The Fourier Transform with piecewise trigonometric kernels and its Applications

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Abstract: In this paper, we develop a theory of the Fourier integral transforms with piecewise-homogeneous trigonometric kernels. Formulas for Integral transforms are obtained in a Hermite-type orthogonal polynomials series form. The resulting formulas are new for both the classical and piecewise- homogeneous cases. In the second part of the article method of Hermite-type orthogonal polynomials series expansion is used to solve the direct and inverse Cauchy problems for the heat equation in a piecewise- homogeneous medium. The inverse Cauchy problem is ill-posed and requires regularization. Formulas for solving direct and inverse Cauchy problems are obtained by the developed method and have advantages: first, they do not contain derivatives, can serve as a basis for regularizing algorithms , and secondly, these formulas are mutually symmetrical.

Key–Words: Fourier integral transform, heat equation, Cauchy problem, Hermite polynomials

1 Introduction

In the first part of the article we prove a new formulas for the direct and inverse Fourier integral transforms on the real axis with n contact points. The proof is based on the theory of orthogonal Hermite polynomials series. In the proof the generating function for the Hermite polynomials playes the key role. Initially, authors deduce a new formulas for the Fourier transforms on the bases of the Hermite-type polynomials series theory. Direct and inverse Fourier integral transforms are obtained as a series of biorthogonal system of Hermite-type polynomials and have a symmetry. Piecewise-homogeneous analogues of the Hermite polynomials played a key role in this construction.Then, new formulas for the direct and inverse Fourier integral transforms on the real axis with n contact points are constructed upon the Sturm - Liouville problem with piecewise-constant coefficients. Hermite- type polynomial is replaced the power function on its piecewise-homogeneous analog. We prove the Hermite-type polynomials and Hermite-type functions form a biorthogonal system.

New methods for direct and inverse Fourier integral transform for piecewise-homogeneous axis are developed in this article. Solutions of the problems are obtained in the form of Hermite- type polynomial series. A well-known classical Fourier integral transform in homogeneous axis are represented in the form of Dirichlet integral. In this case Dirichlet formula is proved on the basis of classical Fourier integral trasform method [14]-[16]. For our main results, we need to develop a Fourier integral trasforms with discontinuous coefficients and based on them to prove the expansion theorems in piecewise-homogeneous axis. Integral transforms with discontinuous coefficients are appeared in the mathematical literature in the 70th of the last century in the works of Uflyand Y.S. [1], Lenuk M.P. [2]; Nayda L.S. [3] , Protsenko V.S. [4], [7].

Direct and adjoint Sturm-Liouville problems with inner contact conditions are considered, their solutions serve as a kernels of direct and inverse Fourier integral transforms with discontinuous coefficients. Expansion theorems are formulated.

We will use required information from the author's work [7]. First note that the structure of integral transforms with the relevant variables are determined by the type of differential equation and the kind of environment where the problem is considered. Therefore decision of integral transforms with discontinuous coefficients are the problem for mathematic modeling in piece-wise homogeneous axis. It is clear this method is an effective for obtaining the exact solution of boundary-value problems for piecewise homogeneous structures mathematical physics. In second part of the article the orthogonal Hermite-type polynomials series are used to solve the direct and inverse Cauchy problem for the heat equation in a piecewise-homogeneous medium. Formulas obtained by authors for inverse Cauchy problems solving in a piecewise homogeneous medium have symmetry with respect to the formulas for the corresponding direct Cauchy problems. In contrast to the classical formulas for the solution of the inverse Cauchy problem, the derivatives are not involved into new formulas. Thus, the new formulas for the inverse problems solving can serve as a basis for regularizing computational algorithms.

2 Integral Fourier transforms at the real axis

In order to prove the new formulas of direct and inverse Fourier transforms, we use well-known theorem [6] on the decomposition of functions into a Hermite polynomials series.

