Two positive almost periodic solutions of harvesting predator-prey model with holling III type functional response and time delays

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Abstract: By means of Mawhin's continuation theorem of coincidence degree theory, some sufficient conditions are obtained for the existence of at least two positive almost periodic solutions of harvesting predator-prey model with Holling III type functional response and time delays. An example and numerical simulations are employed to illustrate the main result of this paper.

Key–Words: Almost periodicity; Coincidence degree; Predator-prey model; Holling III; Harvesting.

1 Introduction

Recently, many authors have paid attention to the study of predator-prey systems [1]-[10]. Some authors have been concerned with the Holling functional response in the study of ecological systems [6]-[10]. Wang and Li [11] considered the following system with Holling III type functional response:

$$\begin{cases} \dot{N}_1(t) = N_1(t)\Big[b_1(t) - a_1(t)N_1(t-\mu_1(t)) \\ \qquad - \frac{\alpha_1(t)N_1(t)}{1+mN_1^2(t)}N_2(t-\nu(t))\Big], \\ \dot{N}_2(t) = N_2(t)\Big[-b_2(t) - a_2(t)N_2(t) \\ \qquad + \frac{\alpha_2(t)N_1^2(t-\mu_2(t))}{1+mN_1^2(t-\mu_2(t))}\Big], \end{cases} \tag{1}$$

where N_1 and N_2 are the densities of the prey population and predator population, m is a nonnegative constant, α_i, a_i, b_i, μ_i and ν are positive T-periodic functions with ecological meaning as follows ($i = 1, 2$):

b_1 : the prey population grows in the absence of predators;

b_2 : the predator population decays in the absence of preys;

a_1 : the prey population decays in the competition among the preys;

a_2 : the predator population decays in the competition among the predators;

α_1 : the prey is fed upon by the predators;

α_2 : the coefficient of transformation from preys to predators.

The Holling type III functional response occurs in predators which increase their search activity with increasing prey density. For example, many predators respond to kairomones (chemicals emitted by prey) and increase their activity. Polyphagous vertebrate predators (e.g., birds) can switch to the most abundant prey species by learning to recognize it visually. Mortality first increases with prey increasing density, and then declines. Thus we also consider the Holling type III functional response in predators. If the predators are more efficient at higher prey densities and less efficient at lower prey densities, then the dynamics of the ecosystem is better described by the Holling type III functional response. In addition, the prey's antipredator efforts may promote predator switching [12]. Since most marine mammals and boreal fish species are considered to be generalist predators (including feeders, grazers, etc.), the Holling type III functional response might be more appropriate [13].

In the real world, any biological or environmental parameters are naturally subject to fluctuation in time, so it is reasonable to study the corresponding nonautonomous system. Considering the biological and environmental periodicity (e.g. seasonal effects of weather, food supplies, mating habits), the authors [11] focused on the existence of a periodic solution with strictly positive components of (1) by using Mawhin's continuation theorem of coincidence degree theory.

In many earlier studies, it has been shown that harvesting has a strong impact on the dynamic evolution of a population, e.g., see [7-9, 14-17]. So the study of the population dynamics with harvesting is becoming a very important subject in mathematical bio-economics. The predator-prey model with Holling III type functional response and harvesting on predator is generally described as:

$$\begin{cases} \dot{N}_1(t) = N_1(t)\Big[b_1(t)-a_1(t)N_1(t-\mu_1(t)) \\ \qquad -\frac{\alpha_1(t)N_1(t)}{1+mN_1^2(t)}N_2(t-\nu(t))\Big], \\ \dot{N}_2(t) = N_2(t)\Big[-b_2(t)-a_2(t)N_2(t) \\ \qquad +\frac{\alpha_2(t)N_1^2(t-\mu_2(t))}{1+mN_1^2(t-\mu_2(t))}\Big]-H(t), \end{cases} \tag{2}$$

where H represents the harvesting term on the predator.

In applications, if the various constituent components of the temporally nonuniform environment are with incommensurable periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. Hence, if we consider the effects of the environmental factors, almost periodicity is sometimes more realistic and more general than periodicity. In recent years, the almost periodic solution of the models in biological populations has been studied extensively (see [18-26] and the references cited therein). During the last decade, by utilizing Mawhin's continuation theorem of coincidence degree theory, many scholars are concerning with the existence of multiple positive periodic solutions for some non-linear ecosystems with harvesting terms, e.g., see [27-33]. Unlike the periodic oscillation, owing to the complexity of the almost periodic oscillation, it is hard to study the existence of positive almost periodic solutions of non-linear ecosystems by using Mawhin's continuation theorem. Therefore, to the best of the authors' knowledge, so far, there are scarcely any papers concerning with the multiplicity of positive almost periodic solutions of system (2). Stimulated by the above reason, the main purpose of this paper is to establish sufficient conditions for the existence of multiple positive almost periodic solutions to system (2) by applying Mawhin's continuation theorem of coincidence degree theory.

Example 1. *Consider the following simple harvesting predator-prey model:*

$$\begin{cases} \dot{N}_1(t) = N_1(t)\Big[1-|\sin\sqrt{3}t|N_1(t)\Big], \\ \dot{N}_2(t) = N_2(t)\Big[-\cos^2(\sqrt{2}t)N_2(t) \\ \qquad +\frac{5N_1^2(t)}{1+N_1^2(t)}\Big]+\frac{1}{(1+e^{18})^2}. \end{cases} \tag{3}$$

In system (3), $\cos^2(\sqrt{2}t)$ is $\frac{\sqrt{2}\pi}{2}$-periodic function and $|\sin(\sqrt{3}t)|$ is $\frac{\sqrt{3}\pi}{3}$-periodic function, which imply that system (3) is with incommensurable periods. Then there is no a priori reason to expect the existence of positive periodic solutions of system (3). Thus, it is significant to study the existence of positive almost periodic solutions of system (3).

Now let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}^+$ denote the sets of real numbers, integers and positive integers, respectively. Related to a continuous function f, we use the following notations:

$$f^l = \inf_{s\in\mathbb{R}} f(s), \quad f^M = \sup_{s\in\mathbb{R}} f(s).$$

Throughout this paper, we always make the following assumption for system (2):

(F_1) All the coefficients in system (2) are continuous nonnegative almost periodic functions with $\bar{a}_i > 0$, $\bar{H} > 0$ and $\bar{b}_1 > 0$, $i = 1, 2$.

The paper is organized as follows. In Section 2, we give some basic definitions and necessary lemmas which will be used in later sections. In Section 3, some sufficient conditions are obtained for the existence of at least two positive almost periodic solutions of system (2) by way of Mawhin's continuous theorem. An example is also given to illustrate our main result.

2 Preliminaries

Definition 1. *[34] $x \in C(\mathbb{R}, \mathbb{R}^n)$ is called almost periodic, if for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval* *such that* $\|x(t+\tau)-x(t)\| < \epsilon$, $\forall t \in \mathbb{R}$, *where* $\|\cdot\|$ *is arbitrary norm of* $\mathbb{R}^n$. τ *is called to the* ϵ*-almost period of* x, $T(x,\epsilon)$ *denotes the set of* ϵ*-almost periods for* x *and* $l(\epsilon)$ *is called to the length of the inclusion interval for* $T(x,\epsilon)$. *The collection of those functions is denoted by* $AP(\mathbb{R},\mathbb{R}^n)$. *Let* $AP(\mathbb{R}) := AP(\mathbb{R},\mathbb{R})$.

Lemma 1. *[34] If* $x \in AP(\mathbb{R})$, *then* x *is bounded and uniformly continuous on* $\mathbb{R}$.

