# Application of Homo-separation of variables method on nonlinear system of PDEs

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*Abstract:* In this study, the Homo-Separation of Variables method is introduced to find the exact solution of some coupled nonlinear partial differential equations. This analytical method is a new combination of two powerful and popular methods; namely the homotopy perturbation method (HPM) and the separation of variables method. The exact solution is constructed by choosing an appropriate initial approximation in addition to retaining only the first term of the series obtained by HPM. The proposed method is capable of reducing the computational load considerably when compared to other classical method.

*Key–Words:* Homo-separation of variables , Homotopy perturbation, Separation of variables, coupled Burgers' equations

## 1 Introduction

Systems of partial differential equations have played a very important role in various fields of science and engineering, such as viscoelasticity, optimal control and relaxation processes. In all scientific fields, it is important to obtain exact or approximate solutions of systems of partial differential equations.

Exact solutions of most of these systems of equations cannot be found easily because there does not exist any general method to solve this class of equations. Therefore, finding efficient methods for solving these equations have been an active research field.

Several analytical and numerical methods have been suggested and developed by many researchers to solve this class of PDEs such as the Adomian decomposition method (ADM) [6, 29], the homotopy perturbation method (HPM) [32] , the modified homotopy perturbation method (MHPM) [5], the reduced differential transform method (RDTM) [31, 16], the variational iteration method (VIM) [2], the homotopy analysis method (HAM) [1] , the first integral method [3, 8], the Sumudu decomposition method (SDM) [7] , the homotopy analysis transform method (HATM) [11], the iterative Laplace transform method (ILTM) [19], the Laplace–Adomian decomposition method (LADM) [10], the tanh-function method [9] and the exp-function method [15]. Some of these methods use specific transformations and others give the solution as a series which converges to the exact solution.

Among these methods, the HPM is the most used one to solve differential and integral equations. This method was originally proposed by Ji-Huan He in 1998 [12, 13, 14]. Liao [26] correctly argues that the HPM is only a special case of the HAM that developed by Shijun Liao in 1992 [27, 28]. Both methods are in principle based on Taylor series in an embedding parameter. The essential difference is that, the HPM has to use a good enough initial approximation, but the initial approximation is not absolutely necessary for the HAM.

The application of the HPM method in linear and nonlinear problems has been devoted by scientists and engineers. The reason is that, this method converts the problem at hand into a simpler problem which is easier and more straight forward to solve. In addition, the HPM has a strong potential to be combined with other methods to get a new method with improved capabilities. These new capabilities may reduce the complexity of the problem or increase the accuracy of the solution of the problem.

Various methods are combined with HPM to solve a system of linear or nonlinear PDEs. An example of existing combination methods is the homotopy perturbation transformation method (HPTM) which is a combination of HPM and the Laplace transform method [25]. Another such combination is the homotopy perturbation Sumudu transform method (HPSTM) which is constructed by combining two powerful methods; namely the homotopy perturbation method and the Sumudu transformation method [21]. In addition, Yang [30] and Karbalaie [20] found the exact solution of one dimensional PDEs with integer and fractional order by using modified homotopy perturbation methods.

In this paper, we extend the method that has been proposed in [22, 23] to solve n-dimensional systems of PDEs and the result suggested to be is called Homo-Separation of Variables. We present a simple and efficient approach by designing and utilizing a proper initial approximation which satisfies the initial condition of the system of PDEs of interest, as follows:

$$
\begin{aligned}
u_{i0}(\overline{x},t) &= u_i(\overline{x},0)\, c_{i1}(t) \\
&\quad + \left(\sum_{j=1}^{n} \frac{\partial u_i(\overline{x},0)}{\partial x_j}\right) c_{i2}(t) \\
&\quad + \left(\sum_{j=1}^{n} \frac{\partial^2 u_i(\overline{x},0)}{\partial x_j^2}\right) c_{i3}(t), \\
&\qquad\qquad i = 1,2,\ldots,n,
\end{aligned}
$$

where $\overline{x} \in \mathbb{R}^{n-1}$, and $u_i(\overline{x},0)$ are the initial conditions of the system of PDEs. We use $u_{i0}(\overline{x},t)$ which has the form of separation of variables ($c_{i1}, c_{i2}$ and $c_{i3}$), for the HPM. By using this method, the system of PDEs to be solved is converted into a system of ODEs and algebraic equations, which is much simpler and consequently much easier to solve.

The structure of the rest of the paper is as follows: In section 2, we introduce and describe the basic principle of the Homo-Separation of Variables method. In section 3, we present four examples to show the efficiency of using the Homo-Separation of Variables method to solve different linear and nonlinear systems of PDEs by simplifying them into systems of ODEs. Finally, relevant conclusions are drawn in section 4.

## 2 The Basic Idea of the Homo-Separation of Variables Method

This approach is a combination of two approaches; namely HPM and separation of variables as indicated by its proposed name. In this section, we will briefly present the algorithm of this method.

