Generalized Inverses of Convolution Type Operators on Bounded Intervals

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Abstract: In this paper, we introduce a study of generalized inverses for convolution type operators defined in bounded intervals Ω.
The main results of this paper are organized as follows: First, the construction of equivalence relations between the convolution type operators and new Wiener-Hopf operators (operators of convolution type defined on the half-line) based on a decomposition method of higher order Wiener-Hopf operators. Second, we introduce the generalized inverses of these operators, when they exist.

Key–Words: Convolution equations, Wiener-Hopf operators, Convolution type operators, Equivalence after extension, Projection methods, Generalized inverse

1 Definitions and Notations

We will start this section by presenting a collection of settings needed for our study. Note, however, that we will not provide the definitions of formal and systematic way, since all of them can be considered as belonging to an elementary part of the functional analysis.

Similarly, we will introduce some notations necessary for the development of this work. Of course we do not want to make a list of notations, because, throughout the article, they will appear in a natural way as they are needed.

We know that a function φ defined on $\mathbb{R}^n$ is rapidly decreasing if there exist constants C_N such that

$$|\varphi| \leq C_N |x|^{-N}, \qquad |x| \to \infty$$

for $N = 1, 2, 3, ...$

Another way to say the same for φ, is when, after multiplying φ by any polynomial $p(x)$, $p(x)\varphi(x)$ must still tend to zero when $|x| \to \infty$.

The Schwartz space of rapidly decreasing functions will be denoted by $\mathcal{S}(\mathbb{R}^n)$.

For functions φ, defined in the space $\mathcal{S}(\mathbb{R}^n)$, the Fourier transformation is given by

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{R}^n} e^{i\xi x} \varphi(x)\, dx$$

and the inverse Fourier transformation by

$$\left(\mathcal{F}^{-1}\varphi\right)(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\xi x} \varphi(x)\, dx,$$

where $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $\xi x = \xi_1 x_1 + \ldots + \xi_n x_n$.

As usual, for $1 < p < \infty$, the Banach space of Lebesgue measurable functions φ on $\mathbb{R}$ such that $|\varphi|^p$ is integrable will be denoted by $L^p(\mathbb{R})$.

For a given domain $\Gamma \subseteq \mathbb{R}_+ =]0, +\infty[$, we denote by $L^p_\Gamma(\mathbb{R})$ the closed subspace of $L^p(\mathbb{R})$ whose elements have support in $\overline{\Gamma}$. The space $L^p_\Gamma(\mathbb{R})$ is endowed with the subspace topology. For simplicity, we write $L^p_+(\mathbb{R})$ instead of $L^p_{\mathbb{R}_+}(\mathbb{R})$.

Throughout this work, we will use repeatedly the projection operators defined by

$$\mathcal{P}_\Gamma : \left[L^p_+(\mathbb{R})\right]^n \to \left[L^p_\Gamma(\mathbb{R})\right]^n$$
$$\varphi(\xi) \mapsto (\mathcal{P}_\Gamma\varphi)(\xi) = \begin{cases} \varphi(\xi) & \text{if } \xi \in \Gamma \\ 0 & \text{if } \xi \notin \Gamma \end{cases}$$

and

$$\mathcal{Q}_\Gamma : \left[L^p_+(\mathbb{R})\right]^n \to \left[L^p_{\mathbb{R}_+ \setminus \overline{\Gamma}}(\mathbb{R})\right]^n$$
$$\varphi(\xi) \mapsto (\mathcal{Q}_\Gamma\varphi)(\xi) = \begin{cases} 0 & \text{if } \xi \in \Gamma \\ \varphi(\xi) & \text{if } \xi \notin \Gamma \end{cases}.$$

In order to simplify the notation we will write $\mathcal{P}_a$ instead of $\mathcal{P}_{[a,+\infty[}$ for $a \in \mathbb{R}$.

Considering Ω a bounded interval, we will study operators of the form

$$W_{\mathsf{K},\Omega} : \left[L^p_\Omega(\mathbb{R})\right]^n \to \left[L^p_\Omega(\mathbb{R})\right]^n$$
$$W_{\mathsf{K},\Omega}\varphi = \mathcal{P}_\Omega(\mathcal{K} * \varphi), \qquad (1)$$

such that

$$W_{\mathsf{K},\Omega} = \mathcal{P}_\Omega \mathcal{F}^{-1}\mathsf{K}\cdot\mathcal{F},$$

where,

- $*$ denotes the convolution operation.
- $\mathcal{F} = \mathcal{F}_{x\mapsto\xi}$ denotes the Fourier transformation in $\mathbb{R}^n$ and $\mathcal{F}^{-1}$ its inverse.
- K is a $n \times n$ matrix valued function, whose components are distributions defined in the dual space of $\mathcal{S}(\mathbb{R})$, denoted by $\mathcal{S}'(\mathbb{R})$, i.e, the space of all linear and continuous operators in $\mathcal{S}(\mathbb{R})$. In this space, with the necessary adaptations, we define the Fourier transformation according to the usual definition for $\mathcal{S}(\mathbb{R})$.
- $\mathcal{F}^{-1}\mathsf{K} = \mathcal{K}$, namely, $\mathsf{K} = \mathcal{F}\mathcal{K}$, with K a $n \times n$ matrix belonging to the Fourier L^p-multiplier algebra (see [16]).

Note that these operators are called convolution type operators due to the fact that

$$\mathcal{P}_\Omega(\mathcal{K} * \varphi)(\xi) = \int_\Omega \mathcal{K}(\xi - x)\,\varphi(x)\,dx, \qquad \xi \in \Omega,$$

where $*$ is the convolution operation and $\mathcal{P}_\Omega$ denotes an operator restriction to Ω.

Analogously we define

$$W_{\mathsf{K},\mathbb{R}_+} : \left[L^p_+(\mathbb{R})\right]^n \to \left[L^p_+(\mathbb{R})\right]^n$$
$$W_{\mathsf{K},\mathbb{R}_+}\varphi = \mathcal{P}_{\mathbb{R}_+}(\mathcal{K} * \varphi), \qquad (2)$$

the usually called Wiener-Hopf operators (or, in related works, pseudodifferencial operators, [5, 16, 17, 24]) defined with projectors on the half-line.

