractioning approach has been proved to be an effective way in [19–21] for addressing the time-delay. Hence, we will employ this approach to give a new stability criterion.

Motivated by the above discussions, in this paper, we aim to investigate the problem of robust stability for a class of uncertain discrete-time state-delayed systems with state saturation nonlinearity. Here, the delay is time-varying with known bounds and the parametric uncertainties are norm-bounded. By using the delay-fractioning approach and constructing an appropriate Lyapunov-Krasovskii functional, the delay-dependent robust stability condition is presented. It is shown that the proposed condition is in term of the solutions of the linear matrix inequalities (LMIs) which can be easily solved by using the standard numerical software. Finally, two illustrative examples are given to demonstrate the effectiveness of the proposed results. The main contribution of this paper lies in that new robust stability criterion is given for uncertain discrete-time state saturation systems with time-varying delays.

The rest of this paper is organized as follows. Section 2 briefly introduces the problem under consideration and gives some useful Lemmas. The criterion of robustly global asymptotic stability is given in Section 3. In Section 4, two numerical examples are presented to illustrate the feasibility and effectiveness of the developed results. This paper is concluded in Section 5.

2 Problem Formulation and Preliminaries

In this paper, we consider the following uncertain discrete-time systems with time-varying delay and state saturated nonlinearities:

$$\begin{aligned} x(k+1) &= f(y(k)) \\ &= [f_1(y_1(k)), \cdots, f_n(y_n(k))]^T \quad (1) \\ y(k) &= [y_1(k), \cdots, y_n(k)]^T \\ &= (A+\Delta A)x(k) \\ &\quad +(A_d+\Delta A_d)x(k-d(k)) \quad (2) \\ x(k) &= \phi(k), k \in [-d_M, 0] \quad (3) \end{aligned}$$

where, $x(k) \in \mathbb{R}^n$ is the state vector; $A, A_d \in \mathbb{R}^{n\times n}$ are known matrices; $\Delta A, \Delta A_d \in \mathbb{R}^{n\times n}$ are the unknown matrices representing parametric uncertainties in the state matrices; $d(k)$ is the positive integer for time delays; $\phi(k) \in \mathbb{R}^n$ is the initial state value at time k; $f(y(k))$ is the vector saturation function.

The saturation functions $f_i(y_i(k))$ are defined as

$$f_i(y_i(k)) = \begin{cases} 1, & y_i(k) > 1 \\ y_i(k), & |y_i(k)| \le 1, i = 1, \cdots, n \\ -1, & y_i(k) < -1 \end{cases} \quad (4)$$

The uncertain matrix satisfies the following condition:

$$\Delta A = HFE, \Delta A_d = HFE_d \quad (5)$$

where H, E and E_d are known constant matrices with appropriate dimensions and F is an unknown matrix satisfying

$$F^T F \le I \quad (6)$$

The time-varying delay satisfies

$$d_m \le d(k) \le d_M \quad (7)$$

where d_m and d_M are known positive integers representing the upper and lower bounds of delay $d(k)$. The lower bound of delay d_m can always be described by $d_m = \tau m$, where τ and m are positive integers.

To proceed, we introduce the following definition and lemmas that will be used in the proofs of the main results.

Definition 1 *[15] The zero solution of the system described by (1)-(7) is globally asymptotically stable if the following holds:*

(i) it is stable in the sense of Lyapunov, i.e., for every $\varepsilon > 0$ there exists a $\delta > 0$ so that $\|x(k)\| < \varepsilon$ for all $k = 1, 2, \cdots$, whenever

$$\|\phi(\cdot)\| = \max_{k\in[-d_M,0]} \|\phi(k)\| < \delta;$$

(ii) it is attractive, i.e., $\lim_{k\to\infty} x(k) = 0$.

Lemma 2 *[16] Let D, E, F and M be real matrices of appropriate dimensions with M satisfying $M = M^T$, then*

$$M + DFE + E^T F^T D^T < 0 \quad (8)$$

for all $F^T F \le I$, if and only if there exists a scalar $\varepsilon > 0$ such that

$$M + \varepsilon^{-1} DD^T + \varepsilon E^T E < 0. \quad (9)$$

Lemma 3 *[16] Let D, E, F and M be real matrices of appropriate dimensions, and P is a symmetric positive definite matrix, then*

$$\begin{aligned} &(M+DFE)^T P(M+DFE) \\ \le\ & M^T(P^{-1}+\varepsilon^{-1}DD^T)^{-1}M+\varepsilon E^T E, \quad (10) \end{aligned}$$

for any $\varepsilon > 0$.