At first, to achieve our goal, we consider the system of PDEs:

$$
\begin{cases}
D_t u_1 + \mathcal{L}_1(u_1,u_2,\cdots,u_n) + \mathcal{N}_1(u_1,u_2,\cdots,u_n) \\
\qquad = f_1(\bar{x},t) \\
D_t u_2 + \mathcal{L}_2(u_1,u_2,\cdots,u_n) + \mathcal{N}_2(u_1,u_2,\cdots,u_n) \\
\qquad = f_2(\bar{x},t) \\
\qquad \cdots \quad \cdots \quad \cdots \\
D_t u_n + \mathcal{L}_n(u_1,u_2,\cdots,u_n) + \mathcal{N}_n(u_1,u_2,\cdots,u_n) \\
\qquad = f_n(\bar{x},t)
\end{cases}
\tag{1}
$$

with the initial conditions

$$
u_i(\overline{x},0) = g_i(\overline{x}), \qquad i = 1,2,\ldots,n, \tag{2}
$$

where $\overline{x} \in \mathbb{R}^{n-1}$, and for $i = 1,2,\ldots,n$, the terms $\mathcal{L}_i, \mathcal{N}_i$ and $f_i$ are linear operators, nonlinear operators and inhomogeneous terms, respectively.

According to the homotopy perturbation technique, we construct the following homotopies:

$$
\begin{cases}
(1-p)(D_t U_1 - D_t u_{10}) + \\
\quad p(D_t U_1 + \tilde{\mathcal{L}_1} + \tilde{\mathcal{N}_1} - f_1(\bar{x},t)) = 0 \\
(1-p)(D_t U_2 - D_t u_{20}) + \\
\quad p(D_t U_2 + \tilde{\mathcal{L}_2} + \tilde{\mathcal{N}_2} - f_2(\bar{x},t)) = 0 \\
\qquad \cdots \quad \cdots \quad \cdots \\
(1-p)(D_t U_n - D_t u_{n0}) + \\
\quad p(D_t U_n + \tilde{\mathcal{L}_n} + \tilde{\mathcal{N}_n} - f_n(\bar{x},t)) = 0
\end{cases}
\tag{3}
$$

where $\tilde{\mathcal{L}_i} = \mathcal{L}_i(U_1,U_2,\cdots,U_n)$, $\tilde{\mathcal{N}_i} = \mathcal{N}_i(U_1,U_2,\cdots,U_n)$ and $p \in [0,1]$. In addition, $u_{i0} = u_{i0}(\overline{x},0)$, $(i = 1,2,\ldots,n,)$ is an initial approximation of the solution of Eq.(1) which satisfies the initial condition in Eq. (2)

We can assume that the solution of Eq. (3) can be expressed as a power series in p, as follows:

$$\begin{aligned} U_i &= \sum_{j=0}^{\infty} p^j U_{ij} \\ &= U_{i0} + pU_{i1} + p^2U_{i2} + p^3U_{i3} + \dots, \\ & \quad i = 1, 2, \dots, n, \end{aligned} \tag{4}$$

By substituting Eq. (4) into (3) and equating the terms with identical powers of $p$, the following set of equations is obtained:

$$p^0 : \begin{cases} D_tU_{10} - D_tu_{10} = 0, \\ D_tU_{20} - D_tu_{20} = 0, \\ \dots\dots \\ D_tU_{n0} - D_tu_{n0} = 0, \end{cases} \tag{5}$$

$$p^1 : \begin{cases} D_tU_{11} + D_tu_{10} + \mathcal{L}_1(U_{10}, U_{20}, \cdots, U_{n0}) \\ +\mathcal{M}_{10} - f_1 = 0, \\ \\ D_tU_{22} + D_tu_{20} + \mathcal{L}_2(U_{10}, U_{20}, \cdots, U_{n0}) \\ +\mathcal{M}_{20} - f_2 = 0, \\ \dots\dots \\ D_tU_{n1} + D_tu_{n0} + \mathcal{L}_n(U_{10}, U_{20}, \cdots, U_{n0}) \\ +\mathcal{M}_{n0} - f_n = 0, \end{cases} \tag{6}$$

$$\vdots$$

$$p^j : \begin{cases} D_tU_{1j} + \mathcal{L}_1(U_{1j-1}, U_{2j-1}, \cdots, U_{nj-1}) + \\ \qquad \mathcal{M}_{1j-1} = 0, \\ \\ D_tU_{2j} + \mathcal{L}_2(U_{1j-1}, U_{2j-1}, \cdots, U_{nj-1}) + \\ \qquad \mathcal{M}_{2j-1} = 0, \\ \qquad \vdots \\ D_tU_{nj} + \mathcal{L}_n(U_{1j-1}, U_{2j-1}, \cdots, U_{nj-1}) + \\ \qquad \mathcal{M}_{nj-1} = 0, \end{cases} \tag{7}$$

where the terms $\mathcal{M}_{ij}$ $(i = 1, 2, \dots, n, \quad j = 1, 2, \dots, n-1,)$ are resulted from Eq. (5-7), by equating the coefficients of the nonlinear operators $\mathcal{N}_{ij}$ $(i = 1, 2, \dots, n, \quad j = 1, 2, \dots, n-1,)$ with the identical powers j of $p$ ; i.e. $p^j$.