Under this considerations, the linear operators $W_{\mathsf{K},\Omega}$ and $W_{\mathsf{K},\mathbb{R}_+}$ are well defined and are the both bounded operators.

To continue, we will need to consider the next important extension operator relations.

Definition 1 *[13] Let us consider two operators*

$$A : X_1 \to Y_1 \qquad \textit{and} \qquad B : X_2 \to Y_2,$$

acting between Banach spaces.

(i) The operators A and B are said to be algebraically equivalent after extension if there exist additional Banach spaces Z_1 and Z_2 and invertible linear operators

$$E : Y_2 \times Z_2 \to Y_1 \times Z_1$$

and

$$F : X_1 \times Z_1 \to X_2 \times Z_2$$

such that

$$\begin{bmatrix} A & 0 \\ 0 & I_{Z_1} \end{bmatrix} = E \begin{bmatrix} B & 0 \\ 0 & I_{Z_2} \end{bmatrix} F. \qquad (3)$$

(ii) If, in addition to (i), the invertible and linear operators E and F in (3) are bounded, then we will say that A and B are topologically equivalent after extension operators or simply say that A and B are equivalent after extension operators. We will denote the equivalence after extension relation between the operators by $A \overset{*}{\sim} B$, [1].

(iii) A and B are said to be equivalent operators, denoted by $A \sim B$, in the particular case when

$$A = E\,B\,F,$$

for some bounded invertible linear operators

$$E : Y_2 \to Y_1 \qquad \textit{and} \qquad F : X_1 \to X_2.$$

We remark that the equivalence between the above notion of topological equivalence after extension relation and the concept of *matricial coupling* was established for the first time in [1]. This last concept is well-known to be very important in solving certain classes of integral equations, and it is also important in (linear algebra) *matrix completion problems*. From the just presented notions it is clear that different consequences can be extracted from these different operator relations. We refer to [3], [4], [13], [21], [22] and [23] for a discussion on the differences between algebraic and topological equivalence after extension relations between convolution type operators and some applications.

2 Extension Method

In this section we mention some properties and characteristics of the operators defined in (1) and (2) for the construction of generalized inverses.

First we relate the convolution type operator $W_{\mathsf{K},\Omega}$ with a higher order Wiener-Hopf operator acting on the half-line, not without introducing some concepts needed for this purpose.

It is said that an operator A is a higher order operator if it consists of addition and/or composition of simple operators (in the sense that these simple operators not exhibit such additions and compositions), i.e., considering two Banach spaces X and Y, $A : X \to Y$ is a higher order operator if

$$A = \sum_{i=1}^{N} \prod_{j=1}^{M_i} a^{i,j}, \qquad N,\ M_i \in \mathbb{N}, \tag{4}$$

where $a^{i,j}$ denotes *simple operators* or also called *non-composite operators*.

Based on the iterative method presented in [12], we know that the extension by the identity of an operator of higher order can be written as a composition of matrices whose components are simple operators. We will address the method of [12] from a new perspective to the extent that will cause decomposition by blocks, i.e., considering A defined like in (4), we will have

$$\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} = EBF,$$

where E, B and F are matrices whose components are 0, I_X, I_Y or $a^{i,j}$, matricial operators. This last identity can sometimes be identified with the identity (3). So, this iterative method is useful to obtain this kind of relations between operators and has been described and used in various publications, e.g. [2], [6], [7]-[11], [14], [22], [23] and the references given there.

In this new approach, we consider the components of each matrix operator $a^{i,j} : X \to X$ as simple operators and $M_i \le 2$, $i = 1, 2, \dots N$. In the case of not having these conditions, we apply the method of [12] in each of the operators $a^{i,j}$, i.e, in each of the blocks. With this new approach, the iterative method is simpler, is best suited to our needs and leads to a more comprehensive construction of the relations between operators.

We will have the following iterative method.

Consider, for the iteration $r \in \mathbb{N}_0$,

$$A_r = \sum_{i=1}^{N} \prod_{j=1}^{M_i} a^{i,j} = \begin{cases} A & \text{if } r = 0 \\ \widetilde{B}_{r-1} & \text{if } r \ge 1 \end{cases}$$

and

$$\widetilde{B}_r = \begin{cases} \displaystyle\sum_{i=1}^{N-1} \prod_{j=1}^{M_i} a^{i,j} & \text{if } N \ge 2 \\ 0 & \text{if } N = 1 \end{cases},$$

where, after setting an ordering of terms in A_r, for example considering $M_1 \le M_2 \le \dots \le M_N$, we define

$$\alpha_r = \begin{cases} a^{N,1} & \text{if } M_N = 2 \\ I & \text{if } M_N = 1 \end{cases},$$

$$\beta_r = -a^{N,M_N}$$

and

$$B_{r+1} = \left[\begin{array}{c|ccccc} \widetilde{B}_r & \alpha_r & \alpha_{r-1} & \cdots & \alpha_1 & \alpha_0 \\ \hline \beta_r & I & 0 & \cdots & 0 & 0 \\ \beta_{r-1} & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_1 & 0 & 0 & \cdots & I & 0 \\ \beta_0 & 0 & 0 & \cdots & 0 & I \end{array}\right].$$

For each iteration the parameters M_N and N are related to the operator A_r and the iteration method ends when $\widetilde{B}_r$ is not a higher order operator.

One of the most important results in this iterative method is the fact that for each iteration we obtain a factorization into simpler structured factors, with invertible outer factors, in the form

$$\begin{bmatrix} A & 0 \\ 0 & I_{X^{r+1}} \end{bmatrix} = E_{r+1} B_{r+1} F_{r+1},$$

where,

$$E_{r+1} = \begin{bmatrix} I & -\alpha_r & \cdots & -\alpha_1 & -\alpha_0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ 0 & 0 & \cdots & 0 & I \end{bmatrix}$$

and

$$F_{r+1} = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ -\beta_r & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\beta_1 & 0 & \cdots & I & 0 \\ -\beta_0 & 0 & \cdots & 0 & I \end{bmatrix},$$

with $r \in \mathbb{N}_0$.

