Bifurcation in a Discrete Two Patch Logistic Metapopulation Model

LI XU
Tianjin University of Commerce
Department of Statistics
Jinba road, Benchen,Tianjin
CHINA
beifang_xl@163.com

ZHONGXIANG CHANG
Tianjin University of Commerce
Department of Mathematics
Jinba road, Benchen,Tianjin
CHINA
zxchang@163.com

FENGXIA MAI
Tianjin University of Commerce
Department of Mathematics
Jinba road, Benchen,Tianjin
CHINA
lxyfxm@tjcu.edu.cn

Abstract: In this paper, local bifurcation of a discrete two patch logistic metapopulation is discussed. By the central manifold method, flip bifurcation can be analyzed at the positive fixed point from the viewpoint of the dynamical system, and the system can not undergo a fold bifurcation. Simulations on this model show the discrete model can have rich dynamical behaviors. The state feedback control is done to stabilize chaotic orbits at an unstable fixed point. Then the eigenvalues of the corresponding diffusion system with Dirichlet boundary conditions are found for future bifurcation analysis.

Key–Words: Logistic metapopulation, Flip bifurcation, Fold bifurcation, Central manifold, State feedback control, Eigenvalue

1 Introduction

The dynamical characteristics of the elementary autonomous differential equation

$$\frac{dx(t)}{dt} = ax(t)(1 - x(t)) \tag{1}$$

in which $a \in (0, +\infty)$ have been utilized in the derivation of a multitude of generalized models to describe the temporal evolution of single species population systems. One assumes that $x(t)$ denotes the density or biomass of a species, a denotes the intrinsic or Malthusian growth rate.

A discrete analogue of (1) can be given by

$$x_{t+1} = x_t \exp(a(1 - x_t)) \tag{2}$$

which has been investigated in the mathematics literatures in its own right as a discrete population model of a single species with non-overlapping generations [1-5]. For example, it was shown in [1, 2] that for some values of the parameter a, solutions of Eq. (2) are “chaotic”. It was also proved in [3] that any solution of Eq. (2) converges to 1 as $n \to \infty$ if and only if $a \leq 2$. Sufficient conditions are obtained for the existence of a unique almost periodic solution which is globally attractive for an almost periodic discrete logistic equation in [4]. Sufficient conditions are obtained for the existence of a positive and globally asymptotically stable w-periodic solution in [5].

Since the pioneering theoretical works of Skellam [6] and Turing [7], the role of spatial structuring in shaping the dynamics of populations has received considerable attention from both theoreticians and experimentalists (for comprehensive reviews, see [8-9]). Many important epidemiological and ecological phenomena are strongly infuenced by spatial heterogeneities because of the localized nature of transmission or other forms of interaction. Thus, spatial models are more suitable for describing the process of population development.

The concept of metapopulation [10] provides a theoretical framework for studying spatially structured populations, and discrete time models on metapopulation are proposed and paid attention. Gyllenberg et al [11] studied a two-patch discrete-time metapopulation model of coupled logistic difference equations. Their work showed that the interaction between local dynamics and symmetric dispersal can lead to the replacement of chaotic local dynamics by periodic dynamics for some initial conditions. Doebeli [12] showed that this stabilizing effect is enhanced if dispersal is asymmetric. Yakubu and Castillo-Chavez [13] studied a more general metapopulation model over N patches. The effects of synchronous dispersal on discrete time metapopulation dynamics with local (patch) dynamics of the same (compensatory or over compensatory) or mixed (compensatory and over compensatory) types were explored. In [14], the period-doubling bifurcation of a discrete metapopulation with delay in the dispersion terms was discussed. By using the central manifold method, the period-doubling bifurcation can be analyzed from the viewpoint of the dynamical system.

In this paper, we consider a two-patch discrete time metapopulation model as follows

$$\left\{\begin{array}{c} u_{t+1} = u_t \exp(a - au_t) + d(-2u_t + v_t) \\ v_{t+1} = v_t \exp(a - av_t) + d(u_t - 2v_t) \end{array}\right., \quad (3)$$

which can be obtained from the following system when $n = 2$

$$x_i^{t+1} = x_i^{t+1} \exp(a(1 - x_i^{t+1})) + d\nabla^2 x_i^{t+1} \quad (4)$$

with the discrete Dirichlet boundary conditions

$$x_0^t = u_{n+1}^t, \quad (5)$$

where n is a positive integer, $i \in \{1, 2, ..., n\} = [1, n]$.