In case the $p$-parameter is considered as small, the best approximation for the solution of Eq. (1) becomes

$$u_i(\overline{x}, t) = \sum_{j=0}^{\infty} p^j U_{ij}(\overline{x}, t) \qquad i = 1, 2, \dots, n,$$

By setting $p = 1$ in above equation we can entirely determined the HPM series solutions

$$\begin{aligned} u_i(\overline{x}, t) &= U_{i0}(\overline{x}, t) + U_{i1}(\overline{x}, t) + U_{i2}(\overline{x}, t) + \dots, \\ & \quad i = 1, 2, \dots, n, \end{aligned} \tag{8}$$

The convergence of the series Eq. (8) has been proved in [16] and the asymptotic behavior of the series was illustrated by Ji-Huan He [17, 18].

If in Eq. (8) there exists some $U_{iN} = 0$ $(N \geq 1, \quad i = 1, 2, \dots, n)$, then the exact solution can be written in the following form:

$$u_i(\overline{x}, t) = \sum_{j=0}^{N-1} U_{ij}(\overline{x}, t), \quad i = 1, 2, \dots, n, \tag{9}$$

For simplicity, we assume that $U_{i1}(\overline{x}, t) \equiv 0$ in Eq. (9), which means that the exact solution of Eq. (1) is simply

$$u_i(\overline{x}, t) = U_{i0}(\overline{x}, t), \quad i = 1, 2, \dots, n, \tag{10}$$

Therefore, the result can be obtained by only solving Eq. (5). thus the result becomes:

$$U_{i0}(\overline{x}, t) = u_{i0}(\overline{x}, t), \quad i = 1, 2, \dots, n, \tag{11}$$

By using the results in Eq.(10) and Eq.(11) we have:

$$\begin{aligned} u_i(\overline{x}, t) &= U_{i0}(\overline{x}, t) = u_{i0}(\overline{x}, t), \\ & \quad i = 1, 2, \dots, n, \end{aligned} \tag{12}$$

Consequently, the core of our new approach is achieved by assuming that the initial approximation of Eq. (1) can be considered as follows (according to the separation of variables method):

$$\begin{aligned} u_i(\overline{x}, t) &= u_{i0}(\overline{x}, t) \\ &= u_i(\overline{x}, 0)\, c_{i1}(t) + \left( \sum_{j=1}^{n} \frac{\partial u_i(\overline{x}, 0)}{\partial x_j} \right) c_{i2}(t) \\ &\quad + \left( \sum_{j=1}^{n} \frac{\partial^2 u_i(\overline{x}, 0)}{\partial x_j^2} \right) c_{i3}(t) \\ &= g_i(\overline{x}) c_{i1}(t) + \left( \sum_{j=1}^{n} \frac{\partial g_i(\overline{x})}{\partial x_j} \right) c_{i2}(t) \\ &\quad + \left( \sum_{j=1}^{n} \frac{\partial^2 g_i(\overline{x})}{\partial x_j^2} \right) c_{i3}(t), \\ & \quad i = 1, 2, \dots, n, \end{aligned} \tag{13}$$

Now the goal is finding the new terms $c_{i1}(t), c_{i2}(t)$ and $c_{i3}(t)$, to be able to get the exact solution of Eq. (1)

Since Eq. (13) satisfies the initial conditions as well as Eq. (2), we get

$$c_{i1}(0) = 1, \quad c_{i2}(0) = 0, \quad c_{i3}(0) = 0, \qquad i = 1, 2, \ldots, n, \quad (14)$$

By substituting Eq. (12) and (13) into Eq. (6), the system of PDEs is changed into a system of ODEs and algebraic equations. It means that it is possible to solve a system of PDEs by solving a family of a systems of ODEs. Consequently, the problem at hand will be simplified considerably. Furthermore, solving a system of ODEs and algebraic equations is much more easier and straightforward in general than solving a system of PDEs. Finally, the target unknowns $c_{i1}(t), c_{i2}(t)$ and $c_{i3}(t)$ in the final exact solution of a PDE's system, can be obtained by solving a system of ODEs and utilizing the initial conditions in Eq. (14). which simplify the task even more.

## 3 Applications

In this section, five examples are presented to illustrate the applicability of the new homo-separation of variables method for solving systems of nonlinear PDEs.

**Example 1.** Consider the following system of linear PDEs to be solved:

$$\begin{cases} u_t = v_x - v - u - 2 \\ v_t = u_x - v - u - 2 \end{cases} \quad (15)$$

with the initial conditions

$$u(x,0) = e^x + 1, \qquad v(x,0) = e^x - 1 \quad (16)$$

To solve Eq. (15) and Eq. (16) by employing the homo-separation of variables method, we choose the initial approximation in Eq. (15) as follows:

$$\begin{cases} u_0(x,t) = u(x,0)\, c_1(t) + \frac{\partial u(x,0)}{\partial x} c_2(t) \\ \qquad + \frac{\partial^2 u(x,0)}{\partial x^2} c_3(t) \\ v_0(x,t) = v(x,0)\, b_1(t) + \frac{\partial v(x,0)}{\partial x} b_2(t) \\ \qquad + \frac{\partial^2 v(x,0)}{\partial x^2} b_3(t) \end{cases}$$

By substituting the two initial conditions given by Eq. (16) into these two initial approximations, we get the following initial approximations:

$$\begin{cases} u_0(x,t) = (e^x + 1) c_1(t) + e^x (c_2(t) + c_3(t)) \\ v_0(x,t) = (e^x - 1) b_1(t) + e^x (b_2(t) + b_3(t)) \end{cases} \quad (17)$$