Theorem 1 *If $f(x) \in L_2(-\infty,\infty)$, then then for each $\alpha > 0$ this function can be expanded into Hermite polynomials series*

$$f(x) = \frac{\exp\left(-\frac{x^2}{8\alpha}\right)}{2\sqrt{\pi\alpha}} \sum_{j=0}^{\infty} \frac{1}{2^j j!} H_j\left(\frac{x}{2\sqrt{\alpha}}\right) f_j,$$

with coefficients:

$$f_j = \int_{-\infty}^{\infty} \exp\left(-\frac{\xi^2}{8\alpha}\right) H_j\left(\frac{\xi}{2\sqrt{\alpha}}\right) f(\xi)\, d\xi.$$

Now we can prove a new formula for the Fourier transform.

Theorem 2 *Let $f(x) \in L_2(-\infty,\infty)$*

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\, dx \tag{1}$$

integral Fourier transform for $f(x)$, then

$$F(\lambda) = 4\sqrt{\pi\alpha}\exp\left(-4\lambda^2\alpha\right)\cdot \cdot\sum_{j=0}^{\infty} \frac{e^{i\frac{\pi}{2}j} f_j}{j!} \left(2\sqrt{\alpha}\right)^j H_j\left(2\sqrt{\alpha}\lambda\right), \tag{2}$$

where

$$f_j = \sum_{0\le 2m\le j} \frac{C_j^{2m}}{(16\alpha)^{2m}} f^{(j-2m)}(0).$$

Proof. Let function

$$e^{\frac{x^2}{16\alpha}} f(x)$$

can be expanded into Taylor series for $x \in (-\infty,\infty)$

$$e^{\frac{x^2}{16\alpha}} f(x) = \sum_{j=0}^{\infty} \frac{f_j}{j!} x^j,$$

and

$$\begin{aligned} f_j &= \left(e^{\frac{x^2}{16\alpha}} f(x)\right)^{(j)}(0) \\ &= \sum_{0\le 2k\le j} \begin{pmatrix} j \\ 2p \end{pmatrix} \frac{(2p)!}{(16\alpha)^{2p}} f^{(j-2p)}(0), \end{aligned}$$

then $\alpha-$any positive. Substitute last formula in (1):

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{16\alpha}} e^{-i\lambda x} \sum_{j=0}^{\infty} \frac{f_j}{j!} x^j\, dx$$

Using the known formula of [6]

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{16\alpha}} e^{-i\lambda x} x^j\, dx = 4\sqrt{\pi\alpha}\exp\left(-4\lambda^2\alpha\right)\cdot \cdot e^{i\frac{\pi}{2}j}\left(2\sqrt{\alpha}\right)^j H_j\left(2\sqrt{\alpha}\lambda\right)$$

as a result we obtain a new formula for the Fourier image

$$\begin{aligned} F(\lambda) &= 4\sqrt{\pi\alpha}\exp\left(-4\lambda^2\alpha\right)\cdot \\ &\cdot\sum_{j=0}^{\infty} \frac{e^{i\frac{\pi}{2}j} f_j}{j!} \left(2\sqrt{\alpha}\right)^j H_j\left(2\sqrt{\alpha}\lambda\right), \end{aligned} \tag{3}$$

where

$$f_j = \left(e^{\frac{x^2}{16\alpha}} f(x)\right)^{(j)}(0).$$

Thus, formula (2) gives the Fourier transform in Hermite functions series form. Coefficients in (2) contain derivatives of the original at the point $x = 0$.

Based on Theorem 1, we prove another formula for the direct Fourier transform. The formula we obtain in Theorem 3, in contrast to (2) does not contain derivatives.

Theorem 3 *Let $f(x) \in L_2(-\infty,\infty)$*

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\, dx \tag{4}$$

integral Fourier transform for $f(x)$, then

$$F(\lambda) = \sqrt{2}e^{-2\lambda^2\alpha} \sum_{j=0}^{\infty} \frac{i^j}{2^j j!} H_j\left(2\sqrt{\alpha}\lambda\right) f_j,$$

where

$$f_j = \int_{-\infty}^{\infty} \exp\left(-\frac{\xi^2}{8\alpha}\right) H_j\left(\frac{\xi}{2\sqrt{\alpha}}\right) f(\xi)\, d\xi.$$

