Exact solutions for fractional partial differential equations by an extended fractional Riccati sub-equation method

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Abstract: In this paper, based on the fractional Riccati equation, we propose an extended fractional Riccati sub-equation method for solving fractional partial differential equations. The fractional derivative is defined in the sense of the modified Riemann-Liouville derivative. By a proposed variable transformation, certain fractional partial differential equations are turned into fractional ordinary differential equations, whose solutions can be expressed in certain forms composed of the solutions of the fractional Riccati equation. As for applications of this method, we apply it to the space-time fractional Whitham-Broer-Kaup (WBK) equations and the space-time fractional Fokas equation. With the aid of the mathematical software Maple, some new exact solutions for the two equations are successfully obtained.

Key–Words: Fractional Riccati sub-equation method; Fractional partial differential equations; Exact solutions; Modified Riemann-Liouville derivative; Fractional Whitham-Broer-Kaup (WBK) equations; Fractional Fokas equation

1 Introduction

Fractional differential equations are generalizations of classical differential equations of integer order. In the last few decades, fractional differential equations have gained much attention as they are widely used to describe various complex phenomena in many fields such as the fluid flow, signal processing, control theory, systems identification, biology and other areas. Among the investigations for fractional differential equations, research for seeking exact solutions and approximate solutions of fractional differential equations is a hot topic. Many powerful and efficient methods have been proposed so far. For example, the fractional variational iteration method [1-5], the Adomian's decomposition method [6-8], the homotopy perturbation method [9-12], the Exp-function method [13], the finite difference method [14], the finite element method [15], the (G'/G)-expansion method [16-18] and so on. Using these methods, solutions with various forms for some given fractional differential equations have been established.

Recently, Zhang et al. [19] first proposed a new direct algebraic method named fractional sub-equation method for solving fractional partial differential equations (FPDEs) based on the homogeneous balance principle, modified Riemann-Liouville derivative by Jumarie [20-23], and the fractional Riccati equation. The main idea of this method lies in that the solutions of certain FPDEs are supposed to have the form $u(\xi) = \sum_{i=0}^{n} a_i\phi^i$, where $\phi = \phi(\xi)$ satisfies the fractional Riccati equation $D_\xi^\alpha\phi = \sigma + \phi^2$, and $D_\xi^\alpha\phi(\xi)$ denotes the modified Riemann-Liouville derivative of order α for $\phi(\xi)$ with respect to ξ. With the aid of mathematical software, the authors established successfully new exact solutions for some FPDEs. Then in [24,25], the authors improved this method to be suitable for more general cases, in which the solutions of certain FPDEs are supposed to have the forms $u(\xi) = \sum_{i=-n}^{n} a_i\phi^i$ and $u(\xi) = a_0 + \sum_{i=1}^{n} a_i(\frac{-\sigma B + D\phi}{D + B\phi})^i$ respectively. In [26,27], the authors Zheng et al. proposed one new fractional sub-equation method, which can be seen as the fractional version of the known (G'/G) method [28-33].

Motivated by the works above, in this paper, we propose an extended fractional sub-equation method to establish exact solutions for

FPDEs in the sense of the modified Riemann-Liouville derivative, in which the solutions of certain FPDEs are supposed to have the forms $u(\xi) = a_0 + \sum_{i=1}^{m}[a_i G^i + b_i G^{i-1}\sqrt{\sigma + G^2}]$, where $G = G(\xi)$ satisfies the fractional Riccati equation

$$D_\xi^\alpha G(\xi) = \sigma + G^2(\xi), \tag{1}$$

and $D_\xi^\alpha G(\xi)$ denotes the modified Riemann-Liouville derivative of order α for $G(\xi)$ with respect to ξ.

The definition and some important properties of the Jumarie's modified Riemann-Liouville derivative of order α are listed as follows (see also in [19,20,24-27])

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, \\ \qquad 0 < \alpha < 1, \\ (f^{(n)}(t))^{(\alpha-n)}, \ n \le \alpha < n+1, \ n \ge 1. \end{cases}$$

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \tag{2}$$

$$D_t^\alpha (f(t)g(t)) = g(t) D_t^\alpha f(t) + f(t) D_t^\alpha g(t), \tag{3}$$

$$\begin{aligned} D_t^\alpha f[g(t)] &= f_g'[g(t)] D_t^\alpha g(t) \\ &= D_g^\alpha f[g(t)](g'(t))^\alpha. \end{aligned} \tag{4}$$

We organize this paper as follows. In Section 2, we give the description of the extended fractional Riccati sub-equation method for solving FPDEs. Then in Section 3 we apply this method to establish exact solutions for the space-time fractional Whitham-Broer-Kaup (WBK) equations and the space-time fractional Fokas equation. Some conclusions are presented at the end of the paper.

2 Description of the extended fractional Riccati sub-equation method

In this section we describe the main steps of the extended fractional Riccati sub-equation method for finding exact solutions for FPDEs.

Suppose that a fractional partial differential equation is given by

$$\begin{aligned} &P(u_1, ...u_k, D_t^\alpha u_1, ..., D_t^\alpha u_k, D_{x_1}^\alpha u_1, ..., D_{x_1}^\alpha u_k, ..., \\ &D_{x_n}^\alpha u_1, ..., D_{x_n}^\alpha u_k, D_t^{2\alpha} u_1, ..., D_t^{2\alpha} u_k, D_{x_1}^{2\alpha} u_1, ...) \\ &= 0, \end{aligned} \tag{5}$$

where $t, x_1, x_2, ..., x_n$ are independent variables, $u_i = u_i(t, x_1, x_2, ..., x_n)$, $i = 1, ..., k$ are unknown functions, P is a polynomial in u_i and their various fractional derivatives.