By substituting Eq. (17) into Eq. (15), we get

$$\begin{cases} D_t u_1 = e^x \left( c_1'(t) + c_2'(t) + c_3'(t) + c_1(t) + c_2(t) \right. \\ \quad \left. + c_3(t) \right) + c_1'(t) + c_1(t) - b_1(t) + 2 \equiv 0, \\ D_t v_1 = e^x \left( b_1'(t) + b_2'(t) + b_3'(t) + b_1(t) + b_2(t) \right. \\ \quad \left. + b_3(t) \right) - b_1'(t) - b_1(t) - c_1(t) + 2 \equiv 0, \end{cases} \quad (18)$$

By solving the latter system of PDEs presented in Eq. (18), we obtain the following system of ODEs:

$$\begin{cases} c_1'(t) + c_2'(t) + c_3'(t) + c_1(t) + c_2(t) + c_3(t) = 0 \\ b_1'(t) + b_2'(t) + b_3'(t) + b_1(t) + b_2(t) + b_3(t) = 0 \\ c_1'(t) + c_1(t) - b_1(t) + 2 = 0 \\ -b_1'(t) - b_1(t) - c_1(t) + 2 = 0 \end{cases} \quad (19)$$

together with the initial conditions

$$\begin{cases} c_1(0) = 1, \\ c_2(0) = 0, \\ c_3(0) = 0, \end{cases} \qquad \begin{cases} b_1(0) = 1, \\ b_2(0) = 0, \\ b_3(0) = 0, \end{cases} \quad (20)$$

Now, solving Eq. (19) and (20) by using the ODEs' properties, we obtain

$$\begin{cases} c_1(t) = e^{-2t}, \\ c_2(t) + c_3(t) = e^{-t} - e^{-2t}, \end{cases}$$
$$\begin{cases} b_1(t) = 2 - e^{-2t}, \\ b_2(t) + b_3(t) = -2 + e^{-2t} + e^{-t}, \end{cases} \quad (21)$$

By substituting Eq. (21) into Eq. (17), we obtain the exact solution of the original system of linear PDEs in Eq. (15) as follows:

$$\begin{cases} u(x,t) = e^{-2t} + e^{x-t}, \\ v(x,t) = e^{-2t} + e^{x-t} - 2, \end{cases} \quad (22)$$

**Example 2.** Consider the following one-dimensional coupled Burgers' equations:

$$\begin{cases} u_t = u_{xx} + 2\, u u_x - \frac{\partial}{\partial x}(uv) \\ v_t = v_{xx} + 2\, v v_x - \frac{\partial}{\partial x}(uv) \end{cases} \quad (23)$$

with the initial conditions

$$u(x,0) = \sin x, \qquad v(x,0) = \sin x, \quad (24)$$

To solve Eq. (23) and Eq. (24) by employing the homo-separation of variables method, we choose the initial approximation in Eq. (23) as follows:

$$\begin{cases} u_0(x,t) = u(x,0)\, c_1(t) + \frac{\partial u(x,0)}{\partial x} c_2(t) \\ \qquad + \frac{\partial^2 u(x,0)}{\partial x^2} c_3(t) \\ v_0(x,t) = v(x,0)\, b_1(t) + \frac{\partial v(x,0)}{\partial x} b_2(t) \\ \qquad + \frac{\partial^2 v(x,0)}{\partial x^2} b_3(t) \end{cases}$$

By substituting the two initial conditions given by Eq. (24) into these two initial approximations, we get the following initial approximations:

$$\begin{cases} u_0(x,t) = (c_1(t) - c_3(t))\sin x + c_3(t)\cos x \\ v_0(x,t) = (c_1(t) - c_3(t))\sin x + c_3(t)\cos x \end{cases} \quad (25)$$

By substituting Eq. (25) into Eq. (23), we get

$$\begin{cases} D_t u_1 = 2\cos^2 x\,(-2\,(c_1(t) + c_3(t))\,c_2(t) \\ -(b_1(t) - b_3(t))\,c_2(t) - b_2(t)\,(c_1(t) - c_3(t))) \\ +2\cos x\sin x\left(-(c_1(t) - c_3(t))^2 + c_2^2(t)\right. \\ \left.-(c_1(t) - c_3(t))\,(b_1(t) - b_3(t)) - b_2(t)\,c_2(t)\right) \\ -2\cos x\,(c_2'(t) + c_2(t)) \\ +\sin x\,((c_1(t) - c_3(t))' + (c_1(t) - c_3(t))) \\ +2\,(c_1(t) - c_3(t))\,c_2(t) - c_2(t)\,(b_1(t) - b_3(t)) \\ -(c_1(t) - c_3(t))\,b_2(t) \equiv 0, \\ \\ D_t v_1 = 2\cos^2 x\,(-2\,(b_1(t) + b_3(t))\,b_2(t) \\ -(b_1(t) - b_3(t))\,c_2(t) - b_2(t)\,(c_1(t) - c_3(t))) \\ +2\cos x\sin x\left(-(b_1(t) - b_3(t))^2 + b_2^2(t)\right. \\ \left.-(c_1(t) - c_3(t))\,(b_1(t) - b_3(t)) - b_2(t)\,c_2(t)\right) \\ -2\cos x\left(b_2'(t) + b_2(t)\right) \\ +\sin x\,((b_1(t) - b_3(t))' + (b_1(t) - b_3(t))) \\ +2\,(b_1(t) - b_3(t))\,b_2(t) - c_2(t)\,(b_1(t) - b_3(t)) \\ -(c_1(t) - c_3(t))\,b_2(t) \equiv 0, \end{cases} \quad (26)$$