To the best of our knowledge, up to now, the dynamics of system (3) has not been discussed. In this paper, we will rigorously prove that this discrete model (3) possesses the flip bifurcation and no fold bifurcation by bifurcation theory and center manifold theory. Meanwhile the numerical simulations not only perfectly show the consistence with the theoretical analysis but exhibit the complex and interesting dynamical behaviors including stable period-one orbit, period-two orbit, period-n orbit or coexistence of several period-orbits, chaotic oscillators. Especially, the effect of the dispersion parameter on bifurcation was shown. The computations of Lyapunov exponents can also confirm the dynamical behaviors. Furthermore, dynamics of the system (4)-(5) will be discuss by eigenvalue analysis. The analysis and results in this paper will be interesting.

This paper is organized as follows. In Section 2, we study the existence and stability of fixed points and give sufficient conditions of existence for flip bifurcation. The numerical simulations including the bifurcation diagrams at neighborhood of critical values and the maximum Lyapunov exponents corresponding to the bifurcation diagrams are given in Section 3. In Section 4, chaos is controlled to an unstable fixed point using the feedback control method. Finally, we give remarks and discussions to conclude this paper and our future work in Section 5.

2 Analysis of equilibria and bifurcations

Clearly, the system (3) has four possible steady states, i.e. $E_0 = (0, 0)$, and nontrivial coexistence point $E_1 = (u^*, v^*)$, where

$$u^* = v^* = 1 - \frac{\ln(1 + d)}{a}. \quad (6)$$

Here $\ln(1 + d) < a$ and $a, d > 0$.

Let

$$F = \{(a, d) : \ln(1 + d) < a,\ a, d > 0\}.$$

The linearized form of (3) is then

$$\left\{\begin{array}{l} u_{t+1} = f_u u_t + f_v v_t \\ v_{t+1} = g_u u_t + g_v v_t \end{array}\right., \quad (7)$$

whose Jacobian matrix is

$$J_{E_i} = \left[\begin{array}{cc} f_u & f_v \\ g_u & g_v \end{array}\right]_{E_i}, \quad i = 0, 1, \quad (8)$$

where

$$\begin{aligned} f_u &= 1 - au - adu - d, \\ f_v &= d, \\ g_u &= 1 - au - adu - d, \\ g_v &= d \end{aligned}$$

and $(u, v) = (0, 0)$ or (u^*, v^*).

The characteristic equation of the Jacobian matrix J is

$$\lambda^2 + p\lambda + q = 0,$$

where $p = -(f_u + g_v), q = f_u g_v - f_v g_u$.

In order to discuss the stability at the fixed points of (3), we also need the following definitions [15]:

Definition 1 *(1) If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then E is called a sink and E is locally asymptotical stable;*

(2) If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, then E is called a source and E is unstable;

(3) If $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$), then E is called a saddle;

(4) If $|\lambda_1| = 1$ and $|\lambda_2| \neq 1$(or $|\lambda_2| = 1$ and $|\lambda_1| \neq 1$), then E is called non-hyperbolic.

The following cases are true.

Case 1: The fixed point $E_0 = (0, 0)$

The linearization of (3) about E_0 has the Jacobian matrix

$$J_{E_0} = \left[\begin{array}{cc} 1 - d & d \\ d & 1 - d \end{array}\right], \quad (9)$$

which has two eigenvalues

$$\lambda_1 = 1, \lambda_2 = 1 - 2d.$$

Then, we have the following results:

(1) $\lambda_1 = 1, \quad |\lambda_2| \neq 1$ if and only if $d \neq -1, 0$.

Case 2: The fixed point $E_1 = (u^*, v^*)$

The linearization of (3) about E_1 has the Jacobian matrix

$$J_{E_1} = \left[\begin{array}{cc} 1 - au^* - adu^* - d & d \\ d & 1 - au^* - adu^* - d \end{array}\right], \quad (10)$$

The eigenvalues of (10) are

$$\begin{aligned}\lambda_1 &= (1-a+\ln(1+d))(1+d)-3d,\\ \lambda_2 &= (1-a+\ln(1+d))(1+d)-d.\end{aligned}$$

Then, we have the following results:

(1) $|\lambda_1|<1, |\lambda_2|<1$ if and only if $0<d<1$ and

$$\ln(1+d)<a<\ln(1+d)-\frac{2d-2}{1+d};$$

(2) $\lambda_1=-1, |\lambda_2|\neq 1$ if and only if $0<d<1$ and

$$a=\ln(1+d)-\frac{2d-2}{1+d};$$

(3) $\lambda_1=1, |\lambda_2|\neq 1$ if and only if

$$a=\ln(1+d)-\frac{2d}{1+d};$$

(4) $\lambda_2=-1, |\lambda_1|\neq 1$ if and only if

$$a=\ln(1+d)+\frac{2}{1+d};$$

(5) $\lambda_2=1, |\lambda_1|\neq 1$ if and only if

$$a=\ln(1+d), d\neq 1;$$

(6) $|\lambda_1|>1, |\lambda_2|<1$ if and only if

$$\ln(1+d)<a<\ln(1+d)+\frac{2}{1+d}, 1<d$$

or $0<d<1$ and

$$\ln(1+d)-\frac{2d-2}{1+d}<a<\ln(1+d)+\frac{2}{1+d};$$

(7) $|\lambda_1|>1, |\lambda_2|>1$ if and only if

$$a>\ln(1+d)+\frac{2}{1+d}.$$

Let

$$\begin{aligned}F_A &= \left\{(a,d): a=\ln(1+d)-\frac{2d}{1+d},\ a,d>0\right\},\\ F_B &= \{(a,d): a=\ln(1+d), d\neq 1,\ a,d>0\}.\end{aligned}$$

Based on the above analysis, we find the fact that $F_A\cap F=\emptyset$, $F_B\cap F=\emptyset$. The following conclusion can be obtained.

Theorem 2 *the system (3) can not undergo a fold bifurcation at positive fixed point E_1.*

Further we can prove the next facts.

Theorem 3 *the positive fixed point E_1 undergoes a flip bifurcation at the threshold $a^*=\ln(1+d)-\frac{2d-2}{1+d}, 0<d<1$.*

Proof: Let $\zeta_n=u_n-u^*, \eta_n=v_n-v^*, \mu_n=a-a^*$,and parameter μ_n is a new and dependent variable, the system (3) becomes:

$$\begin{pmatrix}\zeta_{n+1}\\ \eta_{n+1}\\ \mu_{n+1}\end{pmatrix}=\begin{pmatrix}(\zeta_n+1-\frac{\ln(1+d)}{\mu_n+a^*})\\ \exp(\ln(1+d)-(\mu_n+a^*)\zeta_n)\\ +d(-2\zeta_n+\eta_n-1+\frac{\ln(1+d)}{\mu_n+a^*})\\ -1+\frac{\ln(1+d)}{\mu_n+a^*}\\ (\eta_n+1-\frac{\ln(1+d)}{\mu_n+a^*})\\ \exp(\ln(1+d)-(\mu_n+a^*)\eta_n)\\ +d(-2\eta_n+\zeta_n-1+\frac{\ln(1+d)}{\mu_n+a^*})\\ -1+\frac{\ln(1+d)}{\mu_n+a^*}\\ \mu_n\end{pmatrix}. \quad (11)$$

Let

$$T=\begin{bmatrix}1 & 1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \quad (12)$$

then

$$T^{-1}=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix}, \quad (13)$$

By the following transformation

$$\begin{pmatrix}\zeta_n\\ \eta_n\\ \mu_n\end{pmatrix}=T\begin{pmatrix}x_n\\ y_n\\ \delta_n\end{pmatrix}, \quad (14)$$

the system (11) can be changed into

$$\begin{aligned}\begin{pmatrix}x_{n+1}\\ y_{n+1}\\ \delta_{n+1}\end{pmatrix} &= \begin{pmatrix}-1 & 0 & 0\\ 0 & 2d-1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_n\\ y_n\\ \delta_n\end{pmatrix}\\ &+\begin{pmatrix}f(x_n,y_n,\delta_n)\\ g(x_n,y_n,\delta_n)\\ 0\end{pmatrix}, \end{aligned} \quad (15)$$

where

$$\begin{aligned}& f(x_n,y_n,\delta_n)\\ =\ & -(1+d)x_n\delta_n+[\frac{8d(d-1)}{(1+d)^2}-4d(1+d)]x_ny_n\\ & +[\frac{(5d+1)\ln^2(1+d)}{2}\\ & +\frac{2(1-d)(5d+1)\ln(1+d)}{3(1+d)}\end{aligned}$$

$$
\begin{aligned}
&+\frac{2(1-d)(5d+1)}{3(1+d)}]x_n^3\\
&+[\frac{(5d+1)\ln^2(1+d)}{2}+\frac{2(1-d)(5d+1)}{3(1+d)}\\
&+\frac{2(1-d)(5d+1)\ln(1+d)}{3(1+d)}]x_n y_n^2\\
&+[(1+d)\ln(1+d)+2(1-3d)]x_n y_n\delta_n\\
&+o((|x_n|+|y_n|+|\delta_n|)^3),
\end{aligned}
$$