Example 2 *As example we will apply the iterative method to an operator of the form*

$$A = a^{1,1} + a^{2,1} + a^{3,1}a^{3,2}.$$

To $r = 0$, we have $A_0 = A$. Thus, $N = 3$, $M_1 = 1$, $M_2 = 1$ and $M_3 = 2$.

We obtain $\alpha_0 = a^{3,1}$, $\beta_0 = -a^{3,2}$ *and* $\widetilde{B}_0 = a^{1,1} + a^{2,1}$. *Therefore*

$$B_1 = \begin{bmatrix} \widetilde{B}_0 & \alpha_0 \\ \beta_0 & I \end{bmatrix} = \begin{bmatrix} a^{1,1} + a^{2,1} & a^{3,1} \\ -a^{3,2} & I \end{bmatrix},$$

$$E_1 = \begin{bmatrix} I & -\alpha_0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & -a^{3,1} \\ 0 & I \end{bmatrix}$$

and

$$F_1 = \begin{bmatrix} I & 0 \\ -\beta_0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ a^{3,2} & I \end{bmatrix}.$$

We obtain the equation

$$\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} = E_1 B_1 F_1.$$

The operator $\widetilde{B}_0$ *is not in the form of a simple operator so, we do one more iterations.*

To $r = 1$, *we have* $A_1 = \widetilde{B}_0 = a^{1,1} + a^{2,1}$. *Thus,* $N = 2$, $M_1 = 1$ *and* $M_2 = 1$. *So,* $\alpha_1 = I$, $\beta_1 = -a^{2,1}$ *and* $\widetilde{B}_1 = a^{1,1}$. *Therefore*

$$B_2 = \begin{bmatrix} \widetilde{B}_1 & \alpha_1 & \alpha_0 \\ \beta_1 & I & 0 \\ \beta_0 & 0 & I \end{bmatrix} = \begin{bmatrix} a^{1,1} & I & a^{3,1} \\ -a^{2,1} & I & 0 \\ -a^{3,2} & 0 & I \end{bmatrix},$$

$$E_2 = \begin{bmatrix} I & -\alpha_1 & -\alpha_0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & -I & -a^{3,1} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and

$$F_2 = \begin{bmatrix} I & 0 & 0 \\ -\beta_1 & I & 0 \\ -\beta_0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ a^{2,1} & I & 0 \\ a^{3,2} & 0 & I \end{bmatrix}.$$

We obtain the equation

$$\begin{bmatrix} A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = E_2 B_2 F_2.$$

Observation 3 *The iterative method presented here differs from the method presented in the work [12]. In [12],* A *is a matrix operator whose components are higher order operators. The iterative method will break down each of these components in matrices whose entries are simple operators. For this work, based on the scalar case method in [12], we consider* A *as a composition of matrix operators,* $a^{i,j}$, *whose entries are simple operators. Thus, the operators* $a^{i,j}$ *can be regarded as blocks that do not change.*

3 Equivalence Relations and Generalized Inverses

We will now apply some equivalence relations (see [4]) to our operator defined in (1) considering

$$\Omega =]0, a[, \tag{5}$$

where $0 < a < +\infty$, with $a \in \mathbb{R}$.

Theorem 4 *The convolution type operator* $W_{\mathsf{K},\Omega}$, *presented in (1) with* Ω *defined in (5) is equivalent after extension to the operator*

$$W : \left[L^p_+(\mathbb{R})\right]^n \to \left[L^p_+(\mathbb{R})\right]^n$$
$$W = W_{\mathsf{K},\mathbb{R}_+} - \mathcal{P}_a \left(W_{\mathsf{K},\mathbb{R}_+} - I\right). \tag{6}$$

Proof: It is crucial for our proof to be aware of the following direct sum decomposition,

$$\left[L^p_+(\mathbb{R})\right]^n = \mathcal{P}_\Omega \left[L^p_+(\mathbb{R})\right]^n \oplus \mathcal{Q}_\Omega \left[L^p_+(\mathbb{R})\right]^n = \left[L^p_\Omega(\mathbb{R})\right]^n \oplus \left[L^p_{\mathbb{R}_+\setminus\overline{\Omega}}(\mathbb{R})\right]^n.$$

In this context, taking into account the direct sum decomposition, we will define the new operator

$$\widetilde{W}_{\mathsf{K},\Omega} : \left[L^p_+(\mathbb{R})\right]^n \to \left[L^p_+(\mathbb{R})\right]^n$$
$$\widetilde{W}_{\mathsf{K},\Omega} = \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+} \mathcal{P}_\Omega + \mathcal{Q}_\Omega \tag{7}$$

in the form

$$\widetilde{W}_{\mathsf{K},\Omega} = \begin{bmatrix} W_{\mathsf{K},\Omega} & 0 \\ 0 & I_{\left[L^p_{\mathbb{R}_+\setminus\overline{\Omega}}(\mathbb{R})\right]^n} \end{bmatrix}.$$

We have obviously the operator $W_{\mathsf{K},\Omega}$ equivalent after extension to the operator $\widetilde{W}_{\mathsf{K},\Omega}$. Let us prove now that the operators $\widetilde{W}_{\mathsf{K},\Omega}$ and W are equivalent operators.

Note that,

$$\mathcal{P}_\Omega = I - \mathcal{P}_a \qquad \text{and} \qquad \mathcal{Q}_\Omega = \mathcal{P}_a$$

where

$$\mathcal{P}_a = \mathcal{P}_{\mathbb{R}_+}\mathcal{F}^{-1}\tau_a \cdot \mathcal{F}\mathcal{P}_{\mathbb{R}_+}\mathcal{F}^{-1}\tau_{-a} \cdot \mathcal{F},$$

with τ_a the function defined by $\tau_a(\xi) = e^{ia\xi}$, $\xi \in \mathbb{R}$.