Lemma 2. *[20] Assume that* $x \in AP(\mathbb{R}) \cap C^1(\mathbb{R})$ *with* $x' \in C(\mathbb{R})$, *for* $\forall \epsilon > 0$, *we have the following conclusions*:

(1) *there is a point* $\xi_\epsilon \in [0,+\infty)$ *such that* $x(\xi_\epsilon) \in [x^* - \epsilon, x^*]$ *and* $x'(\xi_\epsilon) = 0$;

(2) *there is a point* $\eta_\epsilon \in [0,+\infty)$ *such that* $x(\eta_\epsilon) \in [x_*, x_* + \epsilon]$ *and* $x'(\eta_\epsilon) = 0$.

Lemma 3. *Assume that* $x \in AP(\mathbb{R}) \cap C^1(\mathbb{R})$ *with* $\dot{x} \in C(\mathbb{R})$. *For an arbitrary interval* $[a,b]$ *with* $b-a = \omega > 0$, *let* $\xi, \eta \in [a,b]$ *and*

$$I_1 = \{s \in [\xi, b] : \dot{x}(s) \geq 0\},$$

$$I_2 = \{s \in [\eta, b] : \dot{x}(s) \leq 0\},$$

then

$$x(t) \leq x(\xi) + \int_{I_1} \dot{x}(s)\,\mathrm{d}s, \quad \forall t \in [\xi, b],$$

$$x(t) \geq x(\eta) + \int_{I_2} \dot{x}(s)\,\mathrm{d}s, \quad \forall t \in [\eta, b].$$

The proof of this lemma is obvious, so we omit it.

Lemma 4. *If* $x \in AP(\mathbb{R})$, *then for an arbitrary interval* $I = [a,b]$ *with* $b - a = \omega > 0$, *there exist* $\xi \in [a,b]$, $\underline{\xi} \in (-\infty, a]$ *and* $\bar{\xi} \in [b, +\infty)$ *such that*

$$x(\underline{\xi}) = x(\bar{\xi}) \quad \textit{and} \quad x(\xi) \leq x(s), \ \forall s \in [\underline{\xi}, \bar{\xi}].$$

Proof. Without loss of generality, we consider $[a,b]$ as $[0,\omega]$. We will present three cases to prove this lemma.

(C_1) $x(0) = x(\omega)$. Let $\xi \in [0,\omega]$ such that $x(\xi) = \min_{s\in[0,\omega]} x(s)$, $\underline{\xi} = 0$ and $\bar{\xi} = \omega$. So this lemma holds.

(C_2) $x(0) > x(\omega)$. Let $x_* = \inf_{s\in\mathbb{R}} x(s)$. From Lemma 1, $-\infty < x_* \leq x(\omega) < x(0)$.

(a) If $x(\omega) > x_*$, then we claim that there exists $\omega_1 \in (-\infty, 0]$ such that $x(\omega_1) = x(\omega)$ and

$$x(s) \geq x(\omega_1) = x(\omega), \ \forall s \in [\omega_1, 0]. \tag{4}$$

In fact, if it is not true, then

$$x(s) > x(\omega), \ \ \forall s \in (-\infty, 0]. \tag{5}$$

By the definition of x_* and the continuity of x, there must exist $t_0 \in \mathbb{R}$ such that $x(t_0) = \frac{x(\omega)+x_*}{2} \in (x_*, x(\omega))$. Since $x \in AP(\mathbb{R})$, then for $\epsilon = x(\omega) - x(t_0) = \frac{x(\omega)-x_*}{2} > 0$, there exists a number $\tau \in T(x,\epsilon) \cap (-\infty, -t_0]$ $(t_0 + \tau \leq 0)$ such that

$$|x(t+\tau) - x(t)| < \epsilon = x(\omega) - x(t_0),$$

which implies from (5) that

$$\begin{aligned} & |x(t_0+\tau) - x(t_0)| \\ = \quad & x(t_0+\tau) - x(t_0) \\ < \quad & x(\omega) - x(t_0) \\ \Longrightarrow \quad & x(t_0+\tau) < x(\omega), \end{aligned}$$

which leads to a contradiction with (5). Therefore, our claim is valid. From (2.1), $\min_{s\in[\omega_1,\omega]} x(s) = \min_{s\in[0,\omega]} x(s)$. So we can choose $\xi \in [0,\omega]$ such that $x(\xi) = \min_{s\in[0,\omega]} x(s)$, $\underline{\xi} = \omega_1$ and $\bar{\xi} = \omega$, we obtain from (4) that

$$x(\underline{\xi}) = x(\bar{\xi}), \quad x(\xi) \leq x(s), \ \ \forall s \in [\underline{\xi}, \bar{\xi}].$$

Therefore, this lemma holds.

(b) If $x(\omega) = x_*$, then we claim that there exist $\omega_1 \in (-\infty, 0)$ and $\omega_2 \in (\omega, +\infty)$ such that

$$\begin{aligned} & x(\omega_1) = x(\omega_2) \\ = \quad & \frac{x(0)+x(\omega)}{2} \in (x(\omega), x(0)). \end{aligned} \tag{6}$$

First, we prove that there exist $\omega_1 \in (-\infty, 0)$ such that

$$x(\omega_1) = \frac{x(0)+x(\omega)}{2}. \tag{7}$$

By the continuity of x, there must exist $t_1 \in \mathbb{R}$ such that $x(t_1) = \frac{x(0)+x(\omega)}{2} \in (x(\omega), x(0))$. If (7) is not true, then

$$\begin{aligned} & x(s) > x(t_1) = \frac{x(0)+x(\omega)}{2} \\ & > x(\omega), \ \ \forall s \in (-\infty, 0]. \end{aligned} \tag{8}$$

Since $x \in AP(\mathbb{R})$, then for $\epsilon = x(t_1) - x(\omega) = \frac{x(0)-x(\omega)}{2} > 0$, there exists a number $\tau \in T(x,\epsilon) \cap (-\infty, -\omega]$ $(\omega + \tau \leq 0)$ such that

$$|x(t+\tau) - x(t)| < \epsilon = x(t_1) - x(\omega),$$

which implies from (8) that

$$\begin{aligned}|x(\omega+\tau)-x(\omega)| &= x(\omega+\tau)-x(\omega)\\ &< x(t_1)-x(\omega)\\ &\Longrightarrow x(\omega+\tau)<x(t_1),\end{aligned}$$

which leads to a contradiction with (8). Therefore, (7) holds. Similar to the above argument, it is not difficult to prove that there exists $\omega_2 \in (\omega,+\infty)$ such that $x(\omega_2)=\frac{x(0)+x(\omega)}{2}=x(\omega_1)$. Hence, (6) is satisfied, which implies that this lemma holds by choosing $\xi=\omega$, $\underline{\xi}=\omega_1$ and $\bar{\xi}=\omega_2$.

(C_3) $x(0)<x(\omega)$. Similar to the argument as that in (C_2), it is not difficult to verify that this lemma holds. So we omit the proof of this case.

This completes the proof of this lemma. □

Similar to the proof of Lemma 4, we can easily show that

Lemma 5. *If $x\in AP(\mathbb{R})$, then for an arbitrary interval $[a,b]$ with $I=b-a=\omega>0$, there exist $\eta\in[a,b]$, $\underline{\eta}\in(-\infty,a]$ and $\bar{\eta}\in[b,+\infty)$ such that*

$$x(\underline{\eta})=x(\bar{\eta}) \quad \textit{and} \quad x(\eta)\geq x(s),\ \forall s\in[\underline{\eta},\bar{\eta}].$$

Lemma 6. *[34] Assume that $f\in AP(\mathbb{R})$ and $\bar{f}=\lim_{T\to\infty}\frac{1}{T}\int_0^T f(s)\,\mathrm{d}s>0$, then for $\forall a\in\mathbb{R}$, there exists a positive constant T_0 independent of a such that*

$$\frac{1}{T}\int_a^{a+T} f(s)\,\mathrm{d}s\in\left[\frac{\bar{f}}{2},\frac{3\bar{f}}{2}\right],\quad \forall T\geq T_0.$$

3 Main results

The method to be used in this paper involves the applications of the continuation theorem of coincidence degree. This requires to introduce a few concepts and results from Gaines and Mawhin [35].