Lemma 4 *[17] Given constant matrices A, B and C of appropriate dimension, with A and C being symmetrical, then*

$$A - BC^{-1}B^T > 0 \; and \; C > 0 \tag{11}$$

if and only if

$$\left[\begin{array}{cc} A & B \\ B^T & C \end{array} \right] > 0 \tag{12}$$

or equivalently

$$\left[\begin{array}{cc} C & B^T \\ B & A \end{array} \right] > 0. \tag{13}$$

Lemma 5 *[15] Suppose matrix $C = (c_{ij}) \in \mathbb{R}^{n\times n}$ is characterized by*

$$\begin{array}{c} c_{ii} = \sum\limits_{j=1,j\neq i}^{n} (\alpha_{ij} + \beta_{ij}), i = 1, 2, \cdots, n \\ c_{ij} = \alpha_{ij} - \beta_{ij}, i, j = 1, 2, \cdots, n(i \neq j) \\ \alpha_{ij} > 0, \beta_{ij} > 0, i, j = 1, 2, \cdots, n \end{array} \tag{14}$$

then the scalar δ is a nonnegative number with

$$\begin{array}{rcl} \delta & = & \sum\limits_{i=1}^{n} 2[y_i(k) - f_i(y_i(k))] \\ & & [\sum\limits_{j=1,j\neq i}^{n} (\alpha_{ij} + \beta_{ij}) f_i(y_i(k)) \\ & & +(\alpha_{ij} - \beta_{ij}) f_j(y_j(k))] \\ & = & y(k)^T C f(y(k)) + f(y(k))^T C^T y(k) \\ & & -f(y(k))^T (C + C^T) f(y(k)). \end{array} \tag{15}$$

3 Main Result

In this section, by using the delay-fractioning approach and the LMI technique, the criterion of robustly global asymptotic stability is given for uncertain discrete-time systems with time-varying delay and state saturation nonlinearities.

Theorem 6 *Consider system (1)-(7). If there exist symmetry positive definite matrices $P > 0$, $Q > 0$, $R > 0$, $S > 0$, the symmetry positive semi-definite matrix $M \geq 0$, $N \geq 0$, real matrices X, Y, Z with appropriate dimensions and matrix C satisfying (14), and positive scalar $\varepsilon > 0$ such that*

$$\left[\begin{array}{cc} \Pi_1 + \Pi_2 + \Pi_2^T + \Pi_3 & \sqrt{2h_2 + 1} C^T H \\ +\Pi_3^T + \Pi_4 + \Pi_5 & \\ * & -\varepsilon I \end{array} \right] < 0 \tag{16}$$

$$\left[\begin{array}{cc} M & X \\ X^T & P \end{array} \right] \geq 0,$$

$$\left[\begin{array}{cc} N & Y \\ Y^T & P \end{array} \right] \geq 0, \tag{17}$$

$$\left[\begin{array}{cc} N & Z \\ Z^T & P \end{array} \right] \geq 0$$

where

$$\begin{array}{rcl} h_1 & = & d_M - d_m + 1, h_2 = d_M - d_m + \tau, \\ \Pi_1 & = & \tau M + (d_M - d_m) N, \\ \Pi_2 & = & \left[\begin{array}{ccc} X & Y & Z \end{array} \right] \times \left[\begin{array}{c} I_{n\times n} \\ O_{n\times mn} \\ O_{n\times(m+1)n} \end{array} \right. \\ & & \left. \begin{array}{cc} -I_{n\times n} & O_{n\times(m+2)n} \\ I_{n\times n}, & -I_{n\times n} \quad O_{n\times 2n} \\ I_{n\times n}, & -I_{n\times n} \quad O_{n\times n} \end{array} \right], \\ \Pi_3 & = & 2(h_2 + 1)(A\theta_1 + A_d\theta_2)^T C \theta_4, \\ \Pi_4 & = & W_R^T \overline{R} W_R, \\ \Pi_5 & = & \theta_1^T [(2h_2 - 1)P + h_1 Q + S]\theta_1 - \theta_2^T Q \theta_2 \\ & & -\theta_3^T Q \theta_3 + (2h_2 + 1)\theta_4^T (P - C - C^T)\theta_4 \\ & & +\varepsilon(2h_2+1)(E\theta_1 + E_d\theta_2)^T (E\theta_1 + E_d\theta_2), \\ W_R & = & \left[\begin{array}{ccc} I_{mn\times mn} & O_{mn\times n} & O_{mn\times 3n} \\ O_{mn\times n} & I_{mn\times mn} & O_{mn\times 3n} \end{array} \right], \\ \overline{R} & = & \left[\begin{array}{cc} R & \\ & -R \end{array} \right], \\ \theta_1 & = & \left[\begin{array}{cc} I_{n\times n} & O_{n\times(m+3)n} \end{array} \right], \\ \theta_2 & = & \left[\begin{array}{ccc} O_{n\times(m+1)n} & I_{n\times n} & O_{n\times 2n} \end{array} \right], \\ \theta_3 & = & \left[\begin{array}{ccc} O_{n\times(m+2)n} & I_{n\times n} & O_{n\times n} \end{array} \right], \\ \theta_4 & = & \left[\begin{array}{cc} O_{n\times(m+3)n} & I_{n\times n} \end{array} \right], \end{array}$$

then the zero solution of system (1)-(7) is globally asymptotically stable.