By solving the latter system of PDEs presented in Eq. (26), we obtain the following system of ODEs and algebraic equations:

$$\begin{cases} 2\,(b_1(t) - b_3(t))\,b_2(t) - c_2(t)\,(b_1(t) - b_3(t)) \\ -(c_1(t) - c_3(t))\,b_2(t) = 0, \\ 2\,(c_1(t) - c_3(t))\,c_2(t) - c_2(t)\,(b_1(t) - b_3(t)) \\ -(c_1(t) - c_3(t))\,b_2(t) = 0, \\ -2\,(b_1(t) + b_3(t))\,b_2(t) - (b_1(t) - b_3(t))\,c_2(t) \\ -b_2(t)\,(c_1(t) - c_3(t)) = 0, \\ -2\,(c_1(t) + c_3(t))\,c_2(t) - (b_1(t) - b_3(t))\,c_2(t) \\ -b_2(t)\,(c_1(t) - c_3(t)) = 0, \\ (c_1(t) - c_3(t))' + (c_1(t) - c_3(t)) = 0, \\ (b_1(t) - b_3(t))' + (b_1(t) - b_3(t)) = 0, \\ b_2'(t) + b_2(t) = 0, \\ c_2'(t) + c_2(t) = 0, \end{cases} \quad (27)$$

together with the initial conditions

$$\begin{cases} c_1(0) = 1, \\ c_2(0) = 0, \\ c_3(0) = 0, \end{cases} \qquad \begin{cases} b_1(0) = 1, \\ b_2(0) = 0, \\ b_3(0) = 0, \end{cases} \quad (28)$$

Now, solving Eq. (27) and (28) by using the ODEs' properties, we obtain

$$\begin{cases} c_1(t) - c_3(t) = e^{-t}, \\ c_2(t) = 0, \\ b_1(t) - b_3(t) = e^{-t}, \\ c_2(t) = 0, \end{cases} \quad (29)$$

By substituting Eq. (29) into Eq. (25), we obtain the exact solution of the original system of linear PDEs in Eq. (23) as follows:

$$\begin{cases} u(x,t) = e^{-t}\sin x, \\ v(x,t) = e^{-t}\sin x, \end{cases} \quad (30)$$

**Example 3.** Consider the following two-dimensional coupled Burgers' equations:

$$\begin{cases} u_t = u_{xx} + u_{yy} + 2\,u\,(u_x + u_y) \\ \quad -\frac{\partial}{\partial x}(uv) - \frac{\partial}{\partial y}(uv) \\ v_t = v_{xx} + v_{yy} + 2\,v\,(v_x + v_y) \\ \quad -\frac{\partial}{\partial x}(uv) - \frac{\partial}{\partial y}(uv) \end{cases} \quad (31)$$

with the initial conditions

$$u(x,y,0) = \cos(x+y),\ v(x,y,0) = \cos(x+y), \quad (32)$$

To solve Eq. (31) and Eq. (32) by using the homo-separation of variables method, we choose the initial approximation in Eq. (31) as follows:

$$\begin{cases} u_0(x,y,t) = u(x,y,0)\,b_1(t) + \left(\frac{\partial u(x,y,0)}{\partial x}\right. \\ \left.+\frac{\partial u(x,y,0)}{\partial y}\right) b_2(t) + \left(\frac{\partial u^2(x,y,0)}{\partial x^2} + \frac{\partial u^2(x,y,0)}{\partial y^2}\right) b_3(t), \\ \\ v_0(x,y,t) = v(x,y,0)\,c_1(t) + \left(\frac{\partial v(x,y,0)}{\partial x}\right. \\ \left.+\frac{\partial v(x,y,0)}{\partial y}\right) c_2(t) + \left(\frac{\partial v^2(x,y,0)}{\partial x^2} + \frac{\partial v^2(x,y,0)}{\partial y^2}\right) c_3(t), \end{cases}$$

By substituting the two initial conditions given by Eq. (32) into these two initial approximations, we get the following initial approximations:

$$\begin{cases} u_0(x,y,t) = \cos(x+y)\,b_1(t) - 2\sin(x+y)\,b_2(t) \\ \quad -2\cos(x+y)\,b_3(t), \\ \\ v_0(x,y,t) = \cos(x+y)\,c_1(t) - 2\sin(x+y)\,c_2(t) \\ \quad -2\cos(x+y)\,c_3(t), \end{cases} \quad (33)$$