Let $\mathbb{X}$ and $\mathbb{Y}$ be real Banach spaces, $L:\mathrm{Dom}L\subseteq\mathbb{X}\to\mathbb{Y}$ be a linear mapping and $N:\mathbb{X}\to\mathbb{Y}$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\mathrm{Im}L$ is closed in $\mathbb{Y}$ and $\dim\mathrm{Ker}L=\mathrm{codim}\mathrm{Im}L<+\infty$. If L is a Fredholm mapping of index zero and there exist continuous projectors $P:\mathbb{X}\to\mathbb{X}$ and $Q:\mathbb{Y}\to\mathbb{Y}$ such that $\mathrm{Im}P=\mathrm{Ker}L$, $\mathrm{Ker}Q=\mathrm{Im}L=\mathrm{Im}(I-Q)$. It follows that $L|_{\mathrm{Dom}L\cap\mathrm{Ker}P}:(I-P)\mathbb{X}\to\mathrm{Im}L$ is invertible and its inverse is denoted by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N:\bar{\Omega}\to\mathbb{X}$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J:\mathrm{Im}Q\to\mathrm{Ker}L$.

Lemma 7. *[35] Let $\Omega\subseteq\mathbb{X}$ be an open bounded set, L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. If all the following conditions hold:*

(a) $Lx\neq\lambda Nx,\ \forall x\in\partial\Omega\cap\mathrm{Dom}L,\ \lambda\in(0,1)$;

(b) $QNx\neq 0,\ \forall x\in\partial\Omega\cap\mathrm{Ker}L$;

(c) $\deg\{JQN,\Omega\cap\mathrm{Ker}L,0\}\neq 0$, *where $J:\mathrm{Im}Q\to\mathrm{Ker}L$ is an isomorphism.*

Then $Lx=Nx$ has a solution on $\bar{\Omega}\cap\mathrm{Dom}L$.

Under the invariant transformation $(N_1,N_2)^T=(e^u,e^v)^T$, system (2) reduces to

$$\begin{cases}\dot{u}(t)=b_1(t)-a_1(t)e^{u(t-\mu_1(t))}\\ \qquad -\frac{\alpha_1(t)e^{u(t)}}{1+me^{2u(t)}}e^{v(t-\nu(t))},\\ \dot{v}(t)=-b_2(t)-a_2(t)e^{v(t)}\\ \qquad +\frac{\alpha_2(t)e^{2u(t-\mu_2(t))}}{1+me^{2u(t-\mu_2(t))}}-\frac{H(t)}{e^{v(t)}}.\end{cases}\tag{9}$$

For $f\in AP(\mathbb{R})$, we denote by

$$\bar{f}=m(f)=\lim_{T\to\infty}\frac{1}{T}\int_0^T f(s)\,\mathrm{d}s,$$

$$\Lambda(f)=\left\{\varpi\in\mathbb{R}:\lim_{T\to\infty}\frac{1}{T}\int_0^T f(s)e^{-\mathrm{i}\varpi s}\mathrm{d}s\neq 0\right\},$$

$$\mathrm{mod}(f)=\left\{\sum_{j=1}^m n_j\varpi_j:n_j\in\mathbb{Z},m\in\mathbb{N},\ \varpi_j\in\Lambda(f),j=1,2\ldots,m\right\}$$

the mean value, the set of Fourier exponents and the module of f, respectively.

Set $\mathbb{X}=\mathbb{Y}=\mathbb{V}_1\bigoplus\mathbb{V}_2$, where

$$\begin{aligned}\mathbb{V}_1 = \Big\{&z=(u,v)^T\in AP(\mathbb{R},\mathbb{R}^2):\\ &\mathrm{mod}(u)\subseteq\mathrm{mod}(L_u),\mathrm{mod}(v)\subseteq\mathrm{mod}(L_v),\\ &\forall\varpi\in\Lambda(u)\cup\Lambda(v),|\varpi|\geq\theta_0\Big\},\end{aligned}$$

$$\mathbb{V}_2=\left\{z=(u,v)^T\equiv(k_1,k_2)^T,k_1,k_2\in\mathbb{R}\right\},$$

where

$$L_u=L_u(t,\varphi)=b_1(t)-a_1(t)e^{\varphi_1(-\mu_1(0))}-\frac{\alpha_1(t)e^{\varphi_1(0)}}{1+me^{2\varphi_1(t)}}e^{\varphi_2(-\nu(0))},$$

$$L_v = L_v(t,\varphi) = -b_2(t) - a_2(t)e^{\varphi_2(0)} + \frac{\alpha_2(t)e^{2\varphi_1(-\mu_2(0))}}{1+me^{2\varphi_1(-\mu_2(0))}} - \frac{H(t)}{e^{\varphi_2(0)}},$$

$$\varphi = (\varphi_1,\varphi_2)^T \in C([-\tau,0],\mathbb{R}^2),$$

$\tau = \max_{i=1,2}\{\mu_i^M, \nu^M\}$, θ_0 is a given positive constant. Define the norm

$$\|z\|_{\mathbb{X}} = \max\left\{\sup_{s\in\mathbb{R}}|u(s)|, \sup_{s\in\mathbb{R}}|v(s)|\right\},$$

where $z=(u,v)^T \in \mathbb{X}=\mathbb{Y}$.

Similar to the proof as that in articles [20,36], it follows that

Lemma 8. *$\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces endowed with $\|\cdot\|_{\mathbb{X}}$.*

Lemma 9. *Let $L:\mathbb{X}\to\mathbb{Y}$, $Lz = L(u,v)^T = (\dot{u},\dot{v})^T$, then L is a Fredholm mapping of index zero.*

Lemma 10. *Define $N:\mathbb{X}\to\mathbb{Y}$, $P:\mathbb{X}\to\mathbb{X}$ and $Q:\mathbb{Y}\to\mathbb{Y}$ by*

$$Nz = N\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} b_1(t) - a_1(t)e^{u(t-\mu_1(t))} \\ -\frac{\alpha_1(t)e^{u(t)}}{1+me^{2u(t)}}e^{v(t-\nu(t))} \\ -b_2(t) - a_2(t)e^{v(t)} \\ +\frac{\alpha_2(t)e^{2u(t-\mu_2(t))}}{1+me^{2u(t-\mu_2(t))}} - \frac{H(t)}{e^{v(t)}} \end{pmatrix},$$

$$\begin{aligned} Pz &= P\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} m(u) \\ m(v) \end{pmatrix} \\ &= \lim_{T\to\infty}\frac{1}{T}\begin{pmatrix} \int_0^T u(s)\,\mathrm{d}s \\ \int_0^T v(s)\,\mathrm{d}s \end{pmatrix} = Qz. \end{aligned}$$

Then N is L-compact on $\bar{\Omega}$(Ω is an open and bounded subset of $\mathbb{X}$).

Now we are in the position to present and prove our result on the existence of at least two positive almost periodic solutions of system (2) by using Lemma 7.