Proof. From Theorem 2

$$\begin{aligned} F(\lambda) &= \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\,dx \\ &= \int_{-\infty}^{\infty} e^{-i\lambda x} \frac{\exp\left(-\frac{x^2}{8\alpha}\right)}{2\sqrt{\pi\alpha}} \cdot \\ &\cdot \sum_{j=0}^{\infty} \frac{1}{2^j j!} H_j\left(\frac{x}{2\sqrt{\alpha}}\right) f_j dx, \end{aligned}$$

where

$$f_j = \int_{-\infty}^{\infty} \exp\left(-\frac{\xi^2}{8\alpha}\right) H_j\left(\frac{\xi}{2\sqrt{\alpha}}\right) f(\xi)\,d\xi.$$

Hermite functions $e^{-\frac{x^2}{2}} H_j(x), j = 0, 1, ...$ are Fourier eigenfunctions with eigenvalues

$$c_j = \sqrt{2\pi} i^j, \quad j = 0, 1, ...,$$

so finally we have

$$F(\lambda) = \sqrt{2} e^{-2\lambda^2\alpha} \sum_{j=0}^{\infty} \frac{i^j}{2^j j!} H_j\left(2\sqrt{\alpha}\lambda\right) f_j.$$

This is desired. □

Next, in Theorem 4 we shall prove a new formula for the inverse Fourier transform. Note that the well-known formula for the inverse Fourier transform (1) has the form:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} F(\lambda)\,d\lambda.$$

Reasoning as in the proof of Theorems 2 and 3 lead to a new inverse Fourier formula.

Theorem 4 *Let $f(x) \in L_2(-\infty, \infty)$ and $F(\lambda)$ its integral Fourier transform. Then the inverse formula*

$$f(x) = \frac{\exp\left(-\frac{x^2}{4\alpha}\right)}{2\sqrt{\pi\alpha}} \sum_{j=0}^{\infty} \frac{e^{i\frac{\pi}{2}j}}{\left(2\sqrt{\alpha}\right)^j j!} H_j\left(\frac{x}{2\sqrt{\alpha}}\right) F_j$$

holds true, where

$$e^{\lambda^2\alpha} F(\lambda) = \sum_{j=0}^{\infty} \frac{F_j}{j!} \lambda^j, \tag{5}$$

$$\begin{aligned} F_j &= \left(e^{\lambda^2\alpha} F(\lambda)\right)^{(j)}(0) = \\ &= \sum_{0 \le 2s \le j} C_j^{2s} \alpha^{2s} F^{(j-2s)}(0). \end{aligned}$$

Corollary 5 *If*

$$m_j = \int_{-\infty}^{\infty} x^j f(x)\,dx \tag{6}$$

$j-$ moment of function $f(x)$, then the solution of the moment problem has the form

$$f(x) = \frac{\exp\left(-\frac{x^2}{4\alpha}\right)}{2\sqrt{\pi\alpha}} \sum_{j=0}^{\infty} \frac{(-1)^j}{\left(2\sqrt{\alpha}\right)^j j!} H_j\left(\frac{x}{2\sqrt{\alpha}}\right) \cdot$$

$$\cdot \sum_{0 \le 2s \le j} (-1)^s C_j^{2s} \alpha^{2s} m_{2s-j}.$$

Example 1. If

$$F(\lambda) = e^{-\lambda^2\alpha} \lambda^m, m = 0, 1, ...,$$

then due to (13), we obtain the value of the original

$$f(x) = \frac{\exp\left(-\frac{x^2}{4\alpha}\right)}{2\sqrt{\pi\alpha}} \frac{e^{i\frac{\pi}{2}m}}{\left(2\sqrt{\alpha}\right)^m} H_m\left(\frac{x}{2\sqrt{\alpha}}\right).$$

In conclusion of this section we give next new formula for the Fourier inversion.