$$
\begin{aligned}
& g(x_n,y_n,\delta_n)\\
=\ & -(1+d)y_n\delta_n\\
&+[\frac{4d(1-d)}{1+d}-2\ln(1+d)]x_n^2\\
&+[\frac{4d(1-d)}{1+d}-2d\ln(1+d)]y_n^2\\
&+[\frac{\ln(1+d)}{2}-(3d-1)]x_n^2\delta_n\\
&+[\frac{\ln(1+d)}{2}-(3d-1)]y_n^2\delta_n\\
&+[\frac{(5d+1)\ln^2(1+d)}{2}\\
&+\frac{2(1+d)(5d+1)\ln(1+d)}{1+d}.\\
&+\frac{2(1-d)^2(5d+1)}{(1+d)^2}]x_n^2 y_n\\
&+[\frac{(5d+1)\ln^2(1+d)}{6}\\
&+\frac{2(1+d)(5d+1)\ln(1+d)}{3(1+d)}\\
&+\frac{2(1-d)^2(5d+1)}{3(1+d)^2}]y_n^3\\
&+o((|x_n|+|y_n|+|\delta_n|)^3).
\end{aligned}
$$

Then, we can consider

$$
\begin{aligned}
y_n\ =\ & h(x_n,\delta_n)=a_1x_n^2+a_2x_n\delta_n+a_3\delta_n^2\\
&+o((|x_n|+|\delta_n|)^3)
\end{aligned}
$$

which must satisfy:

$$
\begin{aligned}
& y_{n+1}=h(x_{n+1},\delta_{n+1})\\
=\ & h(-x_n+f(x_n,y_n,\delta_n),\delta_{n+1})\\
=\ & (2d+1)h(x_n,\delta_n)-(1+d)h(x_n,\delta_n)\delta_n\\
&+[\frac{4d(1-d)}{1+d}-2\ln(1+d)]x_n^2\\
&+[\frac{4d(1-d)}{1+d}-2d\ln(1+d)]h^2(x_n,\delta_n)\\
&+[\frac{\ln(1+d)}{2}-(3d-1)]x_n^2\delta_n\\
&+[\frac{\ln(1+d)}{2}-(3d-1)]h^2(x_n,\delta_n)\delta_n\\
&+[\frac{(5d+1)\ln^2(1+d)}{2}\\
&+\frac{2(1+d)(5d+1)\ln(1+d)}{1+d}\\
&+\frac{2(1-d)^2(5d+1)}{(1+d)^2}]x_n^2 h(x_n,\delta_n)\\
&+[\frac{(5d+1)\ln^2(1+d)}{6}\\
&+\frac{2(1+d)(5d+1)\ln(1+d)}{3(1+d)}\\
&+\frac{2(1-d)^2(5d+1)}{3(1+d)^2}]h^3(x_n,\delta_n)\\
&+o((|x_n|+|y_n|+|\delta_n|)^3).
\end{aligned}
$$

By calculation, we can get that

$$
a_1=\frac{d\ln(1+d)}{d-1}-\frac{2d}{1+d},\ a_2=0,\ a_3=0
$$

And the system (11) is restricted to the center manifold, which is given by:

$$
\begin{aligned}
f\ :\ & x_{n+1}=-x_n-(1+d)x_n\delta_n\\
&+[\frac{8d(d-1)}{(1+d)^2}-4d(1+d)]\\
&[\frac{d(\ln(1+d)}{d-1}-\frac{2d}{1+d}]x_n^3\\
&+[\frac{(5d+1)\ln^2(1+d)}{2}\\
&+\frac{2(1-d)(5d+1)\ln(1+d)}{3(1+d)}\\
&+\frac{2(1-d)(5d+1)}{3(1+d)}]x_n^3\\
&+[\frac{(5d+1)\ln^2(1+d)}{2}\\
&+\frac{2(1-d)(5d+1)(1+\ln(1+d))}{3(1+d)}]\\
&[\frac{d(\ln(1+d)}{d-1}-\frac{2d}{1+d}]^2x_n^5\\
&+[(1+d)\ln(1+d)+2(1-3d)]\\
&[\frac{d(\ln(1+d)}{d-1}-\frac{2d}{1+d}]x_n^3\delta_n\\
&+o((|x_n|^4)
\end{aligned}
$$

Since

$$
\begin{aligned}
& f(0,0)=0,\quad \frac{\partial f}{\partial x}\mid_{(0,0)}=-1,\\
& \frac{\partial^2 f}{\partial x\partial\delta}\Big|_{(0,0)}=-(1+d)\neq 0,
\end{aligned}
$$

$$\frac{\partial^2 f}{\partial \delta^2}\Big|_{(0,0)} = 0, \quad \frac{\partial^2 f}{\partial x^2}\Big|_{(0,0)} = 0,$$
$$\frac{\partial^3 f}{\partial x^3}\Big|_{(0,0)} \neq 0,$$

the system (3) undergoes a flip bifurcation at E_1. The proof is completed. □

Theorem 4 *the positive fixed point E_1 undergoes a flip bifurcation at the threshold $a^* = \ln(1+d) + \frac{2}{1+d}$.*