By substituting Eq. (33) into Eq. (31), we get

$$\begin{cases} D_t u_1 = 8cos^2(x+y)\Big[2(b_1(t)-2b_3(t))b_2(t) \\ -(b_1(t)-2b_3(t))c_2(t)-b_2(t)(c_1(t)-2c_3(t))\Big] \\ +4\cos(x+y)\sin(x+y)\Big[(b_1(t)-2b_3(t))^2 \\ -4b_2^2(t)-(b_1(t)-2b_3(t))(c_1(t)-2c_3(t)) \\ +4b_2(t)c_2(t)\Big]-2\sin(x+y)\Big(b_2'(t)+2b_2(t)\Big) \\ +\cos(x+y)\Big(\ (b_1\,(t)-2b_3(t))' \\ +2\,(b_1\,(t)-2b_3(t))\ \Big)-8b_1(t)b_2(t) \\ +4\,b_2(t)\,(c_1\,(t)-2\,c_3(t)) \\ +4\ (b_1\,(t)-2\,b_3(t))\,c_2(t)\equiv 0, \\ D_t v_1 = 8\cos^2(x+y)\Big(2\ (c_1\,(t)-2\,c_3(t))\,c_2(t)- \\ (b_1\,(t)-2\,b_3(t))\,c_2(t)-b_2(t)\,(c_1\,(t)-2\,c_3(t))\Big) \\ +4\cos(x+y)\sin\,(x+y)\Big(\ (c_1\,(t)-2\,c_3(t))^2 \\ -4\,c_2^2(t)-(c_1\,(t)-2\,c_3(t))\,(b_1\,(t)-2\,b_3(t)) \\ +4\,b_2(t)c_2(t)\Big) \\ -2\,\sin\,(x+y)\,(c_2'\,(t)+2\,c_2(t)) \\ +\cos(x+y)\Big((c_1(t)-2c_3(t))'+2(c_1(t)-2c_3(t))\Big) \\ -8(c_1(t)-2c_3(t))c_2(t)+4c_2(t)(b_1(t)-2\,b_3(t)) \\ +4\ (c_1\,(t)-2\,c_3(t))\,b_2(t)\equiv 0. \end{cases} \tag{34}$$

By solving the latter system of PDEs presented in Eq. (34), we obtain the following system of ODEs and algebraic equations:

$$\begin{cases} 2\,(b_1\,(t)-2\,b_3(t))\,b_2(t)-(b_1\,(t)-2b_3(t))\,c_2(t) \\ -b_2(t)\,(c_1\,(t)-c_3(t))=0, \\ (b_1\,(t)-2\,b_3(t))^2\,(t)-4\,b_2^2(t)-(b_1\,(t)-2\,b_3(t)) \\ \times\,(c_1\,(t)-2\,c_3(t))+4\,b_2(t)c_2(t)=0, \\ 2\ (c_1\,(t)-2\,c_3(t))\,c_2(t)-(b_1\,(t)-2\,b_3(t))\,c_2(t) \\ -b_2(t)\,(c_1\,(t)-2\,c_3(t))=0, \\ (c_1\,(t)-2\,c_3(t))^2-4\,c_2^2(t)-(c_1\,(t)-2\,c_3(t))\,b_1(t) \\ +4\,b_2(t)c_2(t)=0, \\ -8\ (b_1\,(t)-2\,b_3(t))\,b_2(t)+4\,b_2(t)\,(c_1\,(t)-2\,c_3(t)) \\ +4\ (b_1\,(t)-2\,b_3(t))\,c_2(t)=0, \\ -8\ (c_1\,(t)-2\,c_3(t))\,c_2(t)+4\,c_2(t)\,(b_1\,(t)-2\,b_3(t)) \\ +4\ (c_1\,(t)-2\,c_3(t))\,b_2(t)=0, \\ (b_1\,(t)-2\,b_3(t))'+2\ (b_1\,(t)-2\,b_3(t))=0 \\ (c_1\,(t)-2\,c_3(t))'+2\ (c_1\,(t)-2\,c_3(t))=0 \\ c_2'\,(t)+2\,c_2(t)=0, \\ b_2'\,(t)+2\,b_2(t)=0 \end{cases} \tag{35}$$

together with the initial conditions

$$\begin{cases} c_1\,(0)=1, \\ c_2\,(0)=0, \\ c_3\,(0)=0, \end{cases} \qquad \begin{cases} b_1\,(0)=1, \\ b_2\,(0)=0, \\ b_3\,(0)=0, \end{cases} \tag{36}$$

By employing well-known existing methods for solving systems of ODEs, Eq. (35) and (36) can usually be solved more easily as follows:

$$\begin{cases} c_1\,(t)-2\,c_3\,(t)=e^{-2t}, \\ c_2\,(t)=0 \\ b_1\,(t)-2\,b_3\,(t)=e^{-2t}, \\ b_2\,(t)=0 \end{cases} \tag{37}$$

By substituting Eq. (37) into Eq. (33), we obtain the exact solution of the original system of inhomogeneous PDEs in Eq. (31) as follows

$$\begin{cases} u\,(x,y,t)=\cos\,(x+y)\,e^{-2t}, \\ v\,(x,y,t)=\cos\,(x+y)\,e^{-2t}, \end{cases} \tag{38}$$