Step 1. Suppose that

$$\begin{aligned} &u_i(t, x_1, x_2, ..., x_n) = U_i(\xi), \quad i = 1, 2, ..., k; \\ &\xi = ct + k_1 x_1 + k_2 x_2 + ... + k_n x_n + \xi_0. \end{aligned} \tag{6}$$

Then by the second equality in Eq. (4), Eq. (5) can be turned into the following fractional ordinary differential equation with respect to the variable ξ:

$$\begin{aligned} &\widetilde{P}(U_1, ..., U_k, c^\alpha D_\xi^\alpha U_1, ..., c^\alpha D_\xi^\alpha U_k, k_1^\alpha D_\xi^\alpha U_1, \\ &..., k_1^\alpha D_\xi^\alpha U_k, ..., k_n^\alpha D_\xi^\alpha U_1, ..., k_n^\alpha D_\xi^\alpha U_k, c^{2\alpha} D_\xi^{2\alpha} U_1, \\ &..., c^{2\alpha} D_\xi^{2\alpha} U_k, k_1^{2\alpha} D_\xi^{2\alpha} U_1, ...) = 0. \end{aligned} \tag{7}$$

Step 2. Suppose that the solutions of (7) can be expressed by a polynomial in G as follows:

$$\begin{aligned} &U_j(\xi) = a_{j,0} + \sum_{i=1}^{m_j}[a_{j,i} G^i + b_{j,i} G^{i-1}\sqrt{\sigma + G^2}], \\ &j = 1, ..., k \end{aligned} \tag{8}$$

where $G = G(\xi)$ satisfies Eq. (1), and $a_{j,i}$, $i = 0, 1, ..., m$, $j = 1, 2, ..., k$ are constants to be determined later. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (7).

In [34], by using the generalized Exp-function method, Zhang et al. first obtained the following solutions for Eq. (1):

$$G(\xi) = \begin{cases} -\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\xi), & \sigma < 0, \\ -\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\xi), & \sigma < 0, \\ \sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\xi), & \sigma > 0, \\ -\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\xi), & \sigma > 0, \\ -\frac{\Gamma(1+\alpha)}{\xi^\alpha + \omega}, & \sigma = 0, \omega = const, \end{cases} \tag{9}$$

where the generalized hyperbolic and trigonometric functions are defined as

$$\sin_\alpha(\xi) = \frac{E_\alpha(i\xi^\alpha) - E_\alpha(-i\xi^\alpha)}{2i},$$

$$\cos_\alpha(\xi) = \frac{E_\alpha(i\xi^\alpha) + E_\alpha(-i\xi^\alpha)}{2},$$

$$\sinh_\alpha(\xi) = \frac{E_\alpha(\xi^\alpha) - E_\alpha(-\xi^\alpha)}{2},$$

$$\cosh_\alpha(\xi) = \frac{E_\alpha(\xi^\alpha) + E_\alpha(-\xi^\alpha)}{2},$$

$$\tan_\alpha(\xi) = \frac{\sin_\alpha(\xi)}{\cos_\alpha(\xi)},\ \cot_\alpha(\xi) = \frac{\cos_\alpha(\xi)}{\sin_\alpha(\xi)},$$

$$\tanh_\alpha(\xi) = \frac{\sinh_\alpha(\xi)}{\cosh_\alpha(\xi)},\ \coth_\alpha(\xi) = \frac{\cosh_\alpha(\xi)}{\sinh_\alpha(\xi)},$$

where $E_\alpha(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(1+k\alpha)}$, $\alpha > 0$ is the Mittag-Leffler function.

Step 3. Substituting (8) into (7) and using (1), the left-hand side of (7) is converted to another polynomial in $G^j(\sqrt{\sigma+G^2})^i$ after eliminating the denominator. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_{j0},\ a_{j,i},\ b_{j,i},\ i = 1, ..., m,\ j = 1, 2, ..., k$.

Step 4. Solving the equations system in Step 3, and by using the solutions of Eq. (1), we can construct a variety of exact solutions for Eq. (5).

3 Applications for the space-time fractional Whitham-Broer-Kaup (WBK) equations

In this section, we will apply the described method in Section 2 to solve the space-time fractional Whitham-Broer-Kaup (WBK) equations [24]

$$\begin{cases} D_t^\alpha u + uD_x^\alpha u + D_x^\alpha v + \beta D_x^{2\alpha} u = 0, \\ D_t^\alpha v + D_x^\alpha (uv) - \beta D_x^{2\alpha} v + \gamma D_x^{3\alpha} u = 0, \end{cases} \tag{10}$$

with $0 < \alpha \leq 1$. In [24], the authors solved Eqs. (10) by a proposed fractional sub-equation method based on the fractional Riccati equation, and established some exact solutions for them. Now we will apply the described method above to Eqs. (10). To begin with, we suppose $u(x,t) = U(\xi),\ v(x,t) = V(\xi)$, where $\xi = kx + ct + \xi_0$, $k, c \neq 0,\ \xi_0$ are constants. Then by use of the second equality in (4) we have

$$\begin{cases} D_x^\alpha u = D_x^\alpha U(\xi) = (D_\xi^\alpha U)(\xi_x')^\alpha = k^\alpha D_\xi^\alpha U, \\ D_t^\alpha u = D_t^\alpha U(\xi) = (D_\xi^\alpha U)(\xi_t')^\alpha = c^\alpha D_\xi^\alpha U, \end{cases}$$

and similarly we have $D_x^\alpha v = k^\alpha D_\xi^\alpha V,\ D_x^\alpha(uv) = k^\alpha D_\xi^\alpha(UV),\ D_t^\alpha v = c^\alpha D_\xi^\alpha V$. So Eqs. (10) can be turned into

$$\begin{cases} c^\alpha D_\xi^\alpha U + k^\alpha U D_\xi^\alpha U + k^\alpha D_\xi^\alpha V + \beta k^{2\alpha} D_\xi^{2\alpha} U = 0, \\ c^\alpha D_\xi^\alpha V + k^\alpha D_\xi^\alpha (UV) \\ -\beta k^{2\alpha} D_\xi^{2\alpha} V + \gamma k^{3\alpha} D_\xi^{3\alpha} U = 0. \end{cases} \tag{11}$$

Suppose that the solutions of Eqs. (11) can be expressed by

$$\begin{cases} U(\xi) = a_0 + \sum_{i=1}^{m} [a_i G^i + b_i G^{i-1}\sqrt{\sigma + G^2}], \\ V(\xi) = c_0 + \sum_{i=1}^{n} [c_i G^i + d_i G^{i-1}\sqrt{\sigma + G^2}] \end{cases} \tag{12}$$

where $G = G(\xi)$ satisfy Eq. (1).