From (F_1) and Lemma 6, for $\forall k\in\mathbb{R}$, there exists a constant $\omega_0 \in (2\mu_1^M, +\infty)$ independent of k such that

$$\begin{cases} \frac{1}{T}\int_k^{k+T} a_i(s)\,\mathrm{d}s \in \left[\frac{\bar{a}_i}{2}, \frac{3\bar{a}_i}{2}\right], \ i=1,2, \\ \frac{1}{T}\int_k^{k+T} H(s)\,\mathrm{d}s \in \left[\frac{\bar{H}}{2}, \frac{3\bar{H}}{2}\right], \quad \forall T \ge \frac{\omega_0}{2}. \end{cases} \tag{10}$$

Let

$$\rho_1 := \ln\left[\frac{4b_1^M}{\bar{a}_1}\right] + b_1^M\omega_0,$$

$$\rho_2 := \ln\left[\frac{2\alpha_2^M e^{2\rho_1}}{(1+me^{2\rho_1})\bar{a}_2}\right] + \frac{\alpha_2^M e^{2\rho_1}\omega_0}{1+me^{2\rho_1}}.$$

By (F_1), $\bar{b}_1 > 0$. We can take

$$l_0 = \max\left\{\omega_0, \frac{4a_1^M e^{\rho_1}\mu_1^M}{\bar{b}_1}\right\}.$$

Similar to (10), for $\forall k\in\mathbb{R}$, there exists a constant $\pi_0 \in [l_0,+\infty)$ independent of k such that

$$\frac{1}{T}\int_k^{k+T} b_1(s)\,\mathrm{d}s \in \left[\frac{\bar{b}_1}{2}, \frac{3\bar{b}_1}{2}\right], \quad \forall T \ge \pi_0. \tag{11}$$

Theorem 1. *Assume that (F_1) and the following condition hold:*

(F_2) $\Delta > 2\sqrt{a_2^M H^M}$, *where*

$$\Delta := \frac{\alpha_2^l e^{2\rho_3}}{1+me^{2\rho_3}} - b_2^M,$$

$$\rho_3 := \ln\left[\frac{\bar{b}_1}{4a_1^M + \alpha_1^M e^{\rho_2}}\right] - b_1^M\pi_0.$$

Then system (2) admits at least two positive almost periodic solutions.

Proof. It is easy to see that if system (9) has one almost periodic solution $(\bar{u},\bar{v})^T$, then $(\bar{N}_1,\bar{N}_2)^T$=$(e^{\bar{u}},e^{\bar{v}})^T$ is a positive almost periodic solution of system (2). Therefore, to complete the proof it suffices to show that system (9) has two distinct almost periodic solutions.

In order to use the Lemma 7, we set the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ as those in Lemma 8 and L, N, P, Q the same as those defined in Lemmas 9 and 10, respectively. It remains to search for an appropriate open and bounded subset $\Omega\subseteq\mathbb{X}$.

Corresponding to the operator equation $Lz=\lambda z$, $\lambda\in(0,1)$, we have

$$\begin{cases} \dot{u}(t) = \lambda\Big[b_1(t) - a_1(t)e^{u(t-\mu_1(t))} \\ \qquad -\frac{\alpha_1(t)e^{u(t)}}{1+me^{2u(t)}}e^{v(t-\nu(t))}\Big], \\ \dot{v}(t) = \lambda\Big[-b_2(t) - a_2(t)e^{v(t)} \\ \qquad +\frac{\alpha_2(t)e^{2u(t-\mu_2(t))}}{1+me^{2u(t-\mu_2(t))}} - \frac{H(t)}{e^{v(t)}}\Big]. \end{cases} \tag{12}$$

Suppose that $(u,v)^T \in \mathrm{Dom}L \subseteq \mathbb{X}$ is a solution of system (12) for some $\lambda\in(0,1)$, where $\mathrm{Dom}L = \{z=(u,v)^T\in\mathbb{X} : u,v\in C^1(\mathbb{R}), \dot{u},\dot{v}\in C(\mathbb{R})\}$. By the almost periodicity of u and v, there must exist two sequences $\{T_n : n \in \mathbb{N}^+\}$ and $\{P_n : n \in \mathbb{N}^+\}$ such that

$$u(T_n) \in \left[u^* - \tfrac{1}{n}, u^*\right], \quad u^* = \sup_{s\in\mathbb{R}} u(s), \tag{13}$$
$$v(P_n) \in \left[v^* - \tfrac{1}{n}, v^*\right], \quad v^* = \sup_{s\in\mathbb{R}} v(s), \tag{14}$$

where $n \in \mathbb{N}^+$.

For $\forall n_0 \in \mathbb{N}^+$, we consider $[T_{n_0} - \omega_0, T_{n_0}]$ and $[P_{n_0} - \omega_0, P_{n_0}]$, where ω_0 is defined as that in (10). By Lemma 4, there exist $\xi \in [T_{n_0} - \omega_0, T_{n_0}]$, $\underline{\xi} \in (-\infty, T_{n_0} - \omega_0]$ and $\bar{\xi} \in [T_{n_0}, +\infty)$ such that

$$u(\underline{\xi}) = u(\bar{\xi}), \quad u(\xi) \le u(s), \quad \forall s \in [\bar{\xi}, \underline{\xi}]. \tag{15}$$

Integrating the first equation of system (12) from $\underline{\xi}$ to $\bar{\xi}$ leads to

$$\int_{\underline{\xi}}^{\bar{\xi}} \left[b_1(s) - a_1(s)e^{u(s-\mu_1(s))} - \frac{\alpha_1(s)e^{u(s)}}{1+me^{2u(s)}} e^{v(s-\nu(s))} \right] \mathrm{d}s = 0,$$

which yields that

$$\begin{aligned} &\int_{\underline{\xi}+\mu_1^M}^{\bar{\xi}} a_1(s)e^{u(s-\mu_1(s))}\,\mathrm{d}s \\ \le\ & \int_{\underline{\xi}}^{\bar{\xi}} a_1(s)e^{u(s-\mu_1(s))}\,\mathrm{d}s \\ \le\ & \int_{\underline{\xi}}^{\bar{\xi}} b_1(s)\,\mathrm{d}s. \end{aligned} \tag{16}$$

By the integral mean value theorem, (10) and (16), there exists $s_0 \in [\underline{\xi} + \mu_1^M, \bar{\xi}]$ $(s_0 - \mu_1(s_0) \in [\underline{\xi}, \bar{\xi}])$ such that

$$\begin{aligned} &\frac{\bar{a}_1}{4} e^{u(s_0-\mu_1(s_0))} \le \frac{\bar{\xi}-\underline{\xi}-\mu_1^M}{\bar{\xi}-\underline{\xi}} \frac{\bar{a}_1}{2} e^{u(s_0-\mu_1(s_0))} \\ \le\ & \frac{\bar{\xi}-\underline{\xi}-\mu_1^M}{\bar{\xi}-\underline{\xi}} e^{u(s_0-\mu_1(s_0))} \\ & \times \frac{1}{\bar{\xi}-\underline{\xi}-\mu_1^M} \int_{\underline{\xi}+\mu_1^M}^{\bar{\xi}} a_1(s)\,\mathrm{d}s \\ =\ & \frac{1}{\bar{\xi}-\underline{\xi}} \int_{\underline{\xi}+\mu_1^M}^{\bar{\xi}} a_1(s)e^{u(s-\mu_1(s))}\,\mathrm{d}s \\ \le\ & \frac{1}{\bar{\xi}-\underline{\xi}} \int_{\underline{\xi}}^{\bar{\xi}} b_1(s)\,\mathrm{d}s \le b_1^M, \end{aligned}$$

which implies from (15) that

$$u(\xi) \le \ln\left[\frac{4b_1^M}{\bar{a}_1}\right]. \tag{17}$$

Let $I_1 = \{s \in [\xi, T_{n_0}] : \dot{u}(s) \ge 0\}$. It follows from system (12) that

$$\begin{aligned} \int_{I_1} \dot{u}(s)\,\mathrm{d}s &= \int_{I_1} \lambda \Big[b_1(s) - a_1(s)e^{u(s-\mu_1(s))} \\ &\quad - \frac{\alpha_1(s)e^{u(s)}}{1+me^{2u(s)}} e^{v(s-\nu(s))} \Big]\,\mathrm{d}s \\ \le\ & \int_{I_1} \lambda b_1(s)\,\mathrm{d}s \le \int_{T_{n_0}-\omega_0}^{T_{n_0}} b_1(s)\,\mathrm{d}s \\ \le\ & b_1^M \omega_0. \end{aligned} \tag{18}$$