Proof: By using the delay fractioning approach in [19], construct the following Lyapunov-Krasovskii functional

$$V(k) = \sum_{i=1}^{4} V_i(k) \tag{18}$$

where

$$\begin{array}{rcl} V_1(k) & = & x(k)^T P x(k) \\ V_2(k) & = & \sum\limits_{j=k-d(k)}^{k-1} x^T(j) Q x(j) \end{array}$$

$$+\sum_{j=-d_M+1}^{-d_m}\sum_{l=j+k}^{k-1}x^T(l)Qx(l)$$

$$V_3(k) = \sum_{j=k-\tau}^{k-1}\zeta^T(j)R\zeta(j)+\sum_{j=k-d_M}^{k-1}x^T(j)Sx(j)$$

$$V_4(k) = \sum_{j=-\tau}^{-1}\sum_{l=k+j}^{k-1}\eta^T(l)P\eta(l)$$
$$+\sum_{j=-d_M}^{-d_m-1}\sum_{l=k+j}^{k-1}\eta^T(l)P\eta(l)$$

$$\eta(l) = x(k+1)-x(k)$$
$$\zeta^T(j) = [x^T(j),x^T(j-\tau),\cdots,x^T(j-(m-1)\tau)]$$

with $P>0$,$Q>0$,$R>0$, $S>0$ being matrices to be determined.

Denote

$$x_k = x(k),\ \ y_k = y(k),\ \ \eta_l = \eta(l),$$
$$\zeta_j = \zeta(j),\quad d_k = d(k).$$

Calculate the difference of $V(k)$ along systems (1)-(7)

$$\Delta V(k) = V(k+1)-V(k) = \sum_{i=1}^{4}\Delta V_i(k) \quad (19)$$

where

$$\begin{aligned}\Delta V_1(k) &= f(y_k)^TPf(y_k)-x_k^TPx_k\\ &\le f(y_k)^T(P-C-C^T)f(y_k)\\ &\quad +2y_k^TCf(y_k)-x_k^TPx_k \end{aligned}\quad (20)$$

$$\begin{aligned}\Delta V_2(k) &= \sum_{j=k+1-d_{k+1}}^{k}x_j^TQx_j\\ &+\sum_{j=-d_M+1}^{-d_m}\sum_{l=k+1+j}^{k}x_l^TQx_l\\ &-\sum_{j=k-d_k}^{k-1}x_j^TQx_j-\sum_{j=-d_M+1}^{-d_m}\sum_{l=k+j}^{k-1}x_l^TQx_l\\ &\le h_lx_k^TQx_k-x_{k-d_k}^TQx_{k-d_k}\end{aligned}\quad (21)$$

$$\begin{aligned}\Delta V_3(k) &= \sum_{l=k+1-\tau}^{k}\zeta_l^TR\zeta_l+\sum_{l=k+1-d_M}^{k}x_l^TSx_l\\ &\quad -\sum_{l=k-\tau}^{k-1}\zeta_l^TR\zeta_l-\sum_{l=k-d_M}^{k-1}x_l^TSx_l\\ &= \zeta_k^TR\zeta_k-\zeta_{k-\tau}^TR\zeta_{k-\tau}+x_k^TSx_k\\ &\quad -x_{k-d_M}^TSx_{k-d_M}\end{aligned}\quad (22)$$

$$\begin{aligned}\Delta V_4(k) &= \sum_{j=-\tau}^{-1}\sum_{l=k+1+j}^{k}\eta_l^TP\eta_l\\ &+\sum_{j=-d_M}^{-d_m-1}\sum_{l=k+1+j}^{k}\eta_l^TP\eta_l\\ &-\sum_{j=-\tau}^{-1}\sum_{l=k+j}^{k-1}\eta_l^TP\eta_l-\sum_{j=-d_M}^{-d_m-1}\sum_{l=k+j}^{k-1}\eta_l^TP\eta_l\\ &= h_2\eta_k^TP\eta_k-\sum_{j=k-\tau}^{k-1}\eta_j^TP\eta_j\\ &\quad -\sum_{j=k-d_M}^{k-d_m-1}\eta_j^TP\eta_j\end{aligned}\quad (23)$$

Note that

$$\begin{aligned}&\eta_k^TP\eta_k=(x_{k+1}-x_k)^TP(x_{k+1}-x_k)\\ \le\ & 4y_k^TCf(y_k)-2f(y_k)^T\\ &\times(C+C^T-P)f(y_k)+2x_k^TPx_k\\ =\ & 4[(Ax_k+A_dx_{k-d_k})+HF\\ &\times(Ex_k+E_dx_{k-d_k})]^TCf(y_k)\\ &-2f(y_k)^T(C+C^T-P)f(y_k)\\ &+2x_k^TPx_k\\ \le\ & 4(Ax_k+A_dx_{k-d_k})^TCf(y_k)\\ &+2\varepsilon^{-1}f(y_k)^TC^THH^TCf(y_k)\\ &+2\varepsilon(Ex_k+E_dx_{k-d_k})^T\\ &\times(Ex_k+E_dx_{k-d_k})-2f(y_k)^T\\ &\times(C+C^T-P)f(y_k)\\ &+2x_k^TPx_k\end{aligned}\quad (24)$$

Hence, from (20)-(24), we have

$$\begin{aligned}&\Delta V(k)\le x_k^T[(2h_2-1)P+h_1Q+S]x_k\\ &-x_{k-d_k}^TQx_{k-d_k}-x_{k-d_M}^TSx_{k-d_M}\\ &+\zeta_k^TR\zeta_k-\zeta_{k-\tau}^TR\zeta_{k-\tau}\\ &+(2h_2+1)[-f(y_k)^T(C+C^T-P)f(y_k)\\ &+\varepsilon^{-1}f(y_k)^TC^THH^TCf(y_k)\\ &+\varepsilon(Ex_k+E_dx_{k-d_k})^T(Ex_k+E_dx_{k-d_k})\\ &+(Ax_k+A_dx_{k-d_k})^TCf(y_k)\\ &+f(y_k)^TC^T(Ax_k+A_dx_{k-d_k})]\\ &-\sum_{j=k-\tau}^{k-1}\eta_j^TP\eta_j-\sum_{j=k-d_k}^{k-d_m-1}\eta_j^TP\eta_j\\ &-\sum_{j=k-d_M}^{k-d_k-1}\eta_j^TP\eta_j\end{aligned}\quad (25)$$