Balancing the order between the highest order derivative term and nonlinear term in Eqs. (11), we can obtain $m = 1,\ n = 2$. So we have

$$\begin{cases} U(\xi) = a_0 + a_1 G + b_1\sqrt{\sigma + G^2}, \\ V(\xi) = c_0 + c_1 G + c_2 G^2 \\ \qquad +d_1\sqrt{\sigma + G^2} + d_2 G\sqrt{\sigma + G^2}. \end{cases} \tag{13}$$

Substituting (13) along with (1) into (11) and collecting all the terms with the same power of $G\sqrt{\sigma + G^2}$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations, yields:

Case 1:

$$a_0 = -c^\alpha k^{-\alpha},\ a_1 = \pm 2k^\alpha\sqrt{\beta^2+\gamma},\ b_1 = 0,$$

$$c_0 = -2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma} + \beta^2 + \gamma),\ c_1 = 0,$$

$$c_2 = -2k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma} + \beta^2 + \gamma),$$

$$d_1 = 0,\ d_2 = \pm 2k^\alpha\sqrt{\beta^2+\gamma}.$$

Case 2:

$$a_0 = -c^\alpha k^{-\alpha},\ a_1 = 0,\ b_1 = \pm 2k^\alpha\sqrt{\beta^2+\gamma},$$

$$c_0 = -\sigma k^{2\alpha}(\beta^2+\gamma),\ c_1 = 0,$$

$$c_2 = -2k^{2\alpha}(\beta^2+\gamma),$$

$$d_1 = 0,\ d_2 = \mp 2\beta k^{2\alpha}\sqrt{\beta^2+\gamma}.$$

Case 3:

$$a_0 = -c^\alpha k^{-\alpha},\ a_1 = \pm k^\alpha\sqrt{\beta^2+\gamma},$$

$$b_1 = \pm k^\alpha\sqrt{\beta^2+\gamma},$$

$$c_0 = -\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma} + \beta^2 + \gamma),$$

$$c_1 = 0,\ c_2 = -k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma} + \beta^2 + \gamma),$$

$$d_1 = 0,\ d_2 = \mp k^{2\alpha}\sqrt{\beta^2+\gamma}(\beta \pm \sqrt{\beta^2+\gamma}).$$

Substituting the results above into (13), and combining with (9) we can obtain the following exact solutions to the space-time fractional Whitham-Broer-Kaup (WBK) equations.

From Case 1 and (9) we obtain:

When $\sigma < 0$,

$$\left\{\begin{array}{l} U_1(\xi) = -c^\alpha k^{-\alpha} \mp 2\sqrt{-\sigma}k^\alpha\sqrt{\beta^2+\gamma}\tanh_\alpha(\sqrt{-\sigma}\xi), \\ V_1(\xi) = -2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma) \\ \quad +2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma)[\tanh_\alpha(\sqrt{-\sigma}\xi)]^2 \\ \quad \pm 2\sigma k^\alpha\sqrt{\beta^2+\gamma}\tanh_\alpha(\sqrt{-\sigma}\xi) \\ \quad \sqrt{[\tanh_\alpha(\sqrt{-\sigma}\xi)]^2-1}. \end{array}\right. \tag{14}$$

$$\left\{\begin{array}{l} U_2(\xi) = -c^\alpha k^{-\alpha} \mp 2\sqrt{-\sigma}k^\alpha\sqrt{\beta^2+\gamma}\coth_\alpha(\sqrt{-\sigma}\xi), \\ V_2(\xi) = -2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma) \\ \quad +2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma)[\coth_\alpha(\sqrt{-\sigma}\xi)]^2 \\ \quad \pm 2\sigma k^\alpha\sqrt{\beta^2+\gamma}\coth_\alpha(\sqrt{-\sigma}\xi) \\ \quad \sqrt{[\coth_\alpha(\sqrt{-\sigma}\xi)]^2-1}. \end{array}\right. \tag{15}$$

When $\sigma > 0$,

$$\left\{\begin{array}{l} U_3(\xi) = -c^\alpha k^{-\alpha} \pm 2\sqrt{\sigma}k^\alpha\sqrt{\beta^2+\gamma}\tan_\alpha(\sqrt{\sigma}\xi), \\ V_3(\xi) = -2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma) \\ \quad -2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma)[\tan_\alpha(\sqrt{\sigma}\xi)]^2 \\ \quad \pm 2\sigma k^\alpha\sqrt{\beta^2+\gamma}\tan_\alpha(\sqrt{\sigma}\xi)\sqrt{1+[\tan_\alpha(\sqrt{\sigma}\xi)]^2}. \end{array}\right. \tag{16}$$

$$\left\{\begin{array}{l} U_4(\xi) = -c^\alpha k^{-\alpha} \mp 2\sqrt{\sigma}k^\alpha\sqrt{\beta^2+\gamma}\cot_\alpha(\sqrt{\sigma}\xi), \\ V_4(\xi) = -2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma) \\ \quad -2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma)[\cot_\alpha(\sqrt{\sigma}\xi)]^2 \\ \quad \mp 2\sigma k^\alpha\sqrt{\beta^2+\gamma}\cot_\alpha(\sqrt{\sigma}\xi)\sqrt{1+[\cot_\alpha(\sqrt{\sigma}\xi)]^2}. \end{array}\right. \tag{17}$$

When $\sigma = 0$,

$$\left\{\begin{array}{l} U_5(\xi) = -c^\alpha k^{-\alpha} \mp 2k^\alpha\sqrt{\beta^2+\gamma}[\dfrac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}], \\ V_5(\xi) = -2k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma)[\dfrac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}]^2 \\ \quad \mp 2k^\alpha\sqrt{\beta^2+\gamma}[\dfrac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}]|\dfrac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}|, \end{array}\right. \tag{18}$$

where ω is a constant.