By Lemma 3, it follows from (17)-(18) that

$$\begin{aligned} u(t) &\le u(\xi) + \int_{I_1} \dot{u}(s)\,\mathrm{d}s \\ &\le \ln\left[\frac{4b_1^M}{\bar{a}_1}\right] + b_1^M \omega_0 \\ &:= \rho_1, \quad \forall t \in [\xi, T_{n_0}], \end{aligned}$$

which implies that $u(T_{n_0}) \le \rho_1$. In view of (13), letting $n_0 \to +\infty$ in the above inequality leads to

$$u^* = \lim_{n_0 \to +\infty} u(T_{n_0}) \le \rho_1. \tag{19}$$

Also, by Lemma 7, there exist $\zeta \in [P_{n_0} - \omega_0, P_{n_0}]$, $\underline{\zeta} \in (-\infty, P_{n_0} - \omega_0]$ and $\bar{\zeta} \in [P_{n_0}, +\infty)$ such that

$$v(\underline{\zeta}) = v(\bar{\zeta}), \quad v(\zeta) \le v(s), \ \forall s \in [\underline{\zeta}, \bar{\zeta}]. \tag{20}$$

Integrating the second equation of system (12) from $\underline{\zeta}$ to $\bar{\zeta}$ leads to

$$\int_{\underline{\zeta}}^{\bar{\zeta}} \Big[-b_2(s) - a_2(s)e^{v(s)} + \frac{\alpha_2(s)e^{2u(s-\mu_2(s))}}{1+me^{2u(s-\mu_2(s))}} - \frac{H(s)}{e^{v(s)}} \Big]\,\mathrm{d}s = 0, \tag{21}$$

which yields that

$$\int_{\underline{\zeta}}^{\bar{\zeta}} a_2(s)e^{v(s)}\,\mathrm{d}s \le \int_{\underline{\zeta}}^{\bar{\zeta}} \frac{\alpha_2(s)e^{2u(s-\mu_2(s))}}{1+me^{2u(s-\mu_2(s))}}\,\mathrm{d}s,$$

which implies that

$$\begin{aligned} &\frac{\bar{a}_2}{2} e^{v(\zeta)} \le \frac{1}{\bar{\zeta}-\underline{\zeta}} \int_{\underline{\zeta}}^{\bar{\zeta}} a_2(s)e^{v(s)}\,\mathrm{d}s \\ \le\ & \frac{1}{\bar{\zeta}-\underline{\zeta}} \int_{\underline{\zeta}}^{\bar{\zeta}} \frac{\alpha_2(s)e^{2u(s-\mu_2(s))}}{1+me^{2u(s-\mu_2(s))}}\,\mathrm{d}s \\ \le\ & \frac{\alpha_2^M e^{2\rho_1}}{1+me^{2\rho_1}}, \end{aligned}$$

which implies that

$$v(\zeta) \le \ln\left[\frac{2\alpha_2^M e^{2\rho_1}}{(1+me^{2\rho_1})\bar{a}_2}\right]. \tag{22}$$

Let $I_2 = \{s \in [\zeta, P_{n_0}] : \dot{v}(s) \ge 0\}$. It follows from system (12) that

$$\begin{aligned}
\int_{I_2} \dot{v}(s)\,\mathrm{d}s &= \int_{I_2} \lambda\Big[-b_2(s) - a_2(s)e^{v(s)} \\
&\quad + \frac{\alpha_2(s)e^{2u(s-\mu_2(s))}}{1+me^{2u(s-\mu_2(s))}} - \frac{H(s)}{e^{v(s)}}\Big]\,\mathrm{d}s \\
&\le \int_{I_2} \lambda \frac{\alpha_2(s)e^{2u(s-\mu_2(s))}}{1+me^{2u(s-\mu_2(s))}}\,\mathrm{d}s \\
&\le \int_{P_{n_0}-\omega_0}^{P_{n_0}} \frac{\alpha_2(s)e^{2u(s-\mu_2(s))}}{1+me^{2u(s-\mu_2(s))}}\,\mathrm{d}s \\
&\le \frac{\alpha_2^M e^{2\rho_1}\omega_0}{1+me^{2\rho_1}}.
\end{aligned} \tag{23}$$

By Lemma 3, it follows from (22)-(23) that

$$\begin{aligned}
v(t) &\le v(\zeta) + \int_{I_2} \dot{v}(s)\,\mathrm{d}s \\
&\le \ln\left[\frac{2\alpha_2^M e^{2\rho_1}}{(1+me^{2\rho_1})\bar{a}_2}\right] + \frac{\alpha_2^M e^{2\rho_1}\omega_0}{1+me^{2\rho_1}} \\
&:= \rho_2, \quad \forall t \in [\zeta, P_{n_0}],
\end{aligned}$$

which implies that $v(P_{n_0}) \le \rho_2$. In view of (13), letting $n_0 \to +\infty$ in the above inequality leads to

$$v^* = \lim_{n_0 \to +\infty} v(P_{n_0}) \le \rho_2. \tag{24}$$

On the other hand, there exists a sequence $\{H_n : n \in \mathbb{N}^+\}$ such that

$$u(H_n) \in \left[u_*, u_* + \frac{1}{n}\right], \quad u_* = \inf_{s\in\mathbb{R}} u(s), \tag{25}$$

where $n \in \mathbb{N}^+$. For $\forall n_0 \in \mathbb{N}^+$, we consider $[H_{n_0}, H_{n_0}+\pi_0]$, where π_0 is defined as that in (11). By Lemma 5, there exist $\eta \in [H_{n_0}, H_{n_0}+\pi_0]$, $\underline{\eta} \in (-\infty, H_{n_0}]$ and $\bar{\eta} \in [H_{n_0}, +\infty)$ such that

$$u(\underline{\eta}) = u(\bar{\eta}), \quad u(\eta) \ge u(s), \ \forall s \in [\underline{\eta}, \bar{\eta}]. \tag{26}$$

Integrating the first equation of system (12) from $\underline{\eta}$ to $\bar{\eta}$ leads to

$$\begin{aligned}
&\int_{\underline{\eta}}^{\bar{\eta}} \Big[b_1(s) - a_1(s)e^{u(s-\mu_1(s))} \\
&\quad - \frac{\alpha_1(s)e^{u(s)}}{1+me^{2u(s)}}e^{v(s-\nu(s))}\Big]\,\mathrm{d}s = 0,
\end{aligned} \tag{27}$$

which yields from (19) and (24) that

$$\begin{aligned}
&\int_{\underline{\eta}}^{\bar{\eta}} b_1(s)\,\mathrm{d}s \le \int_{\underline{\eta}}^{\bar{\eta}} b_1(s)\,\mathrm{d}s \\
= &\int_{\underline{\eta}}^{\bar{\eta}} a_1(s)e^{u(s-\mu_1(s))}\,\mathrm{d}s \\
&+ \int_{\underline{\eta}}^{\bar{\eta}} \frac{\alpha_1(s)e^{u(s)}}{1+me^{2u(s)}}e^{v(s-\nu(s))}\,\mathrm{d}s \\
\le &\int_{\underline{\eta}+\mu_1^M}^{\bar{\eta}} a_1(s)e^{u(s-\mu_1(s))}\,\mathrm{d}s \\
&+ \int_{\underline{\eta}}^{\underline{\eta}+\mu_1^M} a_1(s)e^{u(s-\mu_1(s))}\,\mathrm{d}s \\
&+ \int_{\underline{\eta}}^{\bar{\eta}} \alpha_1(s)e^{u(s)}e^{v(s-\nu(s))}\,\mathrm{d}s \\
\le &\int_{\underline{\eta}+\mu_1^M}^{\bar{\eta}} a_1(s)e^{u(s-\mu_1(s))}\,\mathrm{d}s \\
&+ a_1^M e^{\rho_1}\mu_1^M + \alpha_1^M e^{\rho_2}e^{u(\eta)}(\bar{\eta} - \underline{\eta}),
\end{aligned}$$