**Example 4.** Consider the following system of PDEs:

$$\begin{cases} u_t = v-u-u_x+q_1\,(x,t)\,, \\ \qquad (0<x\le 1,\ 0<t\le T) \\ v_t = u-v+\lambda e^v+q_2\,(x,t)\,, \\ \qquad (0<x\le 1,\ 0<t\le T) \end{cases} \tag{39}$$

with initial conditions

$$u\,(x,0)=1-x^2, \qquad v\,(x,0)=1-x^2, \tag{40}$$

where $\lambda > 0$ is considered as a parameter, and

$$q_1\,(x,t)=(x^2-2x+1)e^{-t}$$
$$q_2\,(x,t)=\left(1+x^2\right)e^{-t}-(\lambda e^2)e^{-\left(1+x^2\right)e^{-t}}$$

To solve Eq. (39) and Eq. (40) by using the homo-separation of variables method, we choose the initial approximation in Eq. (39) as follows:

$$\begin{cases} u_0\,(x,t)=u\,(x,0)\,c_1\,(t)+\frac{\partial u(x,0)}{\partial x}c_2\,(t) \\ \qquad +\frac{\partial^2 u(x,0)}{\partial x^2}c_3\,(t)\,, \\ v_0\,(x,t)=v\,(x,0)\,b_1\,(t)+\frac{\partial v(x,0)}{\partial x}b_2\,(t) \\ \qquad +\frac{\partial^2 v(x,0)}{\partial x^2}b_3\,(t)\,, \end{cases}$$

By substituting the two initial conditions given by Eq. (40) into these two initial approximations, we get the following initial approximations:

$$\begin{cases} u_0(x,t) = \left(1-x^2\right)c_1(t) - 2xc_2(t) - 2c_3(t) \\ v_0(x,t) = \left(1-x^2\right)b_1(t) - 2xb_2(t) - 2b_3(t) \end{cases} \tag{41}$$

By substituting Eq. (41) into Eq. (39), we get

$$\begin{cases} D_t u_1 = -x^2\left(c_1'(t) + c_1(t) - b_1(t) + e^{-t}\right) \\ \quad -2x\left(c_2'(t) + c_2(t) + c_1(t) - b_1(t)\right) + c_1'(t) \\ \quad -2c_3'(t) + c_1(t) - 2c_2(t) - 2c_3(t) - b_1(t) \\ \quad +3b_3(t) - e^{-t} \equiv 0, \\ \\ D_t v_1 = -x^2\left(b_1'(t) + b_1(t) - c_1(t) + e^{-t}\right) \\ \quad -2x\left(b_2'(t) + b_2(t) - c_2(t)\right) + b_1'(t) - 2b_3'(t) \\ \quad +b_1(t) - 2b_3(t) - c_1(t) + 2c_3(t) - e^{-t} \\ \quad -\lambda e^{-x^2 b_1(t) - 2xb_2(t) + b_1(t) - 2b_3(t)} \\ \quad +\lambda e^{2 - e^{-t} - e^{-t}x^2} \equiv 0, \end{cases} \tag{42}$$

By solving the latter system of PDEs presented in Eq. (42), we obtain the following system of ODEs:

$$\begin{cases} c_1'(t) + c_1(t) - b_1(t) + e^{-t} = 0 \\ c_2'(t) + c_2(t) + c_1(t) - b_1(t) = 0 \\ b_1'(t) + b_1(t) - c_1(t) + e^{-t} = 0 \\ b_2'(t) + b_2(t) - c_2(t) = 0 \\ e^{-x^2 b_1(t) - 2xb_2(t) + b_1(t) - 2b_3(t)} = e^{2 - e^{-t} - e^{-t}x^2} \\ c_1'(t) - 2c_3'(t) + c_1(t) - 2c_2(t) - 2c_3(t) - b_1(t) \\ \quad +3b_3(t) - e^{-t} = 0 \\ b_1'(t) - 2b_3'(t) + b_1(t) - 2b_3(t) - c_1(t) + 2c_3(t) \\ \quad -e^{-t} = 0 \end{cases} \tag{43}$$

together with initial conditions

$$\begin{cases} c_1(0) = 1, \\ c_2(0) = 0, \\ c_3(0) = 0, \end{cases} \qquad \begin{cases} b_1(0) = 1, \\ b_2(0) = 0, \\ b_3(0) = 0, \end{cases} \tag{44}$$

By employing well-known existing methods for solving systems of ODEs, Eq. (43) and (44) can usually be solved more easily as follows:

$$\begin{cases} c_1(t) = e^{-t},\ c_2(t) = 0,\ c_3(t) = e^{-t} - 1, \\ \\ b_1(0) = e^{-t},\ \ b_2(0) = 0,\ \ b_3(0) = e^{-t} - 1, \end{cases} \tag{45}$$

By substituting Eq. (45) into Eq. (41), we obtain the exact solution of the original system of PDEs in Eq. (39) as follows

$$\begin{cases} u(x,t) = 2 - (1+x^2)e^{-t} \\ \\ v(x,t) = 2 - (1+x^2)e^{-t} \end{cases} \tag{46}$$

**Example 5.** Consider the following system of nonlinear PDEs to be solved:

$$\begin{cases} u_t = -u - v_x w_y - v_y w_x, \\ v_t = v - u_x w_y - u_y w_x, \\ w_t = w - u_x v_y - u_y v_x, \end{cases} \tag{47}$$

with the initial conditions

$$\begin{cases} u(x,y,0) = e^{x+y}, \\ v(x,y,0) = e^{x-y}, \\ w(x,y,0) = e^{-x+y}, \end{cases} \tag{48}$$