From Case 2 and (9) we obtain:

When $\sigma < 0$,

$$\left\{\begin{array}{l} U_6(\xi) = -c^\alpha k^{-\alpha} \\ \quad \pm 2\sqrt{-\sigma}k^\alpha\sqrt{\beta^2+\gamma}\sqrt{[\tanh_\alpha(\sqrt{-\sigma}\xi)]^2-1}, \\ V_6(\xi) = -\sigma k^{2\alpha}(\beta^2+\gamma)+2\sigma k^{2\alpha}(\beta^2+\gamma) \\ \quad [\tanh_\alpha(\sqrt{-\sigma}\xi)]^2 \mp 2\sigma\beta k^{2\alpha}\sqrt{\beta^2+\gamma} \\ \quad \tanh_\alpha(\sqrt{-\sigma}\xi)\sqrt{[\tanh_\alpha(\sqrt{-\sigma}\xi)]^2-1}. \end{array}\right. \tag{19}$$

$$\left\{\begin{array}{l} U_7(\xi) = -c^\alpha k^{-\alpha} \\ \quad \pm 2\sqrt{-\sigma}k^\alpha\sqrt{\beta^2+\gamma}\sqrt{[\coth_\alpha(\sqrt{-\sigma}\xi)]^2-1}, \\ V_7(\xi) = -\sigma k^{2\alpha}(\beta^2+\gamma)+2\sigma k^{2\alpha}(\beta^2+\gamma) \\ \quad [\coth_\alpha(\sqrt{-\sigma}\xi)]^2 \mp 2\sigma\beta k^{2\alpha}\sqrt{\beta^2+\gamma} \\ \quad \coth_\alpha(\sqrt{-\sigma}\xi)\sqrt{[\coth_\alpha(\sqrt{-\sigma}\xi)]^2-1}. \end{array}\right. \tag{20}$$

When $\sigma > 0$,

$$\left\{\begin{array}{l} U_8(\xi) = -c^\alpha k^{-\alpha} \\ \quad \pm 2\sqrt{\sigma}k^\alpha\sqrt{\beta^2+\gamma}\sqrt{1+[\tan_\alpha(\sqrt{\sigma}\xi)]^2}, \\ V_8(\xi) = -\sigma k^{2\alpha}(\beta^2+\gamma)-2\sigma k^{2\alpha}(\beta^2+\gamma) \\ \quad [\tan_\alpha(\sqrt{\sigma}\xi)]^2 \mp 2\sigma\beta k^{2\alpha}\sqrt{\beta^2+\gamma} \\ \quad \tan_\alpha(\sqrt{\sigma}\xi)\sqrt{1+[\tan_\alpha(\sqrt{\sigma}\xi)]^2}. \end{array}\right. \tag{21}$$

$$\left\{\begin{array}{l} U_9(\xi) = -c^\alpha k^{-\alpha} \pm 2\sqrt{\sigma}k^\alpha\sqrt{\beta^2+\gamma} \\ \quad \sqrt{1+[\cot_\alpha(\sqrt{\sigma}\xi)]^2}, \\ V_9(\xi) = -\sigma k^{2\alpha}(\beta^2+\gamma) \\ \quad -2\sigma k^{2\alpha}(\beta^2+\gamma)[\cot_\alpha(\sqrt{\sigma}\xi)]^2 \\ \quad \pm 2\sigma\beta k^{2\alpha}\sqrt{\beta^2+\gamma}\cot_\alpha(\sqrt{\sigma}\xi)\sqrt{1+[\cot_\alpha(\sqrt{\sigma}\xi)]^2}. \end{array}\right. \tag{22}$$

When $\sigma = 0$,

$$\left\{\begin{array}{l} U_{10}(\xi) = -c^\alpha k^{-\alpha} \pm 2k^\alpha\sqrt{\beta^2+\gamma}|\dfrac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}|, \\ V_{10}(\xi) = -2k^{2\alpha}(\beta^2+\gamma)[\dfrac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}]^2 \\ \quad \pm 2\beta k^{2\alpha}\sqrt{\beta^2+\gamma}[\dfrac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}]|\dfrac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}|, \end{array}\right. \tag{23}$$

where ω is a constant.

From Case 3 and (9) we obtain:

When $\sigma < 0$,

$$\left\{\begin{array}{l} U_{11}(\xi) = -c^\alpha k^{-\alpha} \mp \sqrt{-\sigma}k^\alpha\sqrt{\beta^2+\gamma}\tanh_\alpha(\sqrt{-\sigma}\xi) \\ \quad \pm\sqrt{-\sigma}k^\alpha\sqrt{\beta^2+\gamma}\sqrt{[\tanh_\alpha(\sqrt{-\sigma}\xi)]^2-1}, \\ V_{11}(\xi) = -\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma) \\ \quad +\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma)[\tanh_\alpha(\sqrt{-\sigma}\xi)]^2 \\ \quad \mp\sigma k^{2\alpha}\sqrt{\beta^2+\gamma}(\beta\pm\sqrt{\beta^2+\gamma}) \\ \quad \tanh_\alpha(\sqrt{-\sigma}\xi)\sqrt{[\tanh_\alpha(\sqrt{-\sigma}\xi)]^2-1}. \end{array}\right. \tag{24}$$

$$\left\{\begin{array}{l} U_{12}(\xi) = -c^\alpha k^{-\alpha} \mp \sqrt{-\sigma}k^\alpha\sqrt{\beta^2+\gamma}\coth_\alpha(\sqrt{-\sigma}\xi) \\ \quad \pm\sqrt{-\sigma}k^\alpha\sqrt{\beta^2+\gamma}\sqrt{[\coth_\alpha(\sqrt{-\sigma}\xi)]^2-1}, \\ V_{12}(\xi) = -\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma) \\ \quad +\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma)[\coth_\alpha(\sqrt{-\sigma}\xi)]^2 \\ \quad \mp\sigma k^{2\alpha}\sqrt{\beta^2+\gamma}(\beta\pm\sqrt{\beta^2+\gamma}) \\ \quad \coth_\alpha(\sqrt{-\sigma}\xi)\sqrt{[\coth_\alpha(\sqrt{-\sigma}\xi)]^2-1}. \end{array}\right. \tag{25}$$