which implies from (11) that

$$\begin{aligned}
\frac{\bar{b}_1}{2} &\le \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}}^{\bar{\eta}} b_1(s)\,\mathrm{d}s \\
&\le \frac{1}{\bar{\eta}-\underline{\eta}}\int_{\underline{\eta}+\mu_1^M}^{\bar{\eta}} a_1(s)e^{u(s-\mu_1(s))}\,\mathrm{d}s \\
&\quad + \frac{a_1^M e^{\rho_1}\mu_1^M}{\bar{\eta}-\underline{\eta}} + \alpha_1^M e^{\rho_2}e^{u(\eta)} \\
&\le \frac{(\bar{\eta}-\underline{\eta}-\mu_1^M)a_1^M e^{u(\eta)}}{\bar{\eta}-\underline{\eta}} \\
&\quad + \frac{a_1^M e^{\rho_1}\mu_1^M}{\pi_0} + \alpha_1^M e^{\rho_2}e^{u(\eta)} \\
&\le (a_1^M + \alpha_1^M e^{\rho_2})e^{u(\eta)} + \frac{\bar{b}_1}{4},
\end{aligned}$$

which implies that

$$u(\eta) \ge \ln\left[\frac{\bar{b}_1}{4a_1^M + \alpha_1^M e^{\rho_2}}\right]. \tag{28}$$

Let $J = \{s \in [H_{n_0}, \eta] : \dot{u}(s) \ge 0\}$. It follows from system (12) that

$$\begin{aligned}
\int_J \dot{u}(s)\,\mathrm{d}s &= \int_J \lambda\Big[b_1(s) - a_1(s)e^{u(s-\mu_1(s))} \\
&\quad - \frac{\alpha_1(s)e^{u(s)}}{1+me^{2u(s)}}e^{v(s-\nu(s))}\Big]\,\mathrm{d}s \\
&\le \int_J \lambda b_1(s)\,\mathrm{d}s \le \int_{H_{n_0}}^{H_{n_0}+\pi_0} b_1(s)\,\mathrm{d}s \\
&\le b_1^M \pi_0.
\end{aligned} \tag{29}$$

By Lemma 3, it follows from (28)-(29) that

$$\begin{aligned} u(t) &\geq u(\eta) - \int_J \dot{u}(s)\,\mathrm{d}s \\ &\geq \ln\left[\frac{\bar{b}_1}{4a_1^M + \alpha_1^M e^{\rho_2}}\right] - b_1^M \pi_0 \\ &:= \rho_3, \quad \forall t \in [H_{n_0}, \eta], \end{aligned}$$

which implies that $u(H_{n_0}) \geq \rho_3$. In view of (25), letting $n_0 \to +\infty$ in the above inequality leads to

$$u_* = \lim_{n_0 \to +\infty} u(H_{n_0}) \geq \rho_3. \tag{30}$$

Also, there exists a sequence $\{L_n : n \in \mathbb{N}^+\}$ such that

$$v(L_n) \in \left[v_*, v_* + \frac{1}{n}\right], \quad v_* = \inf_{s \in \mathbb{R}} v(s), \tag{31}$$

where $n \in \mathbb{N}^+$. For $\forall n_0 \in \mathbb{N}^+$, we consider $[L_{n_0}, L_{n_0} + \pi_0]$. Also, there exist $\varsigma \in [L_{n_0}, L_{n_0} + \pi_0]$, $\underline{\varsigma} \in (-\infty, L_{n_0}]$ and $\bar{\varsigma} \in [L_{n_0} + \pi_0, +\infty)$ such that

$$v(\underline{\varsigma}) = v(\bar{\varsigma}), \quad v(\varsigma) \geq v(s), \quad \forall s \in [\underline{\varsigma}, \bar{\varsigma}]. \tag{32}$$

Integrating the second equation of system (12) from $\underline{\varsigma}$ to $\bar{\varsigma}$ leads to

$$\begin{aligned} &\int_{\underline{\varsigma}}^{\bar{\varsigma}} \bigg[- b_2(s) - a_2(s) e^{v(s)} \\ &+ \frac{\alpha_2(s) e^{2u(s-\mu_2(s))}}{1 + me^{2u(s-\mu_2(s))}} - \frac{H(s)}{e^{v(s)}} \bigg] \mathrm{d}s = 0, \end{aligned}$$

which yields from (10) that

$$\begin{aligned} \frac{\bar{H}(\bar{\varsigma} - \underline{\varsigma})}{2e^{v(\varsigma)}} &\leq \int_{\underline{\varsigma}}^{\bar{\varsigma}} \frac{H(s)}{e^{v(s)}}\,\mathrm{d}s \\ &\leq \int_{\underline{\varsigma}}^{\bar{\varsigma}} \frac{\alpha_2(s) e^{2u(s-\mu_2(s))}}{1 + me^{2u(s-\mu_2(s))}}\,\mathrm{d}s \\ &\leq (\bar{\varsigma} - \underline{\varsigma}) \frac{\alpha_2^M e^{2\rho_1}}{1 + me^{2\rho_1}}, \end{aligned}$$

which implies that

$$v(\varsigma) \geq \ln\left[\frac{\bar{H}(1 + me^{2\rho_1})}{2\alpha_2^M e^{2\rho_1}}\right]. \tag{33}$$

Let $J_2 = \{s \in [L_{n_0}, \varsigma] : \dot{v}(s) \geq 0\}$. It follows from system (12) that

$$\begin{aligned} \int_{J_2} \dot{v}(s)\,\mathrm{d}s &= \int_{J_2} \lambda \bigg[- b_2(s) - a_2(s) e^{v(s)} \\ &\quad + \frac{\alpha_2(s) e^{2u(s-\mu_2(s))}}{1 + me^{2u(s-\mu_2(s))}} - \frac{H(s)}{e^{v(s)}} \bigg] \mathrm{d}s \\ &\leq \int_{J_2} \lambda \frac{\alpha_2(s) e^{2u(s-\mu_2(s))}}{1 + me^{2u(s-\mu_2(s))}}\,\mathrm{d}s \\ &\leq \int_{L_{n_0}}^{L_{n_0}+\pi_0} \frac{\alpha_2(s) e^{2u(s-\mu_2(s))}}{1 + me^{2u(s-\mu_2(s))}}\,\mathrm{d}s \\ &\leq \frac{\alpha_2^M e^{2\rho_1} \pi_0}{1 + me^{2\rho_1}}. \end{aligned} \tag{34}$$

By Lemma 3, it follows from (33)-(34) that

$$\begin{aligned} v(t) &\geq v(\varsigma) - \int_{J_2} \dot{v}(s)\,\mathrm{d}s \\ &\geq \ln\left[\frac{\bar{H}(1 + me^{2\rho_1})}{2\alpha_2^M e^{2\rho_1}}\right] - \frac{\alpha_2^M e^{2\rho_1} \pi_0}{1 + me^{2\rho_1}} \\ r &:= \rho_4, \quad \forall t \in [L_{n_0}, \varsigma], \end{aligned}$$

which implies that $v(L_{n_0}) \geq \rho_4$. In view of (25), letting $n_0 \to +\infty$ in the above inequality leads to

$$v_* = \lim_{n_0 \to +\infty} v(L_{n_0}) \geq \rho_4. \tag{35}$$

By Lemma 2, for $\forall \epsilon \in (0, 1)$, there are two points $\theta = \theta(\epsilon)$ and $\vartheta = \vartheta(\epsilon) \in [0, +\infty)$ such that

$$\begin{cases} \dot{v}(\theta) = 0, \\ v(\theta) \in [v^* - \epsilon, v^*], \end{cases} \quad \begin{cases} \dot{v}(\vartheta) = 0, \\ v(\vartheta) \in [v_*, v_* + \epsilon], \end{cases} \tag{36}$$

where $v^* = \sup_{s \in \mathbb{R}} v(s)$, $v_* = \inf_{s \in \mathbb{R}} v(s)$.