Proof: Let $\zeta_n = u_n - u^*, \eta_n = v_n - v^*, \mu_n = a - a^*$, and parameter μ_n is a new and dependent variable, the system (3) becomes:

$$\begin{pmatrix} \zeta_{n+1} \\ \eta_{n+1} \\ \mu_{n+1} \end{pmatrix} = \begin{pmatrix} (\zeta_n + 1 - \frac{\ln(1+d)}{\mu_n + a^*}) \\ \exp(\ln(1+d) - (\mu_n + a^*)\zeta_n) \\ +d(-2\zeta_n + \eta_n - 1 + \frac{\ln(1+d)}{\mu_n + a^*}) \\ -1 + \frac{\ln(1+d)}{\mu_n + a^*} \\ (\eta_n + 1 - \frac{\ln(1+d)}{\mu_n + a^*}) \\ \exp(\ln(1+d) - (\mu_n + a^*)\eta_n) \\ +d(-2\eta_n + \zeta_n - 1 + \frac{\ln(1+d)}{\mu_n + a^*}) \\ -1 + \frac{\ln(1+d)}{\mu_n + a^*} \\ \mu_n \end{pmatrix} \tag{16}$$

Let

$$T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{17}$$

then

$$T^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{18}$$

By the following transformation

$$\begin{pmatrix} \zeta_n \\ \eta_n \\ \mu_n \end{pmatrix} = T \begin{pmatrix} x_n \\ y_n \\ \delta_n \end{pmatrix} \tag{19}$$

then the system (16) can be changed into

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ \delta_{n+1} \end{pmatrix} = \begin{pmatrix} -1-2d & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ \delta_n \end{pmatrix} + \begin{pmatrix} f(x_n, y_n, \delta_n) \\ g(x_n, y_n, \delta_n) \\ 0 \end{pmatrix} \tag{20}$$

where

$$\begin{aligned} & f(x_n, y_n, \delta_n) \\ = \; & -(1+d)x_n\delta_n - [\frac{2d(1+d)}{(1+d)^2} + d\ln(1+d)]y_n^2 \\ & -[\frac{2d}{1+d} + d\ln(1+d)]x_n^2 \\ & +[(1-d) + \frac{1}{2}(1+d)\ln(1+d)]x_n\delta_n \\ & +[1-d + \frac{1}{2}(1+d)\ln(1+d)]y_n^2\delta_n \\ & +[\frac{2(1+3d)}{(1+d)^2} + \frac{2(1+3d)}{1+d}\ln(1+d) \\ & +\frac{1+3d}{2}\ln^2(1+d)]x_n y_n^2 \\ & +[\frac{2(1+3d)}{3(1+d)^2} + \frac{2(1+3d)}{3(1+d)}\ln(1+d) \\ & +\frac{1+3d}{6}\ln^2(1+d)]x_n^3 \\ & +o((|x_n| + |y_n| + |\delta_n|)^3), \end{aligned}$$

$$\begin{aligned} & g(x_n, y_n, \delta_n) \\ = \; & -(1+d)y_n\delta_n - [2d\ln(1+d) + \frac{4d}{1+d}]x_n y_n \\ & +[(1+d)\ln(1+d) + 2(1-d)]x_n y_n \delta_n \\ & +[\frac{2(1+3d)}{(1+d)^2} + \frac{2(1+3d)}{1+d}\ln(1+d) \\ & +\frac{1+3d}{2}\ln^2(1+d)]x_n^2 y_n \\ & +\frac{1}{6(1+d)^2}[4(1+3d) + 4d(1+3d)\ln(1+d) \\ & +d(5+7d)\ln^2(1+d)]y_n^3 \\ & +o((|x_n| + |y_n| + |\delta_n|)^4). \end{aligned}$$