Thus,

$$\begin{aligned} \widetilde{W}_{\mathsf{K},\Omega} &= \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{P}_\Omega + \mathcal{Q}_\Omega \\ &= \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}(I - \mathcal{Q}_\Omega) + \mathcal{Q}_\Omega \\ &= \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+} I - \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega + \mathcal{Q}_\Omega \\ &= \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+} I + \mathcal{Q}_\Omega - \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega \\ &= \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+} + \mathcal{Q}_\Omega - \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega\mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+} \\ &\quad - \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega\mathcal{Q}_\Omega \\ &= \left(I - \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega\right)\left(\mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+} + \mathcal{Q}_\Omega\right) \\ &= \left(I - \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega\right) W, \end{aligned} \tag{8}$$

since

$$\begin{aligned} W &= W_{\mathsf{K},\mathbb{R}_+} - \mathcal{P}_a(W_{\mathsf{K},\mathbb{R}_+} - I) \\ &= W_{\mathsf{K},\mathbb{R}_+} - \mathcal{P}_a W_{\mathsf{K},\mathbb{R}_+} + \mathcal{P}_a \\ &= (I - \mathcal{P}_a)W_{\mathsf{K},\mathbb{R}_+} + \mathcal{P}_a \\ &= \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+} + \mathcal{Q}_\Omega. \end{aligned}$$

In (8), we have

$$I - \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega : \left[L^p_+(\mathbb{R})\right]^n \to \left[L^p_+(\mathbb{R})\right]^n$$

and is an invertible and bounded operator where the inverse is defined by

$$I + \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega : \left[L^p_+(\mathbb{R})\right]^n \to \left[L^p_+(\mathbb{R})\right]^n,$$

since

$$\begin{aligned} (I - \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega)&(I + \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega) \\ &= I + \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega - \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega \\ &\quad - \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega\mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega \\ &= I, \end{aligned}$$

because $\mathcal{Q}_\Omega\mathcal{P}_\Omega = 0$. Likewise also

$$(I + \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega)(I - \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega) = I.$$

So we have $\widetilde{W}_{\mathsf{K},\Omega}$ equivalent to W.

Is very easy prove that the equivalence after extension and the equivalence are equivalence relations. Thus, by transitivity, if $W_{\mathsf{K},\Omega} \overset{*}{\sim} \widetilde{W}_{\mathsf{K},\Omega}$ and $\widetilde{W}_{\mathsf{K},\Omega} \sim W$, thus $W_{\mathsf{K},\Omega} \overset{*}{\sim} W$. □

Now we introduce the result considering

$$\Omega =]0, a[\cup]b, c[, \tag{9}$$

where $0 < a < b < c < +\infty$, with $a, b, c \in \mathbb{R}$.

Theorem 5 *The convolution type operator $W_{\mathsf{K},\Omega}$, presented in (1) with Ω defined in (9) is equivalent after extension to the operator*

$$\begin{aligned} &W : \left[L^p_+(\mathbb{R})\right]^n \to \left[L^p_+(\mathbb{R})\right]^n \\ &W = W_{\mathsf{K},\mathbb{R}_+} - \mathcal{P}_a\left(W_{\mathsf{K},\mathbb{R}_+} - I\right) + \\ &\qquad \mathcal{P}_b\left(W_{\mathsf{K},\mathbb{R}_+} - I\right) - \mathcal{P}_c\left(W_{\mathsf{K},\mathbb{R}_+} - I\right). \end{aligned} \tag{10}$$

Proof: Proceeding similarly to the proof of the previous theorem, we have $W_{\mathsf{K},\Omega}$ equivalent after extension to the operator $\widetilde{W}_{\mathsf{K},\Omega}$ defined in (7). With

$$\mathcal{P}_\Omega = I - \mathcal{P}_a + \mathcal{P}_b - \mathcal{P}_c$$

and

$$\mathcal{Q}_\Omega = \mathcal{P}_a - \mathcal{P}_b + \mathcal{P}_c,$$

like in (8), we obtain

$$\widetilde{W}_{\mathsf{K},\Omega} = \left(I - \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega\right) W,$$

since

$$\begin{aligned} W &= W_{\mathsf{K},\mathbb{R}_+} - \mathcal{P}_a\left(W_{\mathsf{K},\mathbb{R}_+} - I\right) + \\ &\qquad \mathcal{P}_b\left(W_{\mathsf{K},\mathbb{R}_+} - I\right) - \mathcal{P}_c\left(W_{\mathsf{K},\mathbb{R}_+} - I\right) \\ &= W_{\mathsf{K},\mathbb{R}_+} - \mathcal{P}_a W_{\mathsf{K},\mathbb{R}_+} + \mathcal{P}_a + \mathcal{P}_b W_{\mathsf{K},\mathbb{R}_+} - \\ &\qquad \mathcal{P}_b - \mathcal{P}_c W_{\mathsf{K},\mathbb{R}_+} + \mathcal{P}_c \\ &= (I - \mathcal{P}_a + \mathcal{P}_b - \mathcal{P}_c)W_{\mathsf{K},\mathbb{R}_+} + \mathcal{P}_a - \mathcal{P}_b + \mathcal{P}_c \\ &= \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+} + \mathcal{Q}_\Omega, \end{aligned}$$

with $I - \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega$ an invertible and bounded operator. So $\widetilde{W}_{\mathsf{K},\Omega}$ is equivalent to W.

So, $W_{\mathsf{K},\Omega} \overset{*}{\sim} \widetilde{W}_{\mathsf{K},\Omega}$ and $\widetilde{W}_{\mathsf{K},\Omega} \sim W$ imply the assertion of the theorem. □

We will introduce now the concept of generalized inverse. This concept plays an important role in the study of linear models and in other applications (see [15, 18, 19, 20]).

Recall that L^- denotes a generalized inverse of a bounded linear operator L, acting between Banach spaces, if $LL^-L = L$.

We will now see that the generalized inverse of the operator W, defined in the two previous theorems, generates a generalized inverse to the convolution type operator $W_{\mathsf{K},\Omega}$ defined as in (1).