Let

$$\xi_k^T = [\zeta_k^T, x_{k-d_m}^T, x_{k-d_k}^T, x_{k-d_M}^T, f(y_k)^T] \in \mathbb{R}^{(m+4)n}.$$

It is easy to see that

$$\begin{aligned} x_k &= \theta_1\xi_k, x_{k-d_k} = \theta_2\xi_k, \\ x_{k-d_M} &= \theta_3\xi_k, f(y_k) = \theta_4\xi_k. \end{aligned}$$

Thus

$$\begin{aligned} &\Delta V(k) \le \xi_k^T\{\theta_1^T[(2h_2-1)P + h_1Q+S]\theta_1 \\ &-\theta_2^T Q\theta_2 - \theta_3^T S\theta_3 + W_R^T\overline{R}W_R \\ &+2(h_2+1)[\theta_4^T(P - C - C^T \\ &+\varepsilon^{-1}C^THH^TC)\theta_4 \\ &+\varepsilon(E\theta_1 + E_d\theta_2)^T(E\theta_1 + E_d\theta_2) \\ &+(A\theta_1 + A_d\theta_2)^TC\theta_4 \\ &+\theta_4^TC^T(A\theta_1 + A_d\theta_2)]\}\xi_k \\ &-\sum_{j=k-\tau}^{k-1}\eta_j^TP\eta_j - \sum_{j=k-d_k}^{k-d_m-1}\eta_j^TP\eta_j \\ &-\sum_{j=k-d_M}^{k-d_k-1}\eta_j^TP\eta_j \end{aligned} \tag{26}$$

For any matrices X, Y, Z with appropriate dimensions, we have

$$\begin{aligned} &2\xi_k^TX(x_k - x_{k-\tau} - \sum_{j=k-\tau}^{k-1}\eta_j) = 0, \\ &2\xi_k^TY(x_{k-d_m} - x_{k-d_k} - \sum_{j=k-d_k}^{k-d_m-1}\eta_j) = 0, \\ &2\xi_k^TZ(x_{k-d_k} - x_{k-d_M} - \sum_{j=k-d_M}^{k-d_k-1}\eta_j) = 0. \end{aligned} \tag{27}$$

On the other hand, for any symmetry positive semi-definite matrix $M \ge 0, N \ge 0$ with appropriate dimensions, the following equations always hold

$$\begin{aligned} &\tau\xi_k^TM\xi_k - \sum_{j=k-\tau}^{k-1}\xi_k^TM\xi_k = 0, \\ &(d_M-d_m)\xi_k^TN\xi_k - \sum_{j=k-d_M}^{k-d_m-1}\xi_k^TN\xi_k = 0. \end{aligned} \tag{28}$$

It follows from (27)-(28) that

$$\begin{aligned} &\Delta V(k) \le \xi_k^T[\Pi_1+\Pi_2+\Pi_2^T+\Pi_3+\Pi_3^T+\Pi_4 \\ &+\Pi_5 + \varepsilon^{-1}(2h_2+1)\theta_4^TC^THH^TC\theta_4]\xi_k \\ &-\sum_{j=k-\tau}^{k-1}\xi(k,j)^T\begin{bmatrix} M & X \\ X^T & P \end{bmatrix}\xi(k,j) \\ &-\sum_{j=k-d_M}^{k-d_k-1}\xi(k,j)^T\begin{bmatrix} N & Z \\ Z^T & P \end{bmatrix}\xi(k,j) \\ &-\sum_{j=k-d_k}^{k-d_m-1}\xi(k,j)^T\begin{bmatrix} N & Y \\ Y^T & P \end{bmatrix}\xi(k,j) \end{aligned}$$

Noting (17), we know that $\Delta V(k)$ is negative definite if the following inequality

$$\begin{aligned} \Lambda &= \Pi_1 + \Pi_2 + \Pi_2^T + \Pi_3 + \Pi_3^T + \Pi_4 \\ &\quad +\Pi_5 + \varepsilon^{-1}(2h_2+1)\theta_4^TC^THH^TC\theta_4] < 0 \end{aligned}$$

holds. By using Lemmas 1 and 3, $\Lambda < 0$ is equivalent to (16). This completes the proof of this theorem. □

Remark 7 *Up to know, the problem of the robustly global asymptotic stability has been studied for a class of uncertain discrete-time systems with time-varying delay and state saturation nonlinearities. By using the delay-fractioning approach, an appropriate Lyapunov-Krasovskii functional has been introduced to deal with the time-varying delay. By employing the LMI technique, a new stability criterion has been given to guarantee the robustly global asymptotic stability of the addressed system.*

As a special case, if there is no saturation nonlinearities in system (1), we have the following system:

$$\begin{aligned} x_{k+1} &= (A+\Delta A)x_k + (A_d+\Delta A_d)x_{k-d_k} \\ &= y_k. \end{aligned} \tag{29}$$

In the following corollary, a sufficient condition is proposed to ensure the robustly global asymptotic stability of system (29).