To solve Eq. (47) and Eq. (48) by utilizing the homo-separation of variables method, we choose the initial approximation in Eq. (47) as follows:

$$\begin{cases} u_0(x,y,t) = u(x,y,0)\,b_1(t) + \Big(\frac{\partial u(x,y,0)}{\partial x} \\ \quad + \frac{\partial u(x,y,0)}{\partial y}\Big) b_2(t) + \left(\frac{\partial u^2(x,y,0)}{\partial x^2} + \frac{\partial u^2(x,y,0)}{\partial y^2}\right) b_3(t), \\ \\ v_0(x,y,t) = v(x,y,0)\,c_1(t) + \Big(\frac{\partial v(x,y,0)}{\partial x} \\ \quad + \frac{\partial v(x,y,0)}{\partial y}\Big) c_2(t) + \left(\frac{\partial v^2(x,y,0)}{\partial x^2} + \frac{\partial v^2(x,y,0)}{\partial y^2}\right) c_3(t), \\ \\ w_0(x,y,t) = w(x,y,0)\,d_1(t) + \Big(\frac{\partial w(x,y,0)}{\partial x} \\ \quad + \frac{\partial w(x,y,0)}{\partial y}\Big) d_2(t) + \left(\frac{\partial w^2(x,y,0)}{\partial x^2} + \frac{\partial w^2(x,y,0)}{\partial y^2}\right) d_3(t), \end{cases}$$

By substituting the two initial conditions given by Eq. (48) into these two initial approximations, we get the following initial approximations:

$$\begin{cases} u_0(x,y,t) = e^{x+y}b_1(t) + 2e^{x+y}b_2(t) + 2e^{x+y}b_3(t), \\ v_0(x,y,t) = e^{x-y}c_1(t), \\ w_0(x,y,t) = e^{-x+y}d_1(t), \end{cases} \tag{49}$$

By substituting Eq. (49) into Eq. (47), we get

$$\begin{cases} D_t u_1 = e^{-x+y}\left(b_1'(t) + 2\left(b_2'(t) + b_3'(t)\right)\right. \\ \qquad \left. +b_1(t) + 2\left(b_2(t) + b_3(t)\right)\right) \equiv 0, \\ D_t v_1 = e^{x-y}\left(c_1'(t) - c_1(t)\right) \equiv 0, \\ D_t w_1 = e^{-x+y}\left(d_1'(t) - d_1(t)\right) \equiv 0, \end{cases} \tag{50}$$

By solving the latter system of PDEs presented in Eq. (50), we obtain the following system of ODEs:

$$\begin{cases} \left(\acute{b_1}(t) + 2\left(\acute{b_2}(t) + \acute{b_3}(t)\right)\right) \\ \qquad + \left(b_1(t) + 2\left(b_2(t) + b_3(t)\right)\right) = 0, \\ \acute{c_1}(t) - c_1(t) = 0, \\ \acute{d_1}(t) - d_1(t) = 0, \end{cases} \tag{51}$$

together with the initial conditions

$$\begin{cases} c_1(0) = 1, \\ c_2(0) = 0, \\ c_3(0) = 0, \end{cases} \quad \begin{cases} b_1(0) = 1, \\ b_2(0) = 0, \\ b_3(0) = 0, \end{cases} \quad \begin{cases} d_1(0) = 1, \\ d_2(0) = 0, \\ d_3(0) = 0, \end{cases} \tag{52}$$

By employing well-known existing methods for solving systems of ODEs, Eq. (52) and (**??**) can usually be solved more easily as follows:

$$\begin{cases} b_1(t) - 2\left(b_2(t) + b_3(t)\right) = e^{-t}, \\ c_1(t) = e^{t}, \\ d_1(t) = e^{t}, \end{cases} \tag{53}$$

By substituting Eq. (53) into Eq. (49), we obtain the exact solution of the original nonlinerar system of PDEs in Eq. (47) as follows

$$\begin{cases} u(x,y,t) = e^{x+y-t}, \\ v(x,y,t) = e^{x-y+t}, \\ w(x,y,t) = e^{-x+y+t}, \end{cases} \tag{54}$$

# 4 Conclusion

The fundamental goal of this research work was to propose and develop simple method for solving a system of PDEs. The strategy that was considered to reach this goal, we try to simplify the problem at hand to make it easier to solve. This goal has been achieved by applying Homo-Separation of Variables in addition using suitable initial conditions only. more specifically, this task can be performed by using the first term given by the homotopy perturbation method in addition to a proper initial approximation (given by utilizing separation of variables)which satisfies the initial conditions of the system of PDEs at hand. By using this method the systems of PDEs are converted into systems of ODEs and algebraic equations that are usually much easier and straight-forward to solve then the original PDEs' system.

Generally, this method has a number of attractive advantages. It can be considered as a computationally efficient tool for solving a wide class of linear and nonlinear PDEs. To summaries, the new approach proposed here is simple convenient, easy to understand, straight-forward to apply to various types of problems, and requires much fewer computations which makes it fast requiring low computational load.

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