Theorem 6 *If W^- is a generalized inverse of the operator W defined in (6) (or defined in (10)) then*

$$W^-_{\mathsf{K},\Omega} : \left[L^p_\Omega(\mathbb{R})\right]^n \to \left[L^p_\Omega(\mathbb{R})\right]^n$$

$$W^-_{\mathsf{K},\Omega} = \mathcal{P}_\Omega W^-_{|\mathcal{P}_\Omega\left[L^p_+(\mathbb{R})\right]^n}$$

is a generalized inverse of $W_{\mathsf{K},\Omega}$ with Ω defined by (5) (or with Ω defined by (9)).

Proof: Suppose that W^- is a generalized inverse of W.

By the relation (8), immediately yields a generalized inverse of $\widetilde{W}_{\mathsf{K},\Omega}$. Namely,

$$\widetilde{W}^-_{\mathsf{K},\Omega} = W^-\left(I + \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega\right). \tag{11}$$

On the other hand, writing

$$\widetilde{W}_{\mathsf{K},\Omega}\widetilde{W}^-_{\mathsf{K},\Omega}\widetilde{W}_{\mathsf{K},\Omega} = \widetilde{W}_{\mathsf{K},\Omega}$$

in matrix form, we have

$$\widetilde{W}_{\mathsf{K},\Omega} = \begin{bmatrix} W_{\mathsf{K},\Omega} & 0 \\ 0 & I_{\mathcal{Q}_\Omega\left[L^p_+(\mathbb{R})\right]^n} \end{bmatrix}$$

and

$$\widetilde{W}^-_{\mathsf{K},\Omega} = \begin{bmatrix} \widetilde{W}^-_{\mathsf{K},\Omega,1,1} & \widetilde{W}^-_{\mathsf{K},\Omega,1,2} \\ \widetilde{W}^-_{\mathsf{K},\Omega,2,1} & \widetilde{W}^-_{\mathsf{K},\Omega,2,2} \end{bmatrix},$$

with

$$\widetilde{W}^-_{\mathsf{K},\Omega,1,1} = \mathcal{P}_\Omega\left(\widetilde{W}^-_{\mathsf{K},\Omega}\right)_{|\mathcal{P}_\Omega\left[L^p_+(\mathbb{R})\right]^n},$$

$$\widetilde{W}^-_{\mathsf{K},\Omega,1,2} = \mathcal{P}_\Omega\left(\widetilde{W}^-_{\mathsf{K},\Omega}\right)_{|\mathcal{Q}_\Omega\left[L^p_+(\mathbb{R})\right]^n},$$

$$\widetilde{W}^-_{\mathsf{K},\Omega,2,1} = \mathcal{Q}_\Omega\left(\widetilde{W}^-_{\mathsf{K},\Omega}\right)_{|\mathcal{P}_\Omega\left[L^p_+(\mathbb{R})\right]^n}$$

and

$$\widetilde{W}^-_{\mathsf{K},\Omega,2,2} = \mathcal{Q}_\Omega\left(\widetilde{W}^-_{\mathsf{K},\Omega}\right)_{|\mathcal{Q}_\Omega\left[L^p_+(\mathbb{R})\right]^n}.$$

Thus, we obtain,

$$W_{\mathsf{K},\Omega}\left[\mathcal{P}_\Omega\left(\widetilde{W}^-_{\mathsf{K},\Omega}\right)_{|\mathcal{P}_\Omega\left[L^p_+(\mathbb{R})\right]^n}\right]W_{\mathsf{K},\Omega} = W_{\mathsf{K},\Omega}. \tag{12}$$

Applying (11) in the identity (12), we obtain the following generalized inverse

$$W^-_{\mathsf{K},\Omega} : \left[L^p_\Omega(\mathbb{R})\right]^n \to \left[L^p_\Omega(\mathbb{R})\right]^n$$

$$\begin{aligned} W^-_{\mathsf{K},\Omega} &= \mathcal{P}_\Omega\left[W^-\left(I + \mathcal{P}_\Omega W_{\mathsf{K},\mathbb{R}_+}\mathcal{Q}_\Omega\right)\right]_{|\mathcal{P}_\Omega\left[L^p_+(\mathbb{R})\right]^n} \\ &= \mathcal{P}_\Omega W^-_{|\mathcal{P}_\Omega\left[L^p_+(\mathbb{R})\right]^n}. \end{aligned}$$

□

We are able to apply the iterative method to our operators.

Theorem 7 *The operator W defined in (6) is equivalent after extension to*

$$\mathcal{W} : [L^p_+(\mathbb{R})]^{3n} \longrightarrow [L^p_+(\mathbb{R})]^{3n}$$

$$\mathcal{W} = \begin{bmatrix} W_{\mathsf{K},\mathbb{R}_+} & I & -\mathcal{P}_a \\ -\mathcal{P}_a & I & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & I \end{bmatrix}. \tag{13}$$

Proof: In order to apply the iterative method presented in the previous section for the decomposition of higher order Wiener-Hopf operators, we identify W by

$$\begin{aligned} W &= W_{\mathsf{K},\mathbb{R}_+} - \mathcal{P}_a(W_{\mathsf{K},\mathbb{R}_+} - I) \\ &= W_{\mathsf{K},\mathbb{R}_+} + \mathcal{P}_a + (-\mathcal{P}_a)\,W_{\mathsf{K},\mathbb{R}_+} \\ &= a^{1,1} + a^{2,1} + a^{3,1}a^{3,2}. \end{aligned}$$

We have $N = 3$, $M_1 = 1$, $M_2 = 1$ and $M_3 = 2$.

Applying the iterative method and considering the Example 2, we have, to $r = 0$, $A_0 = W$,

$$B_1 = \begin{bmatrix} \widetilde{B}_0 & \alpha_0 \\ \beta_0 & I \end{bmatrix},$$

where

$$\begin{aligned} \alpha_0 &= a^{3,1} \\ &= -\mathcal{P}_a, \\ \beta_0 &= -a^{3,2} \\ &= -W_{\mathsf{K},\mathbb{R}_+} \end{aligned}$$

and

$$\begin{aligned}\widetilde{B}_0 &= \sum_{i=1}^{2}\prod_{j=1}^{M_i} a^{i,j} \\ &= \prod_{j=1}^{M_1} a^{1,j} + \prod_{j=1}^{M_2} a^{2,j} \\ &= a^{1,1} + a^{2,1} \\ &= W_{\mathsf{K},\mathbb{R}_+} + \mathcal{P}_a.\end{aligned}$$

The operator $\widetilde{B}_0$ is not a simple operator so we do one more iterations.