Then, we can consider

$$\begin{aligned} & x_n = h(y_n, \delta_n) \\ = \; & a_1 y_n^2 + a_2 y_n \delta_n + a_3 \delta_n^2 + o((|y_n| + |\delta_n|)^3), \end{aligned}$$

which must satisfy:

$$\begin{aligned} & x_{n+1} = h(y_n + f(x_n, y_n, \delta_n), \delta_{n+1}) \\ & = -(1+2d)h(y_n, \delta_n) - (1+d)h(y_n, \delta_n)\delta_n \\ & -[\frac{2d(1-d)}{(1+d)^2} + d\ln(1+d)]y_n^2 \\ & -[\frac{2d}{1+d} + d\ln(1+d)]h^2(y_n, \delta_n) \\ & +[(1-d) + \frac{1}{2}(1+d)\ln(1+d)]h^2(y_n, \delta_n)\delta_n \\ & +[(1-d) + \frac{1}{2}(1+d)\ln(1+d]y_n^2\delta_n \end{aligned}$$

$$
\begin{aligned}
&+[\frac{2(1+3d)}{(1+d)^2}+\frac{2(1+3d)}{1+d}\ln(1+d)\\
&+\frac{1+3d}{2}\ln^2(1+d)]h(y_n,\delta_n)y_n^2\\
&+[\frac{2(1+3d)}{3(1+d)^2}+\frac{2(1+3d)}{3(1+d)}\ln(1+d)\\
&+\frac{1+3d}{6}\ln^2(1+d)]h^3(y_n,\delta_n)\\
&+o((|x_n|+|y_n|+|\delta_n|)^3).
\end{aligned}
$$

By calculating, we can get that

$$
a_1=\frac{d(d-1)}{(1+d)^3}-\frac{d\ln(1+d)}{2(1+d)}, a_2=0, a_3=0.
$$

And the system (16) is restricted to the center manifold, which is given by:

$$
\begin{aligned}
g\ :\ & y_{n+1}=-y_n-(1+d)y_n\delta_n\\
&-[2d\ln(1+d)+\frac{4d}{1+d}]\\
&[\frac{d(d-1)}{(1+d)^3}-\frac{d\ln(1+d)}{2(1+d)}]y_n^3\\
&+[(1+d)\ln(1+d)+2(1-d)]\\
&[\frac{d(d-1)}{(1+d)^3}-\frac{d\ln(1+d)}{2(1+d)}]y_n^3\delta_n\\
&+\frac{1}{6(1+d)^2}[4(1+3d)+4d(1+3d)\ln(1+d)\\
&+d(5+7d)\ln^2(1+d)]y_n^3\\
&+[\frac{2(1+3d)}{(1+d)^2}+\frac{2(1+3d)}{1+d}\ln(1+d)\\
&+\frac{1+3d}{2}\ln^2(1+d)]\\
&[\frac{d(d-1)}{(1+d)^3}-\frac{d\ln(1+d)}{2(1+d)}]y_n^5\\
&+o(|y_n|+|\delta_n|)^3).
\end{aligned}
$$

Since

$$
\begin{aligned}
&g(0,\delta_n)=0,\quad \frac{\partial g}{\partial y}\Big|_{(0,0)}=-1,\\
&\frac{\partial^2 g}{\partial y\partial\delta}\Big|_{(0,0)}=-(1+d)\neq 0,\\
&\frac{\partial^2 g}{\partial \delta^2}=0,\quad \frac{\partial^2 g}{\partial y^2}=0,\\
&\frac{\partial^3 g}{\partial y^3}\Big|_{(0,0)}=\frac{1}{(1+d)^2}[4(1+3d)\\
&+4d(1+3d)\ln(1+d)+d(5+7d)\ln^2(1+d)]\\
&\neq 0,
\end{aligned}
$$

the system (3) undergoes a flip bifurcation at E_1. The proof is completed. □

3 Numerical simulation

As is known to all that the bifurcation diagram provides a general view of the evolution process of the dynamical behaviors by plotting a state variable with the abscissa being one parameter. As a parameter varies, the dynamics of the system we concerned change through a local or global bifurcation which leads to the change of stability at the same time. In this section, we use the bifurcation diagrams, Lyapunov exponents and phase portraits to illustrating the above analytic results and finding new dynamics of map as the parameters varying.

Now, a is considered as a parameter with the range (0.5-3.5). The bifurcation diagram of map (3) in $(a-u)$ plane is given in Fig. 1 for $d=0.002$.

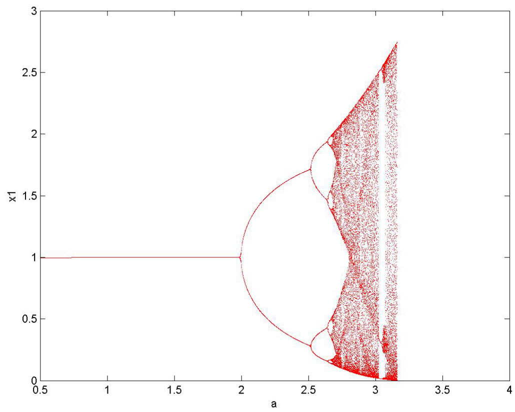

Figure 1: Bifurcation diagram for the system (3)in $a-u$ planes when $d=0.002$.