Corollary 8 *Consider the system (29). If there exist the symmetry positive definite matrices $P > 0, Q > 0, R > 0, S > 0$, the symmetry positive semi-definite matrix $M \ge 0, N \ge 0$, real matrices X, Y, Z, and positive scalar $\varepsilon > 0$ such that*

$$\begin{bmatrix} W_{11} & W_{12}^T & W_{13}^T \\ * & -P - \varepsilon HH^T & 0 \\ * & * & -\varepsilon I \end{bmatrix} < 0 \tag{30}$$

$$\begin{aligned}\left[\begin{array}{cc} M & X \\ X^T & P \end{array}\right] &\geq 0,\\ \left[\begin{array}{cc} N & Y \\ Y^T & P \end{array}\right] &\geq 0, \qquad (31)\\ \left[\begin{array}{cc} N & Z \\ Z^T & P \end{array}\right] &\geq 0.\end{aligned}$$

where

$$\begin{aligned} W_{11} &= \Pi_1+\Pi_2^{'}+\Pi_2^{'T}+\Pi_4^{'}+\Pi_5^{'}\\ W_{12} &= \sqrt{2h_2+1}(A\theta_1^{'}+A_d\theta_2^{'})\\ W_{13} &= \sqrt{2h_2+1}(E\theta_1^{'}+E_d\theta_2^{'}) \end{aligned}$$

and $h_1, h_2, \Pi_1, \overline{R}$ *are same as in the Theorem 6, and*

$$\begin{aligned} \Pi_2^{'} &= \left[\begin{array}{ccc} X & Y & Z \end{array}\right]\times\left[\begin{array}{cc} I_{n\times n} & \\ O_{n\times mn} & \\ O_{n\times(m+1)n} & \end{array}\right.\\ &\quad \left.\begin{array}{cc} -I_{n\times n} & O_{n\times(m+1)n}\\ I_{n\times n}, -I_{n\times n} & O_{n\times n}\\ I_{n\times n} & -I_{n\times n}\end{array}\right],\\ \Pi_4^{'} &= W_R^{'T}\overline{R}W_R^{'}\\ \Pi_5^{'} &= \theta_1^{'T}[(2h_2-1)P+h_1Q+S]\theta_1^{'}-\theta_2^{'T}Q\theta_2^{'}\\ &\quad -\theta_3^{'T}Q\theta_3^{'}+\varepsilon(2h_2+1)(E\theta_1^{'}+E_d\theta_2^{'})^T\\ &\quad \times(E\theta_1^{'}+E_d\theta_2^{'})\\ W_R^{'} &= \left[\begin{array}{ccc} I_{mn\times mn} & O_{mn\times n} & O_{mn\times 2n}\\ O_{mn\times n} & I_{mn\times mn} & O_{mn\times 2n}\end{array}\right]\\ \theta_1^{'} &= \left[\begin{array}{cc} I_{n\times n} & O_{n\times(m+2)n}\end{array}\right],\\ \theta_2^{'} &= \left[\begin{array}{ccc} O_{n\times(m+1)n} & I_{n\times n} & O_{n\times n}\end{array}\right],\\ \theta_3^{'} &= \left[\begin{array}{cc} O_{n\times(m+2)n} & I_{n\times n}\end{array}\right] \end{aligned}$$

then the zero solution of system(29) is globally asymptotically stable.

Proof: Constructing the Lyapunov-Krasovskii functional as in (18), we have

$$\Delta V_1(k) = y_k^TPy_k - x_k^TPx_k$$

and $\Delta V_2(k), \Delta V_3(k), \Delta V_4(k)$ are same as in the Theorem 6. Note that

$$\begin{aligned} \eta_k^TP\eta_k &= (y_k-x_k)^TP(y_k-x_k)\\ &\leq 2y_k^TPy_k+2x_k^TPx_k\\ y_k^TPy_k &= [(A+\Delta A)x_k+(A_d+\Delta A_d)x_{k-d_k}]^TP\\ &\quad \times[(A+\Delta A)x_k+(A_d+\Delta A_d)x_{k-d_k}] \end{aligned}$$

then

$$\begin{aligned} &\Delta V(k) \leq x_k^T[(2h_2-1)P+h_1Q+S\\ &+(2h_2+1)(A+\Delta A)^TP(A+\Delta A)]x_k\\ &+x_{k-d_k}^T[-Q+(2h_2+1)(A_d+\Delta A_d)^T\\ &\times P(A_d+\Delta A_d)]x_{k-d_k}+2(2h_2+1)x_k^T\\ &\times(A+\Delta A)^TP(A_d+\Delta A_d)x_{k-d_k}\\ &-x_{k-d_M}^TSx_{k-d_M}+\zeta_k^TR\zeta_k\\ &-\zeta_{k-\tau}^TR\zeta_{k-\tau}-\sum_{j=k-\tau}^{k-1}\eta_j^TP\eta_j\\ &-\sum_{j=k-d_k}^{k-d_m-1}\eta_j^TP\eta_j-\sum_{j=k-d_M}^{k-d_k-1}\eta_j^TP\eta_j \end{aligned}$$