To $r = 1$, we have $A_1 = \widetilde{B}_0$. Thus, $N = 2$, $M_1 = 1$ and $M_2 = 1$. So, $\alpha_1 = I$, $\beta_1 = -a^{2,1}$ and $\widetilde{B}_1 = a^{1,1}$. Therefore

$$\begin{aligned}B_2 &= \begin{bmatrix} \widetilde{B}_1 & \alpha_1 & \alpha_0 \\ \beta_1 & I & 0 \\ \beta_0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} W_{\mathsf{K},\mathbb{R}_+} & I & \mathcal{P}_a \\ -\mathcal{P}_a & I & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & I \end{bmatrix} \\ &= \mathcal{W}, \\ E_2 &= \begin{bmatrix} I & -\alpha_1 & -\alpha_0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} I & -I & \mathcal{P}_a \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \qquad (14)\end{aligned}$$

and

$$\begin{aligned}F_2 &= \begin{bmatrix} I & 0 & 0 \\ -\beta_1 & I & 0 \\ -\beta_0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} I & 0 & 0 \\ \mathcal{P}_a & I & 0 \\ W_{\mathsf{K},\mathbb{R}_+} & 0 & I \end{bmatrix} \qquad (15)\end{aligned}$$

with E_2 and F_2 two invertible bounded linear operators which allows us to write the equivalence after extension relation in the explicit way

$$\begin{bmatrix} W & 0 & 0 \\ 0 & I_{\mathcal{P}_\Omega\left[L_+^p(\mathbb{R})\right]^n} & 0 \\ 0 & 0 & I_{\mathcal{P}_\Omega\left[L_+^p(\mathbb{R})\right]^n} \end{bmatrix} = E_2 \mathcal{W} F_2.$$

□

Theorem 8 *The operator W defined in* (10) *is equivalent after extension to*

$$\mathcal{W} : [L_+^p(\mathbb{R})]^{7n} \longrightarrow [L_+^p(\mathbb{R})]^{7n}$$

$$\mathcal{W} = \begin{bmatrix} W_{\mathsf{K},\mathbb{R}_+} & I & I & I & -\mathcal{P}_a & \mathcal{P}_b & -\mathcal{P}_c \\ -\mathcal{P}_a & I & 0 & 0 & 0 & 0 & 0 \\ \mathcal{P}_b & 0 & I & 0 & 0 & 0 & 0 \\ -\mathcal{P}_c & 0 & 0 & I & 0 & 0 & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & 0 & I & 0 & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & 0 & 0 & I & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}. \qquad (16)$$

Proof: In order to apply the iterative method we identify W by

$$\begin{aligned}W &= W_{\mathsf{K},\mathbb{R}_+} - \mathcal{P}_a\left(W_{\mathsf{K},\mathbb{R}_+} - I\right) + \\ &\qquad \mathcal{P}_b\left(W_{\mathsf{K},\mathbb{R}_+} - I\right) - \mathcal{P}_c\left(W_{\mathsf{K},\mathbb{R}_+} - I\right) \\ &= W_{\mathsf{K},\mathbb{R}_+} - \mathcal{P}_a W_{\mathsf{K},\mathbb{R}_+} + \mathcal{P}_a + \\ &\qquad \mathcal{P}_b W_{\mathsf{K},\mathbb{R}_+} - \mathcal{P}_b - \mathcal{P}_c W_{\mathsf{K},\mathbb{R}_+} + \mathcal{P}_c \\ &= W_{\mathsf{K},\mathbb{R}_+} + \mathcal{P}_a - \mathcal{P}_b + \mathcal{P}_c + (-\mathcal{P}_a)W_{\mathsf{K},\mathbb{R}_+} + \\ &\qquad \mathcal{P}_b W_{\mathsf{K},\mathbb{R}_+} + (-\mathcal{P}_c)W_{\mathsf{K},\mathbb{R}_+} \\ &= a^{1,1} + a^{2,1} + a^{3,1} + a^{4,1} + \\ &\qquad a^{5,1}a^{5,2} + a^{6,1}a^{6,2} + a^{7,1}a^{7,2}.\end{aligned}$$

To $r = 0$, we have $A_0 = W$. Thus, $N = 7$, $M_1 = M_2 = M_3 = M_4 = 1$ and $M_5 = M_6 = M_7 = 2$. So, applying the iterative method we obtain

$$\begin{aligned}B_1 &= \begin{bmatrix} \widetilde{B}_0 & -\mathcal{P}_c \\ -W_{\mathsf{K},\mathbb{R}_+} & I \end{bmatrix}, \\ E_1 &= \begin{bmatrix} I & \mathcal{P}_c \\ 0 & I \end{bmatrix}\end{aligned}$$

and

$$F_1 = \begin{bmatrix} I & 0 \\ W_{\mathsf{K},\mathbb{R}_+} & I \end{bmatrix}$$

with

$$\begin{aligned}\widetilde{B}_0 &= W_{\mathsf{K},\mathbb{R}_+} + \mathcal{P}_a - \mathcal{P}_b + \mathcal{P}_c + (-\mathcal{P}_a)W_{\mathsf{K},\mathbb{R}_+} + \\ &\qquad \mathcal{P}_b W_{\mathsf{K},\mathbb{R}_+}.\end{aligned}$$

We obtain

$$\begin{bmatrix} W & 0 \\ 0 & I_{\mathcal{P}_\Omega\left[L_+^p(\mathbb{R})\right]^n} \end{bmatrix} = E_1 B_1 F_1.$$

The element $\widetilde{B}_0$ is not a simple operator so, we continue the iterative method until $\widetilde{B}_r$ to be a simple element.