When $\sigma > 0$,

$$\left\{\begin{array}{l} U_{13}(\xi) = -c^\alpha k^{-\alpha} \pm \sqrt{\sigma}k^\alpha\sqrt{\beta^2+\gamma}\tan_\alpha(\sqrt{\sigma}\xi) \\ \quad \pm\sqrt{\sigma}k^\alpha\sqrt{\beta^2+\gamma}\sqrt{1+[\tan_\alpha(\sqrt{\sigma}\xi)]^2}, \\ V_{13}(\xi) = -\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma) \\ \quad -\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma)[\tan_\alpha(\sqrt{\sigma}\xi)]^2 \\ \quad \mp\sigma k^{2\alpha}\sqrt{\beta^2+\gamma}(\beta\pm\sqrt{\beta^2+\gamma}) \\ \quad \tan_\alpha(\sqrt{\sigma}\xi)\sqrt{1+[\tan_\alpha(\sqrt{\sigma}\xi)]^2}. \end{array}\right. \tag{26}$$

$$\begin{cases} U_{14}(\xi) = -c^\alpha k^{-\alpha} \mp \sqrt{\sigma} k^\alpha \sqrt{\beta^2+\gamma} \cot_\alpha(\sqrt{\sigma}\xi) \\ \quad \pm\sqrt{\sigma} k^\alpha \sqrt{\beta^2+\gamma}\sqrt{\{1+[\cot_\alpha(\sqrt{\sigma}\xi)]^2\}}, \\ V_{14}(\xi) = -\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma) \\ \quad -\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma)[\cot_\alpha(\sqrt{\sigma}\xi)]^2 \\ \quad \pm\sigma k^{2\alpha}\sqrt{\beta^2+\gamma}(\beta\pm\sqrt{\beta^2+\gamma}) \\ \quad \cot_\alpha(\sqrt{\sigma}\xi)\sqrt{1+[\cot_\alpha(\sqrt{\sigma}\xi)]^2}. \end{cases} \tag{27}$$

When $\sigma = 0$,

$$\begin{cases} U_{15}(\xi) = -c^\alpha k^{-\alpha} \mp k^\alpha \sqrt{\beta^2+\gamma}[\frac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}] \\ \quad \pm k^\alpha \sqrt{\beta^2+\gamma}|\frac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}|, \\ V_{15}(\xi) = -k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma)[\frac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}]^2 \\ \quad \pm k^{2\alpha}\sqrt{\beta^2+\gamma}(\beta\pm\sqrt{\beta^2+\gamma}) \\ \quad [\frac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}]|\frac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}|, \end{cases} \tag{28}$$

where ω is a constant.

Remark 1 *Compared with the results in [24], the established solutions in Eqs. (14)-(28) are new exact solutions for the space-time fractional Whitham-Broer-Kaup (WBK) equations, and have not been reported by other authors in the literature.*

4 Application for the space-time fractional Fokas equation

We consider the space-time fractional Fokas equation [19]

$$4\frac{\partial^{2\alpha}q}{\partial t^\alpha \partial x_1^\alpha} - \frac{\partial^{4\alpha}q}{\partial x_1^{3\alpha}\partial x_2^\alpha} + \frac{\partial^{4\alpha}q}{\partial x_2^{3\alpha}\partial x_1^\alpha} + 12\frac{\partial^\alpha q}{\partial x_1^\alpha}\frac{\partial^\alpha q}{\partial x_2^\alpha} + 12q\frac{\partial^{2\alpha}q}{\partial x_1^\alpha \partial x_2^\alpha} - 6\frac{\partial^{2\alpha}q}{\partial y_1^\alpha \partial y_2^\alpha} = 0,\ 0<\alpha\le 1. \tag{29}$$

In [19], the authors solved Eq. (29) by a fractional Riccati sub-equation method, and obtained some exact solutions for it. Now we will apply the described method in Section 3 to Eq. (29).

Suppose $q(x,y,t) = U(\xi)$, where $\xi = ct + k_1x_1 + k_2x_2 + l_1y_1 + l_2y_2 + \xi_0$, $k_1,\ k_2,\ l_1,\ l_2,\ c,\ \xi_0$ are all constants with $k_1,\ k_2,\ l_1,\ l_2,\ c \neq 0$. Then by use of the second equality in Eq. (4), Eq. (29) can be turned into

$$4c^\alpha k_1^\alpha D_\xi^{2\alpha}U - k_1^{3\alpha}k_2^\alpha D_\xi^{4\alpha}U + k_2^{3\alpha}k_1 D_\xi^{4\alpha}U + 12k_1^\alpha k_2^\alpha(D_\xi^\alpha U)^2 + 12k_1^\alpha k_2^\alpha U D_\xi^{2\alpha}U - 6l_1^\alpha l_2^\alpha D_\xi^{2\alpha}U = 0. \tag{30}$$

Suppose that the solution of Eq. (30) can be expressed by

$$U(\xi) = a_0 + \sum_{i=1}^{m}[a_iG^i + b_iG^{i-1}\sqrt{\sigma+G^2}], \tag{31}$$

where $G = G(\xi)$ satisfies Eq. (1). By Balancing the order between the highest order derivative term and nonlinear term in Eq. (30), we can obtain $m = 2$. So we have

$$U(\xi) = a_0 + a_1G + a_2G^2 + b_1\sqrt{\sigma+G^2} + b_2G\sqrt{\sigma+G^2}. \tag{32}$$

Substituting (32) along with (1) into (30) and collecting all the terms with the same power of $G\sqrt{\sigma+G^2}$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations, yields:

Case 1:

$$a_0 = \frac{4k_1^{3\alpha}k_2\sigma - 4k_1^\alpha k_2^{3\alpha}\sigma - 2c^\alpha k_1^\alpha + 3l_1^\alpha l_2^\alpha}{6k_1^\alpha k_2^\alpha},$$

$$a_1 = 0,\ a_2 = k_1^{2\alpha} - k_2^{2\alpha},\ b_1 = 0,\ b_2 = 0.$$

Case 2:

$$a_0 = \frac{3l_2^\alpha l_1^\alpha - 2c^\alpha k_1^\alpha}{6k_2^\alpha k_1^\alpha},\ a_1 = 0,\ a_2 = \frac{9}{40}(k_1^{2\alpha} - k_2^{2\alpha}),$$

$$b_1 = 0,\ b_2 = \pm\frac{3}{40}\sqrt{31}(k_1^{2\alpha} - k_2^{2\alpha}).$$

Substituting the results above into Eq. (32), and combining with (9) we can obtain the following exact solutions to Eq. (29).