Denote

$$\xi_k^{'T}=[\zeta_k^T, x_{k-d_m}^T, x_{k-d_k}^T, x_{k-d_M}^T]\in\mathbb{R}^{(m+3)n},$$

then it yields

$$x_k=\theta_1^{'}\xi_k^{'}, x_{k-d_k}=\theta_2^{'}\xi_k^{'}, x_{k-d_M}=\theta_3^{'}\xi_k^{'}$$

Therefore,

$$\begin{aligned} &\Delta V(k) \leq \xi_k^{'T}\{\theta_1^{'T}[(2h_2-1)P+h_1Q+S\\ &+(2h_2+1)(A+\Delta A)^TP(A+\Delta A)]\theta_1^{'}\\ &+\theta_2^{'T}[-Q+(2h_2+1)(A_d+\Delta A_d)^TP\\ &\times(A_d+\Delta A_d)]\theta_2^{'}+(2h_2+1)\theta_1^{'T}(A\\ &+\Delta A)^TP(A_d+\Delta A_d)\theta_2^{'}+(2h_2+1)\\ &\times\theta_2^{'T}(A_d+\Delta A_d)^TP(A+\Delta A)\theta_1^{'}\\ &-\theta_3^{'T}S\theta_3^{'}+W_R^{'T}\overline{R}W_R^{'}\}\xi_k^{'}\\ &-\sum_{j=k-\tau}^{k-1}\eta_j^TP\eta_j-\sum_{j=k-d_k}^{k-d_m-1}\eta_j^TP\eta_j\\ &-\sum_{j=k-d_M}^{k-d_k-1}\eta_j^TP\eta_j \end{aligned}$$

By using Lemmas 3 and the method as in the Theorem 6, the proof of this corollary is complete. □

Remark 9 *The sufficient condition has been proposed in Corollary 8 to guarantee the global asymptotic stability of the addressed system without state saturation. Compared with [3–5], it can be easily*

seen that a new Lyapunov functional has been constructed based on the delay-fractioning approach and more information of the time-varying delay has been considered. Hence, the stability condition presented in this paper is less conservative than the results in [3–5].

If time delay is time-invariant in system (1), that is, $d(k) = d$, then we can have the following corollary.

Corollary 10 *Consider the time-invariant delay $d(k) = d$. If there exist symmetric positive definite matrices $P > 0, Q > 0$ and $R > 0$, a symmetry positive semi-definite matrix $M \geq 0$, real matrix X with appropriate dimensions, matrix C satisfying (14), and positive scalar $\varepsilon > 0$ such that*

$$\begin{bmatrix} \Pi_1^\circ + \Pi_2^\circ + \Pi_2^{\circ T} + \Pi_3^\circ \\ +\Pi_3^{\circ T} + \Pi_4^\circ + \Pi_5^\circ & \sqrt{2\tau+1}C^T H \\ * & -\varepsilon I \end{bmatrix} < 0 \tag{32}$$

$$\begin{bmatrix} M & X \\ X^T & P \end{bmatrix} \geq 0 \tag{33}$$

where

$$\begin{aligned}
\Pi_1^\circ &= \tau M, \\
\Pi_2^\circ &= X \begin{bmatrix} I_{n\times n} & O_{n\times(m-1)n} & -I_{n\times n} & O_{n_{\times n}} \end{bmatrix}, \\
\Pi_3^\circ &= (2\tau+1)(A\theta_1^\circ + A_d\theta_2^\circ)^T C\theta_4^\circ, \\
\Pi_4^\circ &= W_R^{\circ T}\overline{R}W_R^\circ, \\
\Pi_5^\circ &= \theta_1^{\circ T}[(2\tau-1)P+Q]\theta_1^\circ - \theta_2^{\circ T}Q\theta_2^\circ \\
&\quad +(2\tau+1)[\theta_4^{\circ T}(P-C-C^T)\theta_4^\circ \\
&\quad +\varepsilon(E\theta_1^\circ + E_d\theta_2^\circ)^T(E\theta_1^\circ + E_d\theta_2^\circ)], \\
W_R^\circ &= \begin{bmatrix} I_{mn\times mn} & O_{mn\times n} & O_{mn\times n} \\ O_{mn\times n} & I_{mn\times mn} & O_{mn\times n} \end{bmatrix},
\end{aligned}$$

and $\overline{R}$ is same as in the Theorem 6, and

$$\begin{aligned}
\theta_1^\circ &= \begin{bmatrix} I_{n\times n} & O_{n\times(m+1)n} \end{bmatrix}, \\
\theta_2^\circ &= \begin{bmatrix} O_{n\times mn} & I_{n\times n} & O_{n\times n} \end{bmatrix}, \\
\theta_4^\circ &= \begin{bmatrix} O_{n\times(m+1)n} & I_{n\times n} \end{bmatrix},
\end{aligned}$$

then, the origin $x = 0$ of system (1) with time-invariant delay is globally asymptotically stable.