In view of the second equation of system (12), it follows from (36) that

$$b_2(\theta) + a_2(\theta) e^{v(\theta)} - \frac{\alpha_2(\theta) e^{2u(\theta-\mu_2(\theta))}}{1 + me^{2u(\theta-\mu_2(\theta))}} = -\frac{H(\theta)}{e^{v(\theta)}}$$

and

$$b_2(\vartheta) + a_2(\vartheta) e^{v(\vartheta)} - \frac{\alpha_2(\vartheta) e^{2u(\vartheta-\mu_2(\vartheta))}}{1 + me^{2u(\vartheta-\mu_2(\vartheta))}} = -\frac{H(\vartheta)}{e^{v(\vartheta)}},$$

which imply that

$$a_2^M e^{2v(\theta)} - \left[\frac{\alpha_2^l e^{2\rho_3}}{1 + me^{2\rho_3}} - b_2^M\right] e^{v(\theta)} \geq -H^M \tag{37}$$

and

$$a_2^M e^{2v(\vartheta)} - \left[\frac{\alpha_2^l e^{2\rho_3}}{1 + me^{2\rho_3}} - b_2^M\right] e^{v(\vartheta)} \geq -H^M. \tag{38}$$

In view of (37)-(38), we have from (36) that

$$a_2^M e^{2v^*} - \left[\frac{\alpha_2^l e^{2\rho_3}}{1+me^{2\rho_3}} - b_2^M\right] e^{v^*-\epsilon} \geq -H^M$$

and

$$a_2^M e^{2(v_*+\epsilon)} - \left[\frac{\alpha_2^l e^{2\rho_3}}{1+me^{2\rho_3}} - b_2^M\right] e^{v_*} \geq -H^M.$$

Letting $\epsilon \to 0$ in the above inequalities leads to

$$a_2^M e^{2v^*} - \left[\frac{\alpha_2^l e^{2\rho_3}}{1+me^{2\rho_3}} - b_2^M\right] e^{v^*} + H^M \geq 0 \quad (39)$$

and

$$a_2^M e^{2v_*} - \left[\frac{\alpha_2^l e^{2\rho_3}}{1+me^{2\rho_3}} - b_2^M\right] e^{v_*} + H^M \geq 0. \quad (40)$$

By (F_2), it follows from (39)-(40) that

$$\begin{aligned} &v^* \geq \ln l_+ \text{ or } v^* \leq \ln l_-, \\ &v_* \geq \ln l_+ \text{ or } v_* \leq \ln l_-, \end{aligned} \quad (41)$$

where

$$l_\pm := \frac{\Delta \pm \sqrt{\Delta^2 - 4a_2^M H^M}}{2a_2^M}.$$

From (24), (35) and (41), it follows that

$$\begin{aligned} &\rho_4 \leq v(t) \leq \ln l_-, \\ &\ln l_+ \leq v(t) \leq \rho_2, \quad \forall t \in \mathbb{R}. \end{aligned} \quad (42)$$

Obviously, $\ln l_\pm$, ρ_1, ρ_2, ρ_3 and ρ_4 are independent of λ. Let $\varepsilon = \frac{\ln l_+ - \ln l_-}{4}$ and

$$\begin{aligned} \Omega_1 &= \Big\{z = (u,v)^T \in \mathbb{X} : \rho_3 - 1 < u(t) < \rho_1 + 1, \\ &\quad \rho_4 - 1 < v(t) < \ln l_- + \varepsilon, \forall t \in \mathbb{R}\Big\}, \\ \Omega_2 &= \Big\{z = (u,v)^T \in \mathbb{X} : \rho_3 - 1 < u(t) < \rho_1 + 1, \\ &\quad \varepsilon - \ln l_+ < v(t) < \rho_2 + 1, \forall t \in \mathbb{R}\Big\}. \end{aligned}$$

Then Ω_1 and Ω_2 are bounded open subsets of $\mathbb{X}$, $\Omega_1 \cap \Omega_2 = \emptyset$. Therefore, Ω_1 and Ω_2 satisfy condition (a) of Lemma 7.

Now we show that condition (b) of Lemma 7 holds, i.e., we prove that $QNz \neq 0$ for all $z = (u,v)^T \in \partial\Omega_i \cap \mathrm{Ker}L = \partial\Omega_i \cap \mathbb{R}^2$, $i = 1, 2$. If it is not true, then there exists at least one constant vector $z_0 = (u^0, v^0)^T \in \partial\Omega_i\,(i = 1, 2)$ such that

$$\begin{cases} \bar{b}_1 - \bar{a}_1 e^{u^0} - \frac{\bar{\alpha}_1 e^{u^0}}{1+me^{2u^0}} e^{v^0} = 0, \\ -\bar{b}_2 - \bar{a}_2 e^{v^0} + \frac{\bar{\alpha}_2 e^{2u^0}}{1+me^{2u^0}} - \frac{\bar{H}}{e^{v^0}} = 0. \end{cases}$$

Similar to the arguments as that in (19), (30) and (42), it follows that

$$\rho_3 \leq u^0 \leq \rho_1, \quad \rho_4 \leq v^0 \leq \ln l_-, \quad \ln l_+ \leq v^0 \leq \rho_2.$$

Then $z_0 \in \Omega_1 \cap \mathbb{R}^2$ or $z_0 \in \Omega_2 \cap \mathbb{R}^2$. This contradicts the fact that $z_0 \in \partial\Omega_i\,(i = 1, 2)$. This proves that condition (b) of Lemma 7 holds.

Finally, we will show that condition (c) of Lemma 7 is satisfied. Let us consider the homotopy

$$H(\iota, z) = \iota QNz + (1-\iota)Fz, \ (\iota, z) \in [0,1] \times \mathbb{R}^2,$$

where

$$Fz = F\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \bar{b}_1 - \bar{a}_1 e^u \\ -\bar{b}_2 - \bar{a}_2 e^v + \frac{\bar{\alpha}_2 e^{2u}}{1+me^{2u}} - \frac{\bar{H}}{e^v} \end{bmatrix}.$$

From the above discussion it is easy to verify that $H(\iota, z) \neq 0$ on $\partial\Omega \cap \mathrm{Ker}L$. By the invariance property of homotopy, we have

$$\begin{aligned} &\deg\big(JQN, \Omega_i \cap \mathrm{Ker}L, 0\big) \\ = &\deg\big(QN, \Omega_i \cap \mathrm{Ker}L, 0\big) \\ = &\deg\big(F, \Omega_i \cap \mathrm{Ker}L, 0\big), \ i = 1, 2, \end{aligned}$$

where $\deg(\cdot,\cdot,\cdot)$ is the Brouwer degree and J is the identity mapping since $\mathrm{Im}Q = \mathrm{Ker}L$.