From Case 1 and (9) we obtain same results as in [19], while from Case 2 and (9) we obtain the following results:

When $\sigma < 0$,

$$\begin{aligned} U_1(\xi) &= \frac{3l_2^\alpha l_1^\alpha - 2c^\alpha k_1^\alpha}{6k_2^\alpha k_1^\alpha} \\ &-\frac{9}{40}(k_1^{2\alpha} - k_2^{2\alpha})\sigma[\tanh_\alpha(\sqrt{-\sigma}\xi)]^2 \\ &+b_1\sqrt{\sigma\{1-[\tanh_\alpha(\sqrt{-\sigma}\xi)]^2\}} \\ &\mp\frac{3}{40}\sqrt{31}(k_1^{2\alpha} - k_2^{2\alpha})\sqrt{-\sigma}[\tanh_\alpha(\sqrt{-\sigma}\xi)]\times \\ &\sqrt{\sigma\{1-[\tanh_\alpha(\sqrt{-\sigma}\xi)]^2\}}. \end{aligned} \tag{33}$$

$$U_2(\xi) = \frac{3l_2^\alpha l_1^\alpha - 2c^\alpha k_1^\alpha}{6k_2^\alpha k_1^\alpha}$$

$$-\frac{9}{40}(k_1^{2\alpha}-k_2^{2\alpha})\sigma[\coth_\alpha(\sqrt{-\sigma}\xi)]^2$$
$$+b_1\sqrt{\sigma\{1-[\coth_\alpha(\sqrt{-\sigma}\xi)]^2\}}$$
$$\mp\frac{3}{40}\sqrt{31}(k_1^{2\alpha}-k_2^{2\alpha})\sqrt{-\sigma}[\coth_\alpha(\sqrt{-\sigma}\xi)]\times$$
$$\sqrt{\sigma\{1-[\coth_\alpha(\sqrt{-\sigma}\xi)]^2\}}. \quad (34)$$

When $\sigma>0$,

$$U_3(\xi)=\frac{3l_2^\alpha l_1^\alpha-2c^\alpha k_1^\alpha}{6k_2^\alpha k_1^\alpha}$$
$$+\frac{9}{40}(k_1^{2\alpha}-k_2^{2\alpha})\sigma[\tan_\alpha(\sqrt{\sigma}\xi)]^2$$
$$+b_1\sqrt{\sigma\{1+[\tan_\alpha(\sqrt{\sigma}\xi)]^2\}}$$
$$\pm\frac{3}{40}\sqrt{31}(k_1^{2\alpha}-k_2^{2\alpha})\sqrt{\sigma}[\tan_\alpha(\sqrt{\sigma}\xi)]\times$$
$$\sqrt{\sigma\{1+[\tan_\alpha(\sqrt{\sigma}\xi)]^2\}}. \quad (35)$$

$$U_4(\xi)=\frac{3l_2^\alpha l_1^\alpha-2c^\alpha k_1^\alpha}{6k_2^\alpha k_1^\alpha}$$
$$+\frac{9}{40}(k_1^{2\alpha}-k_2^{2\alpha})\sigma[\cot_\alpha(\sqrt{\sigma}\xi)]^2$$
$$+b_1\sqrt{\sigma\{1+[\cot_\alpha(\sqrt{\sigma}\xi)]^2\}}$$
$$\mp\frac{3}{40}\sqrt{31}(k_1^{2\alpha}-k_2^{2\alpha})\sqrt{\sigma}[\cot_\alpha(\sqrt{\sigma}\xi)]\times$$
$$\sqrt{\sigma\{1+[\cot_\alpha(\sqrt{\sigma}\xi)]^2\}}. \quad (36)$$

When $\sigma=0$,

$$U_5(\xi)=\frac{3l_2^\alpha l_1^\alpha-2c^\alpha k_1^\alpha}{6k_2^\alpha k_1^\alpha}$$
$$+\frac{9}{40}(k_1^{2\alpha}-k_2^{2\alpha})[\frac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}]^2+b_1|\frac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}|$$
$$\mp\frac{3}{40}\sqrt{31}(k_1^{2\alpha}-k_2^{2\alpha})[\frac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}]|\frac{\Gamma(1+\alpha)}{\xi^\alpha+\omega}|, \quad (37)$$

where ω is a constant.

Remark 2 *As one can see, the established solutions (33)-(37) for the space-time fractional Fokas equation above are different from the results in [19], and are new exact solutions so far to our best knowledge.*

5 Some further considerations

In this section, we will deduce some new general solutions for Eq. (1). Suppose $G(\xi)=H(\eta)$, and a nonlinear fractional complex transformation $\eta=\frac{\xi^\alpha}{\Gamma(1+\alpha)}$. Then by Eq. (2) and the first equality in Eq. (4), we have $D_\xi^\alpha G(\xi)=D_\xi^\alpha H(\eta)=H'(\eta)D_\xi^\alpha\eta=H'(\eta)$. So Eq. (1) can be turned into the following ordinary differential equation

$$H'(\eta)=\sigma+H^2(\eta), \quad (38)$$

which admits the following solutions

$$H(\eta)=\begin{cases}-\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\eta),\ \sigma<0,\\ -\sqrt{-\sigma}\coth(\sqrt{-\sigma}\eta),\ \sigma<0,\\ \sqrt{\sigma}\tan(\sqrt{\sigma}\eta),\ \sigma>0,\\ -\sqrt{\sigma}\cot(\sqrt{\sigma}\eta),\ \sigma>0,\\ -\frac{1}{\eta+\omega},\ \omega=const,\ \sigma=0,\end{cases} \quad (39)$$

So we can obtain some new solutions for Eq. (1):

$$G(\xi)=\begin{cases}-\sqrt{-\sigma}\tanh(\frac{\sqrt{-\sigma}\xi^\alpha}{\Gamma(1+\alpha)}),\ \sigma<0,\\ -\sqrt{-\sigma}\coth(\frac{\sqrt{-\sigma}\xi^\alpha}{\Gamma(1+\alpha)}),\ \sigma<0,\\ \sqrt{\sigma}\tan(\frac{\sqrt{\sigma}\xi^\alpha}{\Gamma(1+\alpha)}),\ \sigma>0,\\ -\sqrt{\sigma}\cot(\frac{\sqrt{\sigma}\xi^\alpha}{\Gamma(1+\alpha)}),\ \sigma>0,\\ -\frac{\Gamma(1+\alpha)}{\xi^\alpha+\omega\Gamma(1+\alpha)},\ \omega=const,\ \sigma=0.\end{cases} \quad (40)$$

By use of the solutions of Eq. (1) denoted in (40), we can obtain a series of new solutions for the space-time fractional Whitham-Broer-Kaup (WBK) equations and the space-time fractional Fokas equation.