From Fig. 1 we see that equilibrium E_1 is stable for $a<1.9980$, and loses its stability when $a=1.9980$. Further, when $a>1.9980$, there is the period-doubling bifurcation. We also observe that there is a cascade of period doubling. Moreover, a chaotic set is emerged with the increasing of a. But, when a increases to some fixed value, we see u extinct. A powerful numerical tool to investigate whether the dynamical behavior is chaotic is a plot of the largest Lyapunov exponent, as a function of one of the model parameters. The largest Lyapunov exponent is the average growth rate of an infinitesimal state perturbation along a typical trajectory (orbit) showed in Fig. 2.

Furthermore, we show the effect of the dispersion parameter on bifurcation, the diffusion coefficient is increased for $d=0.005$.

Fig. 3 exhibits the detail bifurcation diagram and Fig. 4 is the corresponding largest Lyapunov exponent. Comparing Fig. 1 with Fig. 3, we also find that the bigger d becomes, the easier it becomes for the periodic orbits to lose stability and chaos to arise; and

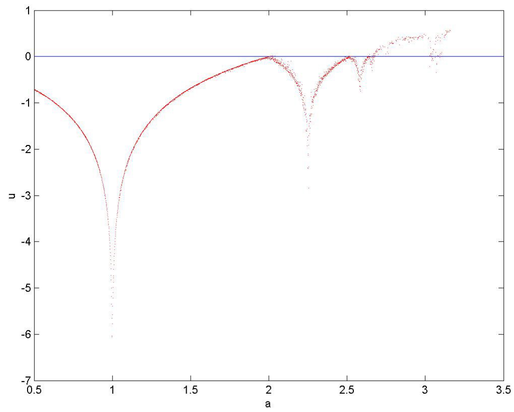

Figure 2: The maximum Lyapunov exponent corresponding to Fig. 1.

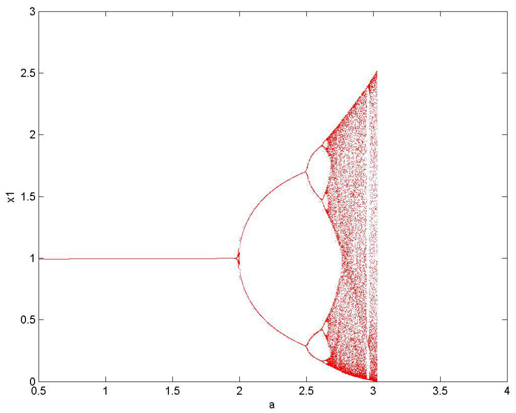

Figure 3: Bifurcation diagram for the system (3)in $a-u$ planes when $d = 0.005$.

some solutions will approach infinity at the end. In other words, the dispersion parameter d will destabilize the system when d is relatively large.

4 Choas control

In this section, we apply the state feedback control method [16–19] to stabilize chaotic orbits at an unstable fixed point of (3). Consider the following controlled form of system (3):

$$\begin{cases} u_{t+1} = u_t \exp(a - au_t) + d(-2u_t + v_t) + \varepsilon_n \\ v_{t+1} = v_t \exp(a - av_t) + d(u_t - 2v_t) \end{cases} \tag{21}$$

with the following feedback control law as the control force:

$$\varepsilon_n = -k_1(u_n - u^*) - k_2(v_n - v^*) \tag{22}$$

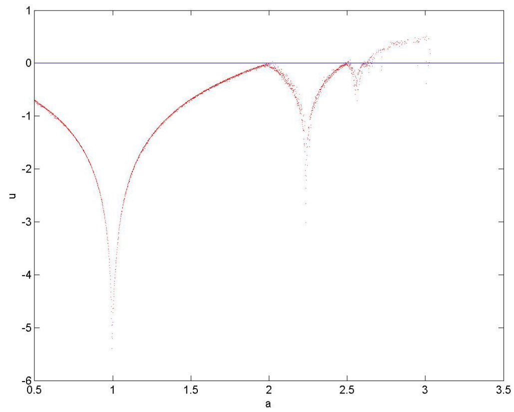

Figure 4: The maximum Lyapunov exponent corresponding to Fig. 3.

where k_1 and k_2 are the feedback gain, (u^*, v^*) is the positive fixed point of (3).