To $r = 1$, we have $A_1 = \widetilde{B}_0$. Thus, $N = 6$, $M_1 = M_2 = M_3 = M_4 = 1$ and $M_5 = M_6 = 2$. Therefore

$$B_2 = \begin{bmatrix} \widetilde{B}_1 & \mathcal{P}_b & -\mathcal{P}_c \\ -W_{\mathsf{K},\mathbb{R}_+} & I & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & I \end{bmatrix},$$

$$E_2 = \begin{bmatrix} I & -\mathcal{P}_b & \mathcal{P}_c \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and

$$F_2 = \begin{bmatrix} I & 0 & 0 \\ W_{\mathsf{K},\mathbb{R}_+} & I & 0 \\ W_{\mathsf{K},\mathbb{R}_+} & 0 & I \end{bmatrix}$$

with

$$\widetilde{B}_1 = W_{\mathsf{K},\mathbb{R}_+} + \mathcal{P}_a - \mathcal{P}_b + \mathcal{P}_c + (-\mathcal{P}_a)W_{\mathsf{K},\mathbb{R}_+}.$$

We obtain

$$\begin{bmatrix} W & 0 \\ 0 & I_{\mathcal{P}_\Omega\left[L^p_+(\mathbb{R})\right]^{2n}} \end{bmatrix} = E_2 B_2 F_2.$$

To $r = 2$, we have $A_2 = \widetilde{B}_1$. Thus, $N = 5$, $M_1 = M_2 = M_3 = M_4 = 1$ and $M_5 = 2$. Therefore

$$B_3 = \begin{bmatrix} \widetilde{B}_2 & -\mathcal{P}_a & \mathcal{P}_b & -\mathcal{P}_c \\ -W_{\mathsf{K},\mathbb{R}_+} & I & 0 & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & I & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & I \end{bmatrix},$$

$$E_3 = \begin{bmatrix} I & \mathcal{P}_a & -\mathcal{P}_b & \mathcal{P}_c \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and

$$F_3 = \begin{bmatrix} I & 0 & 0 & 0 \\ W_{\mathsf{K},\mathbb{R}_+} & I & 0 & 0 \\ W_{\mathsf{K},\mathbb{R}_+} & 0 & I & 0 \\ W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & I \end{bmatrix}$$

with

$$\widetilde{B}_2 = W_{\mathsf{K},\mathbb{R}_+} + \mathcal{P}_a - \mathcal{P}_b + \mathcal{P}_c.$$

We obtain

$$\begin{bmatrix} W & 0 \\ 0 & I_{\mathcal{P}_\Omega\left[L^p_+(\mathbb{R})\right]^{3n}} \end{bmatrix} = E_3 B_3 F_3.$$

To $r = 3$, we have $A_1 = \widetilde{B}_2$. Thus, $N = 4$ and $M_1 = M_2 = M_3 = M_4 = 1$. Therefore

$$B_4 = \begin{bmatrix} \widetilde{B}_3 & I & -\mathcal{P}_a & \mathcal{P}_b & -\mathcal{P}_c \\ -\mathcal{P}_c & I & 0 & 0 & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & I & 0 & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & I & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & 0 & I \end{bmatrix},$$

$$E_4 = \begin{bmatrix} I & -I & \mathcal{P}_a & -\mathcal{P}_b & \mathcal{P}_c \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

and

$$F_4 = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ \mathcal{P}_c & I & 0 & 0 & 0 \\ W_{\mathsf{K},\mathbb{R}_+} & 0 & I & 0 & 0 \\ W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & I & 0 \\ W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & 0 & I \end{bmatrix}$$

with

$$\widetilde{B}_3 = W_{\mathsf{K},\mathbb{R}_+} + \mathcal{P}_a - \mathcal{P}_b.$$

We obtain

$$\begin{bmatrix} W & 0 \\ 0 & I_{\mathcal{P}_\Omega\left[L^p_+(\mathbb{R})\right]^{4n}} \end{bmatrix} = E_4 B_4 F_4.$$

To $r = 4$, we have $A_4 = \widetilde{B}_3$. Thus, $N = 3$ and $M_1 = M_2 = M_3 = 1$. Therefore

$$B_5 = \begin{bmatrix} \widetilde{B}_4 & I & I & -\mathcal{P}_a & \mathcal{P}_b & -\mathcal{P}_c \\ \mathcal{P}_b & I & 0 & 0 & 0 & 0 \\ -\mathcal{P}_c & 0 & I & 0 & 0 & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & I & 0 & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & 0 & I & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & 0 & 0 & I \end{bmatrix},$$

$$E_5 = \begin{bmatrix} I & -I & -I & \mathcal{P}_a & -\mathcal{P}_b & \mathcal{P}_c \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

and

$$F_5 = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ -\mathcal{P}_b & I & 0 & 0 & 0 & 0 \\ \mathcal{P}_c & 0 & I & 0 & 0 & 0 \\ W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & I & 0 & 0 \\ W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & 0 & I & 0 \\ W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

with

$$\widetilde{B}_4 = W_{\mathsf{K},\mathbb{R}_+} + \mathcal{P}_a.$$

We obtain

$$\begin{bmatrix} W & 0 \\ 0 & I_{\mathcal{P}_\Omega\left[L_+^p(\mathbb{R})\right]^{5n}} \end{bmatrix} = E_5 B_5 F_5.$$

And finally, the last iteration, to $r = 5$ we have $A_5 = \widetilde{B}_4$. Thus, $N = 2$ and $M_1 = M_2 = 1$. Therefore

$$B_6 = \begin{bmatrix} W_{\mathsf{K},\mathbb{R}_+} & I & I & I & -\mathcal{P}_a & \mathcal{P}_b & -\mathcal{P}_c \\ -\mathcal{P}_a & I & 0 & 0 & 0 & 0 & 0 \\ \mathcal{P}_b & 0 & I & 0 & 0 & 0 & 0 \\ -\mathcal{P}_c & 0 & 0 & I & 0 & 0 & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & 0 & I & 0 & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & 0 & 0 & I & 0 \\ -W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},$$

$$E_6 = \begin{bmatrix} I & -I & -I & -I & \mathcal{P}_a & -\mathcal{P}_b & \mathcal{P}_c \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \quad (17)$$

and

$$F_6 = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathcal{P}_a & I & 0 & 0 & 0 & 0 & 0 \\ -\mathcal{P}_b & 0 & I & 0 & 0 & 0 & 0 \\ \mathcal{P}_c & 0 & 0 & I & 0 & 0 & 0 \\ W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & 0 & I & 0 & 0 \\ W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & 0 & 0 & I & 0 \\ W_{\mathsf{K},\mathbb{R}_+} & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}. \quad (18)$$