For example, from Case 1 in Section 3 and (40) we have the following solutions for the space-time fractional Whitham-Broer-Kaup (WBK) equations:

When $\sigma<0$,

$$\begin{cases}U_{16}(\xi)=-c^\alpha k^{-\alpha}\mp2\sqrt{-\sigma}k^\alpha\sqrt{\beta^2+\gamma}\\ \quad\tanh(\frac{\sqrt{-\sigma}\xi^\alpha}{\Gamma(1+\alpha)}),\\ V_{16}(\xi)=-2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma)\\ \quad+2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma)\\ \quad[\tanh(\frac{\sqrt{-\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]^2\\ \quad\pm2\sigma k^\alpha\sqrt{\beta^2+\gamma}\tanh(\frac{\sqrt{-\sigma}\xi^\alpha}{\Gamma(1+\alpha)})\\ \quad\sqrt{[\tanh(\frac{\sqrt{-\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]^2-1}.\end{cases} \quad (41)$$

$$\begin{cases} U_{17}(\xi) = -c^{\alpha}k^{-\alpha} \mp 2\sqrt{-\sigma}k^{\alpha}\sqrt{\beta^2+\gamma} \\ \quad \coth(\frac{\sqrt{-\sigma}\xi^{\alpha}}{\Gamma(1+\alpha)}), \\ V_{17}(\xi) = -2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma) \\ \quad +2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma) \\ \quad [\coth(\frac{\sqrt{-\sigma}\xi^{\alpha}}{\Gamma(1+\alpha)})]^2 \\ \quad \pm 2\sigma k^{\alpha}\sqrt{\beta^2+\gamma}\coth(\frac{\sqrt{-\sigma}\xi^{\alpha}}{\Gamma(1+\alpha)}) \\ \quad \sqrt{[\coth(\frac{\sqrt{-\sigma}\xi^{\alpha}}{\Gamma(1+\alpha)})]^2-1}. \end{cases} \tag{42}$$

When $\sigma > 0$,

$$\begin{cases} U_{18}(\xi) = -c^{\alpha}k^{-\alpha} \pm 2\sqrt{\sigma}k^{\alpha}\sqrt{\beta^2+\gamma}\tan(\frac{\sqrt{\sigma}\xi^{\alpha}}{\Gamma(1+\alpha)}), \\ V_{18}(\xi) = -2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma) \\ \quad -2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma)[\tan(\frac{\sqrt{\sigma}\xi^{\alpha}}{\Gamma(1+\alpha)})]^2 \\ \quad \pm 2\sigma k^{\alpha}\sqrt{\beta^2+\gamma}\tan(\frac{\sqrt{\sigma}\xi^{\alpha}}{\Gamma(1+\alpha)}) \\ \quad \sqrt{1+[\tan(\frac{\sqrt{\sigma}\xi^{\alpha}}{\Gamma(1+\alpha)})]^2}. \end{cases} \tag{43}$$

$$\begin{cases} U_{19}(\xi) = -c^{\alpha}k^{-\alpha} \mp 2\sqrt{\sigma}k^{\alpha}\sqrt{\beta^2+\gamma}\cot(\frac{\sqrt{\sigma}\xi^{\alpha}}{\Gamma(1+\alpha)}), \\ V_{19}(\xi) = -2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma) \\ \quad -2\sigma k^{2\alpha}(\pm\beta\sqrt{\beta^2+\gamma}+\beta^2+\gamma)[\cot(\frac{\sqrt{\sigma}\xi^{\alpha}}{\Gamma(1+\alpha)})]^2 \\ \quad \mp 2\sigma k^{\alpha}\sqrt{\beta^2+\gamma}\cot(\frac{\sqrt{\sigma}\xi^{\alpha}}{\Gamma(1+\alpha)}) \\ \quad \sqrt{1+[\cot(\frac{\sqrt{\sigma}\xi^{\alpha}}{\Gamma(1+\alpha)})]^2}. \end{cases} \tag{44}$$

In particular, if we take $\sigma = -1,\ k = c = 1,\ \beta = 1,\ \gamma = 3,\ \alpha = \frac{1}{2},\ \xi_0 = 0$, then we obtain the following solitary wave solutions, which are shown in Figs. 1-2.

$$\begin{cases} u_{20}(x,t) = -1 - 4\tanh(\frac{(x+t)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}), \\ v_{20}(\xi) = 12 - 12[\tanh(\frac{(x+t)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})})]^2 \\ \quad -4\tanh(\frac{(x+t)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})})\sqrt{[\tanh(\frac{(x+t)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})})]^2-1}. \end{cases} \tag{45}$$

If we take $\sigma = 1,\ k = c = 1,\ \beta = 1,\ \gamma = 3,\ \alpha = \frac{1}{2},\ \xi_0 = 0$, then we obtain the following periodic wave solutions, which are shown in Figs. 3-4.

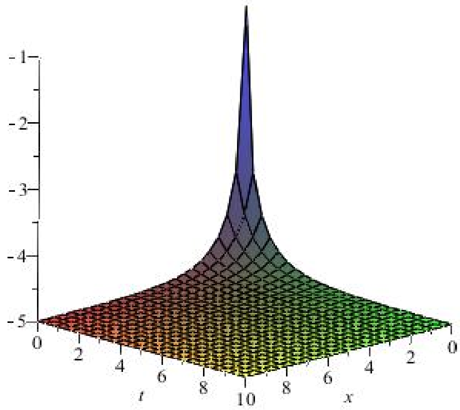

Fig 1. The solitary solution $u_{20}(x,\ t)$ with $\sigma = -1,\ k = c = 1,\ \xi_0 = 0,\ \alpha = \frac{1}{2},\ \beta = 1,\ \gamma = 3$

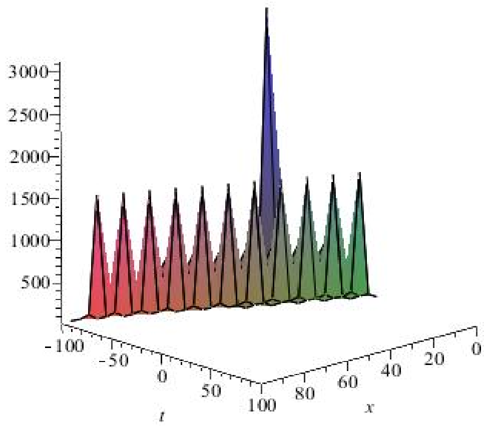

Fig 2. The solitary solution $v_{20}(x,\ t)$ with $\sigma = -1,\ k = c = 1,\ \xi_0 = 0,\ \alpha = \frac{1}{2},\ \beta = 1,\ \gamma = 3$

$$\begin{cases} u_{21}(\xi) = -1 + 4\tan(\frac{(x+t)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}), \\ v_{21}(\xi) = -12 - 12[\tan(\frac{(x+t)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})})]^2 \\ \quad +4\tan(\frac{(x+t)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})})\sqrt{1+[\tan(\frac{(x+t)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})})]^2}. \end{cases} \tag{46}$$