The Jacobian matrix J of the controlled system (21), (22) evaluated at the fixed point (u^*, v^*) is given by

$$J = \begin{bmatrix} a_{11} - k_1 & a_{12} - k_2 \\ a_{21} & a_{22} \end{bmatrix}, \tag{23}$$

where

$$\begin{aligned} a_{11} &= 1 - au^* - adu^* - d, \\ a_{12} &= d, \\ a_{21} &= d, \\ a_{22} &= 1 - au^* - adu^* - d. \end{aligned} \tag{24}$$

The characteristic equation of the Jacobian matrix J is

$$\begin{aligned} &\lambda^2 - (a_{11} + a_{22} - k_1)\lambda + a_{22}(a_{11} - k_1) \\ &\quad -a_{21}(a_{12} - k_2) = 0. \end{aligned} \tag{25}$$

Assume that the eigenvalues are given by λ_1 and λ_2, then $\lambda_{1,2}$ satisfy the condition $|\lambda_{1,2}| < 1$ if and only if

$$\begin{aligned} &-(a_{11} + a_{22} - k_1) - 1 \\ &< a_{22}(a_{11} - k_1) - a_{21}(a_{12} - k_2), \\ &a_{11} + a_{22} - k_1 - 1 \\ &< a_{22}(a_{11} - k_1) - a_{21}(a_{12} - k_2), \\ &a_{22}(a_{11} - k_1) - a_{21}(a_{12} - k_2) < 1. \end{aligned} \tag{26}$$

Parameter values k_1, k_2 are selected in the above space, a chaotic trajectory can be stabilized at the fixed point (u^*, v^*).

5 Discussion and conclusion

In this paper, the behaviors of the discrete two patch metapopulation model were investigated, and some complex and interesting dynamical phenomena were shown. As the parameters vary, the model exhibits a variety of dynamical behaviors, which include stable period-one orbit, period-two orbit, period-n orbit or coexistence of several period-orbits, chaotic oscillators, even onset of chaos suddenly and the chaotic dynamics approach to the period-orbits. The effect of the dispersion parameter on bifurcation was shown. Then the chaotic orbits at an unstable fixed point were stabilized by the feedback control method.

It is well known that reaction-diffusion systems have been playing a significant role in different fields of science such as chemical reactions, electronic devices, combustion processes neuron structures, population of organisms etc. The couplings between the diffusion of species and non-linear population dynamics could give rise to instabilities of the inhomogeneous steady state and hence transition to a new space dependent regime. Bifurcation such as hopf bifurcation, turing bifurcation, transcritical and pitchfork bifurcation in reaction-diffusion systems have been extensively studied in recent years (for example see [20-21] and reference therein). For discrete reaction-diffusion systems, Turing instability has been studied by means of linearization method and inner product technique or other methods (see [22-24] and its listed reference). To the best of our knowledge, up to now, hopf bifurcation, transcritical and pitchfork bifurcation in discrete reaction-diffusion systems have been not extensively studied. In this paper, we also expect to make some preparations for further bifurcation analysis of discrete reaction-diffusion systems.

Simply, the authors expect to explore the bifurcation analysis of the following system

$$x_i^{t+1} = x_i^{t+1}\exp(a(1-x_i^{t+1})) + d\nabla^2 x_i^{t+1} \quad (27)$$

with the discrete Dirichlet boundary conditions

$$x_0^t = u_{n+1}^t, \quad (28)$$

where n is a positive integer, $i \in \{1,2,...,n\} = [1,n]$.

Assume x^* is the fixed point to the above system, then the linearized system of (27) is

$$x_i^{t+1} = f_x(x^*)x_i^t + d\nabla^2 x_i^t, \quad (29)$$

$$x_0^t = u_{n+1}^t. \quad (30)$$

It is well known that the following eigenvalue problem has the eigenvalue

$$\lambda_k = 4\sin^2\frac{k\pi}{2(n+1)}, k = 1,2,\ldots,n,$$

and the corresponding eigenfunctions are

$$\eta_s^k = \sin\frac{sk\pi}{n+1}, k, s = 1,2,\ldots,n$$

or

$$\eta^k = (\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \ldots, \sin\frac{nk\pi}{n+1})^T.$$

The system (29) can be rewritten as

$$x^{t+1} = -dAx^t + f_x^{'}(x^*)x^t,$$

where

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & 0 \\ 0 & -1 & 2 & -1 & 0 \\ \vdots & & \ddots & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix},$$

and its coefficient matrix has the eigenvalues

$$\mu_k = f_x^{'}(x^*) - 4\sin^2\frac{k\pi}{2(n+1)},$$

$k = 1,2,\ldots,n$.

Although eigenvalues analysis of system (27)-(28) is put forward in this paper, bifurcation analysis deserves attention and the mathematical mechanism should be explained in our future work.

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