Note that the equations of the following system

$$\begin{cases} \bar{b}_1 - \bar{a}_1 e^u = 0, \\ -\bar{b}_2 - \bar{a}_2 e^v + \frac{\bar{\alpha}_2 e^{2u}}{1+me^{2u}} - \frac{\bar{H}}{e^v} = 0 \end{cases}$$

has two distinct solutions:

$$(u_1, v_1) = \left[\ln\left[\frac{\bar{b}_1}{\bar{a}_1}\right], \ln\left[\frac{\bar{\Delta} - \sqrt{\bar{\Delta}^2 - 4\bar{a}_2\bar{H}}}{2\bar{a}_2}\right]\right],$$

$$(u_1, v_2) = \left[\ln\left[\frac{\bar{b}_1}{\bar{a}_1}\right], \ln\left[\frac{\bar{\Delta} + \sqrt{\bar{\Delta}^2 - 4\bar{a}_2\bar{H}}}{2\bar{a}_2}\right]\right].$$

where

$$\bar{\Delta} := \frac{\bar{\alpha}_2 e^{2u_1}}{1+me^{2u_1}} - \bar{b}_2.$$

A direct computation gives

$$\begin{aligned} &\deg\big(F, \Omega_1 \cap \mathrm{Ker}L, 0\big) \\ = &\ \mathrm{sgn}\left|\begin{matrix} -\bar{a}_1 e^u & 0 \\ \frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{\bar{\alpha}_2 e^{2u}}{1+me^{2u}}\right] & -\bar{a}_2 e^v + \frac{\bar{H}}{e^v} \end{matrix}\right|_{(u,v)=(u_1,v_1)} \\ = &\ \mathrm{sgn}\big[\bar{a}_1 e^{u_1}(\bar{a}_2 e^{v_1} - \bar{H}e^{-v_1})\big] \\ = &\ \mathrm{sgn}\left[2\bar{a}_2 e^{v_1} + \bar{b}_2 - \frac{\bar{\alpha}_2 e^{2u_1}}{1+me^{2u_1}}\right] \\ = &\ -1. \end{aligned}$$

Similarly,

$$\begin{aligned} &\deg\big(F, \Omega_2 \cap \mathrm{Ker}L, 0\big) \\ = &\ \mathrm{sgn}\begin{vmatrix} -\bar{a}_1 e^u & 0 \\ \frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{\bar{\alpha}_2 e^{2u}}{1+me^{2u}}\right] & -\bar{a}_2 e^v + \frac{\bar{H}}{e^v} \end{vmatrix}_{(u,v)=(u_1,v_2)} \\ = &\ \mathrm{sgn}\left[2\bar{a}_2 e^{v_2} + \bar{b}_2 - \frac{\bar{\alpha}_2 e^{2u_1}}{1+me^{2u_1}}\right] \\ = &\ 1. \end{aligned}$$

So

$$\begin{aligned} &\deg\big(JQN, \Omega_i \cap \mathrm{Ker}L, 0\big) \\ = &\ \deg\big(F, \Omega_i \cap \mathrm{Ker}L, 0\big) \neq 0,\ i = 1, 2. \end{aligned}$$

Obviously, all the conditions of Lemma 7 are satisfied. Therefore, system (9) has two distinct almost periodic solutions, that is, system (2) has two distinct positive almost periodic solutions. This completes the proof. □

From the proof of Theorem 1, we can show the following results are true.

Corollary 1. *Assume that (F_1)-(F_2) hold. Suppose further that a_i, b_i, α_i, H, μ_i and ν of system (2) are continuous nonnegative periodic functions with different periods, $i = 1, 2$, then system (2) admits at least two positive almost periodic solutions.*

Corollary 2. *Assume that (F_1)-(F_2) hold. Suppose further that all the coefficients of system (2) are continuous nonnegative ω-periodic functions, then system (2) admits at least two positive ω-periodic solutions.*

Remark 1. *By Corollary 1, it is easy to obtain the existence of at least four positive almost periodic solutions of system (3) in Example 1, although the positive periodic solution of system (3) is nonexistent.*

4 An example and numerical simulations

Example 2. *Consider the following delayed almost periodic predator-prey system:*

$$\begin{cases} \dot{N}_1(t) = N_1(t)\Big[1 - |\sin\sqrt{3}t| N_1(t - e^{-10}) \\ \qquad - \frac{N_1(t)}{e^{38}[1+N_1^2(t)]} N_2(t - e^{-10})\Big], \\ \dot{N}_2(t) = N_2(t)\Big[-\frac{1}{1+e^{18}} - \cos^2(\sqrt{2}t) N_2(t) \\ \qquad + \frac{5N_1^2(t-e^{-10})}{1+N_1^2(t-e^{-10})}\Big] + \frac{\sin^2(\sqrt{2}t)}{(1+e^{18})^2}. \end{cases} \tag{43}$$

Corresponding to system (2), we have $b_1 \equiv 1$, $b_2 \equiv \frac{1}{1+e^{18}}$, $\bar{a}_1 = \frac{2}{\pi}$, $\bar{a}_2 = \frac{1}{2}$, $\alpha_1 \equiv \frac{1}{e^{38}}$, $\alpha_2 \equiv 5$, $\bar{H} = \frac{1}{2(1+e^{18})^2}$, $\mu_1^M = e^{-10}$, $m = 1$. Further, for $\forall k \in \mathbb{R}$, we can choose $\omega_0 = \frac{4\sqrt{3}\pi}{3}$ so that (10) holds, that is,

$$\frac{1}{T}\int_k^{k+T} H(s)\,\mathrm{d}s \in \left[\frac{1}{4}, \frac{3}{4}\right], \quad \forall T \geq \frac{\omega_0}{2} = \frac{2\sqrt{3}\pi}{3}$$

and

$$\frac{1}{T}\int_k^{k+T} a_1(s)\,\mathrm{d}s \in \left[\frac{1}{\pi}, \frac{3}{\pi}\right],$$

$$\frac{1}{T}\int_k^{k+T} a_2(s)\,\mathrm{d}s \in \left[\frac{1}{4}, \frac{3}{4}\right], \quad \forall T \geq \frac{\omega_0}{2} = \frac{2\sqrt{3}\pi}{3}.$$

By an easy calculation, we obtain that

$$\rho_1 \approx 8.84, \quad \rho_2 < 38.$$

Let $\pi_0 = \omega_0 = \frac{4\sqrt{3}\pi}{3}$. Then (11) holds and $\rho_3 > -8.6$, which yields that

$$\Delta \approx \frac{5}{1+e^{17.2}} - \frac{1}{1+e^{18}} > \frac{2}{1+e^{18}} > 2\sqrt{a_2^M H^M},$$

which implies that (F_2) holds. Therefore, all the conditions of Theorem 1 are satisfied. By Theorem 1, system (43) admits at least two positive almost periodic solutions (see Figures 1-2 in Appendix).

5 Appendix

In this section, we shall corroborate the analytical findings with the help of numerical simulations accomplished with dde23 function and Matlab.

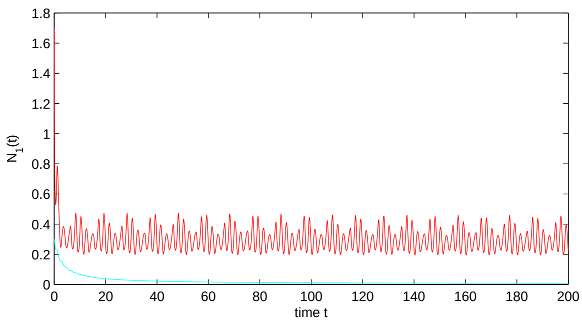

Figure 1 Two positive almost periodicity of state variable N_1 of system (43)

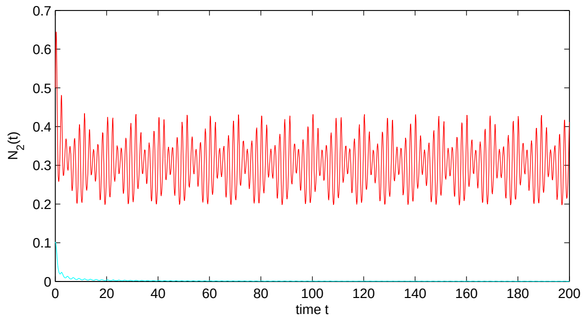

Figure 2 Two positive almost periodicity of state variable N_2 of system (43)

6 Conclusions

By using a fixed point theorem of coincidence degree theory, some criterions for the existence of multiple positive almost periodic solutions to a harvesting predator-prey model with Holling III type functional response and time delays are obtained. Theorem 1 gives the sufficient conditions for the existence of two positive almost periodic solutions of system (2). Therefore, the method used in this paper provides a possible method to study the existence of multiple positive almost periodic solutions of the models in biological populations [27-33, 37-39].

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