From Case 2 in Section 4 and (40) we have the following solutions for the space-time fractional Fokas equation:

When $\sigma < 0$,

$$U_6(\xi) = \frac{3l_2^{\alpha}l_1^{\alpha} - 2c^{\alpha}k_1^{\alpha}}{6k_2^{\alpha}k_1^{\alpha}}$$

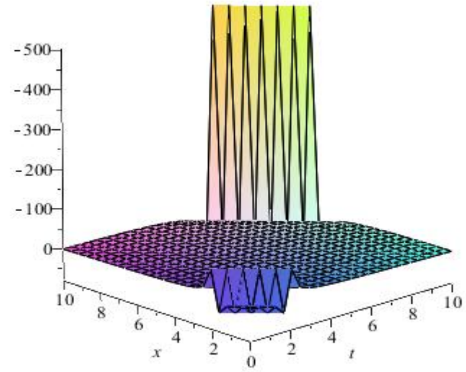

Fig 3. The periodic solution $u_{21}(x,\ t)$ with $\sigma=1,\ k=c=1,\ \xi_0=0,\ \alpha=\frac{1}{2},\ \beta=1,\ \gamma=3$

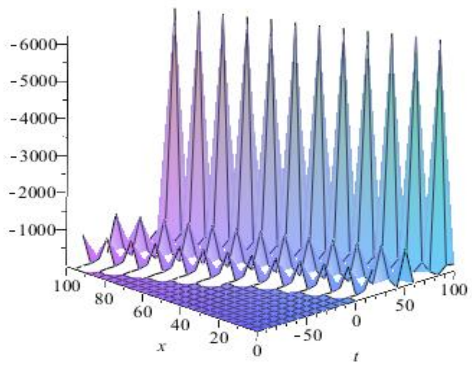

Fig 4. The periodic solution $v_{21}(x,\ t)$ with $\sigma=1,\ k=c=1,\ \xi_0=0,\ \alpha=\frac{1}{2},\ \beta=1,\ \gamma=3$

$$-\frac{9}{40}(k_1^{2\alpha}-k_2^{2\alpha})\sigma[\tanh(\frac{\sqrt{-\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]^2$$

$$+b_1\sqrt{\sigma\{1-[\tanh(\frac{\sqrt{-\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]^2\}}$$

$$\mp\frac{3}{40}\sqrt{31}(k_1^{2\alpha}-k_2^{2\alpha})\sqrt{-\sigma}[\tanh(\frac{\sqrt{-\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]\times$$

$$\sqrt{\sigma\{1-[\tanh(\frac{\sqrt{-\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]^2\}}. \qquad (47)$$

$$U_7(\xi)=\frac{3l_2^\alpha l_1^\alpha-2c^\alpha k_1^\alpha}{6k_2^\alpha k_1^\alpha}$$

$$-\frac{9}{40}(k_1^{2\alpha}-k_2^{2\alpha})\sigma[\coth(\frac{\sqrt{-\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]^2$$

$$+b_1\sqrt{\sigma\{1-[\coth(\frac{\sqrt{-\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]^2\}}$$

$$\mp\frac{3}{40}\sqrt{31}(k_1^{2\alpha}-k_2^{2\alpha})\sqrt{-\sigma}[\coth(\frac{\sqrt{-\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]\times$$

$$\sqrt{\sigma\{1-[\coth(\frac{\sqrt{-\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]^2\}}. \qquad (48)$$

When $\sigma>0$,

$$U_8(\xi)=\frac{3l_2^\alpha l_1^\alpha-2c^\alpha k_1^\alpha}{6k_2^\alpha k_1^\alpha}$$

$$+\frac{9}{40}(k_1^{2\alpha}-k_2^{2\alpha})\sigma[\tan(\frac{\sqrt{\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]^2$$

$$+b_1\sqrt{\sigma\{1+[\tan(\frac{\sqrt{\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]^2\}}$$

$$\pm\frac{3}{40}\sqrt{31}(k_1^{2\alpha}-k_2^{2\alpha})\sqrt{\sigma}[\tan(\frac{\sqrt{\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]\times$$

$$\sqrt{\sigma\{1+[\tan(\frac{\sqrt{\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]^2\}}. \qquad (49)$$

$$U_9(\xi)=\frac{3l_2^\alpha l_1^\alpha-2c^\alpha k_1^\alpha}{6k_2^\alpha k_1^\alpha}$$

$$+\frac{9}{40}(k_1^{2\alpha}-k_2^{2\alpha})\sigma[\cot(\frac{\sqrt{\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]^2$$

$$+b_1\sqrt{\sigma\{1+[\cot(\frac{\sqrt{\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]^2\}}$$

$$\mp\frac{3}{40}\sqrt{31}(k_1^{2\alpha}-k_2^{2\alpha})\sqrt{\sigma}[\cot(\frac{\sqrt{\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]\times$$

$$\sqrt{\sigma\{1+[\cot(\frac{\sqrt{\sigma}\xi^\alpha}{\Gamma(1+\alpha)})]^2\}}. \qquad (50)$$

Similarly, combining the rest cases in Sections 3 and 4 with (40) we can also obtain abundant exact solutions for the space-time fractional Whitham-Broer-Kaup (WBK) equations and the space-time fractional Fokas equation, which are omitted here.

6 Conclusions

By introducing a new ansatz, we have proposed an extended fractional Riccati sub-equation method for solving FPDEs. It is worthy to note that the variable transformation $\xi=ct+k_1x_1+k_2x_2+\ldots+k_nx_n+\xi_0$ used here plays an important role in the process of establishing exact solutions, which ensures that a certain FPDE can be turned into another fractional ordinary differential equation. Being concise and powerful, the proposed method can be applied to solve other fractional partial differential equations.

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