The matricial operators E_6 and F_6 are two invertible bounded linear operators which allows us to write the equivalence after extension relation in the explicit way

$$\begin{bmatrix} W & 0 \\ 0 & I_{\mathcal{P}_\Omega\left[L_+^p(\mathbb{R})\right]^{6n}} \end{bmatrix} = E_6 \mathcal{W} F_6.$$

□

The next statements are a direct consequence of Theorem 4 and Theorem 7 and Theorem 5 and Theorem 8 respectively, since the relation of equivalence after extension enjoys the transitive property.

Corollary 9 *The operator $W_{\mathsf{K},\Omega}$ defined in (1) with Ω defined in (5) is equivalent after extension to the operator $\mathcal{W}$ defined in* (13).

Corollary 10 *The operator $W_{\mathsf{K},\Omega}$ defined in (1) with Ω defined in (9) is equivalent after extension to the operator $\mathcal{W}$ defined in* (16).

Finally, we present a generalized inverse of $W_{\mathsf{K},\Omega}$ in terms of a generalized inverse of our Wiener-Hopf operators $\mathcal{W}$.

Let consider

$$(A)_{nn} : \left[L_+^p(\mathbb{R})\right]^n \to \left[L_+^p(\mathbb{R})\right]^n$$

as the restriction of the first $n \times n$ components of the operator

$$A : \left[L_+^p(\mathbb{R})\right]^{n\times\cdots\times n} \to \left[L_+^p(\mathbb{R})\right]^{n\times\cdots\times n},$$

in terms of its matrix representation.

Theorem 11 *If $\mathcal{W}^-$ is a generalized inverse of the operator $\mathcal{W}$ defined in* (13), *then*

$$W_{\mathsf{K},\Omega}^- : \left[L_\Omega^p(\mathbb{R})\right]^n \to \left[L_\Omega^p(\mathbb{R})\right]^n$$

$$W_{\mathsf{K},\Omega}^- = \mathcal{P}_\Omega \left[\left(F_2^{-1}\mathcal{W}^- E_2^{-1}\right)_{nn}\right]_{|\mathcal{P}_\Omega\left[L_+^p(\mathbb{R})\right]^n}$$

is a generalized inverse of the convolution type operator $W_{\mathsf{K},\Omega}$ with Ω defined by (5), where E_2^{-1} and F_2^{-1} are the inverses of E_2 and F_2 presented in (14) *and* (15), *respectively.*

Proof: From the proof of Theorem 7 we know that

$$\begin{bmatrix} W & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = E_2 \mathcal{W} F_2.$$

Since E_2 and F_2 are invertible operators, we have

$$\mathcal{W} = E_2^{-1} \begin{bmatrix} W & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} F_2^{-1}. \quad (19)$$

Therefore, we can define a generalized inverse of $\mathcal{W}$ in the form

$$\mathcal{W}^- = F_2 \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix} E_2. \quad (20)$$

Consequently, using (19) and (20) in

$$\mathcal{W}\mathcal{W}^-\mathcal{W} = \mathcal{W},$$

we obtain

$$\begin{bmatrix} WA_{1,1}W & WA_{1,2} & WA_{1,3} \\ A_{2,1}W & A_{2,2} & A_{2,3} \\ A_{3,1}W & A_{3,2} & A_{3,3} \end{bmatrix} = \begin{bmatrix} W & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

From (20) we can also conclude that

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix} = F_2^{-1}\mathcal{W}^- E_2^{-1},$$

and thus,

$$A_{1,1} = (F_2^{-1}\mathcal{W}^- E_2^{-1})_{nn}.$$

Then, we have $W^- = (F_2^{-1}\mathcal{W}^- F_2^{-1})_{nn}$.
Applying Theorem 6 we get the final result

$$\begin{aligned} W^-_{\mathsf{K},\Omega} &= \mathcal{P}_\Omega W^-_{|\mathcal{P}_\Omega\left[L^p_+(\mathbb{R})\right]^n} \\ &= \mathcal{P}_\Omega \left[\left(F_2^{-1}\mathcal{W}^- E_2^{-1}\right)_{nn}\right]_{|\mathcal{P}_\Omega\left[L^p_+(\mathbb{R})\right]^n} . \end{aligned}$$

□

Theorem 12 *If $\mathcal{W}^-$ is a generalized inverse of the operator $\mathcal{W}$ defined in* (16)*, then*

$$W^-_{\mathsf{K},\Omega} : \left[L^p_\Omega(\mathbb{R})\right]^n \to \left[L^p_\Omega(\mathbb{R})\right]^n$$

$$W^-_{\mathsf{K},\Omega} = \mathcal{P}_\Omega \left[\left(F_6^{-1}\mathcal{W}^- E_6^{-1}\right)_{nn}\right]_{|\mathcal{P}_\Omega\left[L^p_+(\mathbb{R})\right]^n}$$

is a generalized inverse of the convolution type operator $W_{\mathsf{K},\Omega}$ with Ω defined by (9), where E_6^{-1} and F_6^{-1} are the inverses of E_6 and F_6 presented in (17) *and* (18)*, respectively.*

The proof is omitted because is similar to that of Theorem 11.

Before finalizing the present work, let us say that is clear that we can apply the present methods to show the existence of generalized inverses for operators defined in other geometries of Ω. Our finite interval Ω can be changed by the union of one finite interval and one infinite interval or by the finite union of several finite intervals. We plan to present a generalization in a future work.

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