Remark 11 *The sufficient condition has been investigated in Corollary 10 to ensure the global asymptotic stability of the addressed systems with time-invariant delay and state saturation. The sufficient condition in Corollary 10 is dependent on the delay-fractioning information τ. Hence, the proposed results are delay-dependent. Note that the criteria of global asymptotic stability have been presented in [15] for uncertain discrete state saturation systems with time invariant delays. The results in [15] are delay-independent. Therefore, the stability condition given in [15] is more conservative than the one in this paper.*

Remark 12 *To the best of authors' knowledge, few results based on delay-fractioning approach have been proposed for state saturation 2-D (two-dimensional) discrete systems with time-varying delays. Motivated by the results in this paper, we are now extending the results of robust stability for a class of uncertain discrete systems with state saturation and time-delays to 2-D discrete systems. The corresponding criteria will appear in the near future.*

4 Numerical Examples

In this section, two numerical examples are given to illustrate the usefulness of the proposed results.

Example 13 *Consider the system (1)-(7) with the following parameters:*

$$d_M = 3, d_m = 1, \tau = 1, m = 1$$

$$A = \begin{bmatrix} 1.01 & -2.5 \\ 0.1 & 0 \end{bmatrix}, A_d = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.002 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, E = \begin{bmatrix} 0.01 & 0 \end{bmatrix}, E_d = \begin{bmatrix} 0 & 0.01 \end{bmatrix}$$

By using the Matlab LMI Toolbox, it turns out that LMI (16)-(17) is feasible. We obtain the following values:

$$P = \begin{bmatrix} 14.87 & 0 \\ 0 & 14.87 \end{bmatrix}, Q = \begin{bmatrix} 26.76 & 0 \\ 0 & 22.3 \end{bmatrix}$$

$$R = \begin{bmatrix} 22.3 & 0 \\ 0 & 17.84 \end{bmatrix}, S = \begin{bmatrix} 22.3 & 0 \\ 0 & 22.3 \end{bmatrix}$$

$$X = \begin{bmatrix} O_{2\times4}, & X_1^T, & O_{2\times4} \end{bmatrix}^T,$$

$$X_1^T = \begin{bmatrix} 0 & 0 \\ -4.46 & 0 \end{bmatrix}$$

$$\begin{aligned}
Y &= -Z \\
&= \begin{bmatrix} -44.6 & 0 & -8.92 & -40.14 & -4.62 \\ -4.46 & 0 & -360.91 & -4.46 & 4.46 \\ -4.46 & 4.46 & 104.15 & -8.92 & 4.46 \\ -492.88 & -4.46 & 8.94 & 0 & -8.92 \end{bmatrix}^T
\end{aligned}$$

$$C = \begin{bmatrix} 35.68 & 8.92 \\ 0 & 44.6 \end{bmatrix}, \varepsilon = 22.3$$

Thus, according to Theorem 6, the system under consideration is robust globally asymptotical stable which confirms the feasibility of the proposed stability criterion.

Example 14 *Consider the system (29) with the following parameters:*

$$d_M = 3, d_m = 1, h_1 = 1, h_2 = 3$$

$$A = \begin{bmatrix} 1.01 & -3 \\ 0.1 & 0.78 \end{bmatrix}, A_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, E = \begin{bmatrix} 0.1 & 0 \end{bmatrix}, E_d = \begin{bmatrix} 0 & 0.1 \end{bmatrix}$$

Solving the inequalities(30)and (31) in the Matlab environment, we can obtain

$$P = \begin{bmatrix} 144.6923 & 0 \\ 0 & 144.4029 \end{bmatrix},$$

$$Q = \begin{bmatrix} 144.6923 & 0 \\ 0 & 144.6923 \end{bmatrix},$$

$$R = \begin{bmatrix} 94.2441 & -8.8682 \\ -8.8682 & 54.8486 \end{bmatrix},$$

$$S = \begin{bmatrix} 144.6923 & 0 \\ 0 & 144.6923 \end{bmatrix},$$

$$\begin{aligned} X &= \begin{bmatrix} -63.7164 & 1.9707 & 5.091 & 5.091 \\ 1.9707 & -89.7999 & 5.091 & 5.091 \\ -19.7878 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T, \\ Y &= \begin{bmatrix} 0 & 0 & -31.8582 & 0 \\ 0 & 0 & 0 & 1.9707 \\ 0 & -44.9 & 0 & 0 \\ 0 & 0 & 5.091 & 0 \end{bmatrix}^T, \\ Z &= \begin{bmatrix} 5.091 & 0 & 0 & 5.091 \\ 0 & -6.6341 & 0 & 0 \\ 0 & 0 & 10.8389 & 0 \\ 5.091 & 0 & 0 & -19.7659 \end{bmatrix}^T, \\ \varepsilon &= 144.6865. \end{aligned}$$

Then, we can see that there exist the required matrices satisfying LMIs (30) and (31). As such, according to Corollary 8, the system under consideration is globally asymptotical stable which confirms the effectiveness of the presented stability criterion.

5 Conclusion

In this paper, a new criterion of the robust stability has been given for a class of uncertain discrete-time systems with time-varying delay and state saturation nonlinearities. The norm-bounded parametric uncertainties have been considered. The delay-dependent stability criterion has been given by integrating the delay-fractioning approach and the LMI technique. It is worth mentioning that the proposed results are less conservative than the existing results if there is no saturation nonlinearity in the addressed system. Also, both the constructed Lyapunov-Krasovskii functional based on the delay fractioning approach and the nonnegative scalar δ are important. It has been shown that the proposed scheme can be easily checked by using the standard numerical software. Finally, two numerical examples have been given to illustrate the feasibility of the proposed results.

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References:

[1] M. Malek-Zavarei, M. Jamshidi, *Time-Delay Systems: Analysis, Optimization and Applications,* North-Holland, Amsterdam, 1987.

[2] M. S. Mahmoud, A. A. Bahnasawi, Asymptotic Stability for A Class of Linear Discrete Systems with Bounded Uncertainties, *IEEE Trans. Automat. Control,* Vol. 33, No. 6, 1988, pp. 572–575.

[3] M. S. Mahmoud, *Robust Control and Filtering for Time-Delay Systems,* Marcel-Dekker, New York, 2000.

[4] S. Xu, J. Lam, C. Yang, Quadratic Stability and Stabilization of Uncertain Linear Discrete-Time Systems with State Delay, *Systems Control Letter,* Vol. 43, No. 2, 2001, pp. 77–84.

[5] M. S. Mahmoud, New Results on Robust Control Design of Discrete-Time Uncertain Systems, *IEEE Proc. Control Theory Appl.,* Vol. 152, No. 4, 2005, pp. 453–459.

[6] J. H. Li, A. N. Michel, and W. Porod, Analysis and Synthesis of A Class of Neural Networks: Linear Systems Operating on A Closed Hypercube, *IEEE Trans. Circuits Syst.,* Vol. 36, 1989, pp. 1405–1422.

[7] A. N. Michel, Jie Si, and Gune Yen, Analysis and Synthesis of A Class of Discrete-time Neural Networks Described on Hypercube, *IEEE Trans. Neural Networks,* Vol. 2, 1991, pp. 32–46.

[8] D. Liu, A. N. Michel, Asymptotic Stability of Discrete-Time Systems with Saturation Nonlinearities with Applications to Digital Filters, *IEEE Trans. Circuits Syst.* I, Vol. 39, 1992, pp. 798–807.

[9] D. Liu, A. N. Michel, Asymptotic Stability of Systems with Partial State Saturation Nonlinearities, *Proceedings of the 33rdConference on Decision and Control,* Lake Buena Vista, 1994, pp. 1311–1316.

[10] H. Kar, V. Singh, A New Criterion for the Overflow Stability of Second-Order State-Space Digital Filters Using Saturation Arithmetic, *IEEE Trans. Circuits Syst. I: Fundamental Theory and Applications,* Vol. 45, No. 3, 1998, pp. 311–313.

[11] T. Hu, Z. Lin, A Complete Stability Analysis of Planar Discrete-Time Linear Systems Under Saturation, *IEEE Trans. Circuits Syst. I: Fundamental Theory and Applications,* Vol. 48, No. 6, 2001, pp. 710–725.

[12] H. Kar, V. Singh, Stability Analysis of Discrete-time Systems in A State-space Realization with Partial State Saturation Nonlinearities, *IEEE Proceedings,* 2003, pp. 205–208.

[13] V. Singh, Stability Analysis of Discrete-time Systems in A State-space Realization with State Saturation Nonlinearities: Linear Matrix Inequality Approach, *IEEE Proceedings,* 2005, pp. 9–12.

[14] V. Singh, Stability of Discrete-time Systems Joined with A Saturation Operator on the State-space: Generalized Form of Liu-Michels Criterion, *Automatica,* Vol. 47, 2011, pp. 634–637.

[15] V. Krishna, R. Kandavli, H. Kar, Robust Stability of Discrete-time State-delayed Systems with Saturation Nonlinearities: Linear Matrix Inequality Approach, *Signal Processing,* Vol. 89, 2009, pp. 161–173.

[16] L. Xie, M. Fu, C. E. D. Souza, H_∞ Control and Quadratic Stabilization of Systems with Parameter Uncertainty via Output Feedback, *IEEE Trans. Automat. Control,* Vol. 37, No. 8, 1992, pp. 1253–1256.

[17] P. Gahinet, A. Nemirovski, A. J. Laub, M. Chilali, *LMI Control Toolbox-for Use with Matlab,* The MATH Works Inc., Natic, MA, 1995.

[18] J. H. Kim, S. J. Ahn, S. Ahn, Guaranteed Cost and Filtering for Discrete-Time Polytopic Uncertain Systems with Time Delay, *J. Franklin Inst.,* Vol. 342, No. 4, 2005, pp. 365–378.

[19] J. Hu, Z. Wang, H. Gao, A Delay Fractioning Approach to Robust Sliding Mode Control for Discrete-time Stochastic Systems with Randomly Occurring Non-linearities, *IMA J. Mathematical Control and Information,* Vol. 28, 2011, pp. 345–363.

[20] J. Hu, Z. Wang, Y. Niu, L. K. Stergioulas, H_∞ Sliding Mode Observer Design for a Class of Nonlinear Discrete Time-delay Systems: A Delay-Fractioning Approach, *Int. J. Robust. Nonlinear Control,* Vol. 22, No. 16, 2012, pp. 1806–1826.

[21] D. Peaucelle, D. Arzelier, D. Henrion and F. Gouaisbaut, Quadratic Separation for Feedback Connection of an Uncertain Matrix and an Implicit Linear Transformation, *Automatica,* Vol. 43, No. 5, 2007, pp. 795-804.