Theorem 6 *Let $f(x) \in L_2(-\infty, \infty)$ and $F(\lambda)$ its integral Fourier transform then the Fourier inversion formula*

$$f(x) = \frac{1}{\pi\sqrt{2}} e^{-\frac{x^2}{8\alpha}} \sum_{j=0}^{\infty} \frac{e^{-i\frac{\pi}{2}m}}{2^j j!} H_j\left(\frac{x}{2\sqrt{\alpha}}\right) F_j,$$

holds true, where

$$F_j = \int_{-\infty}^{\infty} \exp\left(-2\lambda^2\alpha\right) H_j\left(2\sqrt{\alpha}\lambda\right) F(\lambda)\,d\lambda.$$

3 Fourier integral transforms with non- separated variables. New expansion theorems

The author's has proposed integral transforms with non-separated variables for solving multidimensional problems [7]. Let V from R^{n+1} be the half-space

$$V = \left\{(y_1, ..., y_n, x) \in R^{n+1} : x > 0\right\},$$

then solution of the Dirichlet's problem for the half-space is expressed by Poisson formula: [17]

$$u(x, y) = \Gamma\left(\frac{n+1}{2}\right) \pi^{-\frac{n+1}{2}} \cdot$$

$$\cdot \int\limits_{y=0} \frac{x}{\left[(y-\eta)^2+x^2\right]^{\frac{n+1}{2}}} f(\eta) d\eta.$$

Obviously Poisson's kernel has the form of integral Laplace transform:

$$\Gamma\left(\frac{n+1}{2}\right)\pi^{-\frac{n+1}{2}} \frac{x}{\left[x^2+(y-\eta)^2\right]^{\frac{n+1}{2}}} =$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_0^\infty \lambda^{\frac{n}{2}} e^{-\lambda x} \frac{J_{\frac{n-1}{2}}(\lambda|y-\eta|)}{|y-\eta|^{\frac{n-2}{2}}} d\lambda,$$

here J_ν is Bessel's function of order ν [18]. Reproduce property of the Poisson kernel is obtained from the expansion of the function $f(y)$ by the Laplace operator Δ eigenfunctions:

$$f(y) = \int\limits_0^\infty \frac{\lambda^{\frac{n}{2}}}{\left(\sqrt{2\pi}\right)^n} \int\limits_{R^n} \frac{J_{\frac{n-2}{2}}(\lambda|y-\eta|)}{|y-\eta|^{\frac{n-2}{2}}} f(\eta)\, d\eta d\lambda. \tag{7}$$

On the basis of this expansion we may conclude that integral transforms with non- separated variables are defined as follows [19]:

Direct integral Fourier transform has the form

$$F[f](y,\lambda) \equiv \hat{f}(y,\lambda) =$$

$$= \frac{1}{\left(\sqrt{2\pi}\right)^n} \int\limits_{R^n} \frac{J_{\frac{n-2}{2}}(\lambda|y-\eta|)}{|y-\eta|^{\frac{n-2}{2}}} f(\eta)\, d\eta, \tag{8}$$

inverse Fourier integral transform has the form

$$F^{-1}[\hat{f}](y) = \int\limits_0^\infty \lambda^{\frac{n}{2}} \hat{f}(y;\lambda) d\lambda \equiv f(y). \tag{9}$$

Now we get new formula. To do this, formula (7) can be written in the form

$$f(y) = \sqrt{\frac{2}{\pi}} \frac{1}{(\sqrt{2\pi})^n} \int_0^\infty e^{-\lambda^2\beta} \lambda^{n-1} \cdot$$

$$\cdot \int_{R^n} e^{\lambda^2\beta} j_{\frac{n-2}{2}}(\lambda|y-\eta|) f(\eta)\, d\eta d\lambda,$$

where $\beta > 0, j_\alpha(z) = \sqrt{\frac{2}{\pi}} \frac{J_\alpha(z)}{z^\alpha}$.

Use a Taylor series expansion with respect to λ

$$e^{\lambda^2\beta} j_\alpha(\lambda x) = \sum_{j=0}^\infty \frac{(-1)^j \lambda^{2j}}{(2j)!} \beta^j H_{2j}^\alpha\left(\frac{x}{2\sqrt{\beta}}\right). \tag{10}$$

where

$$H_{2j}^\alpha(x) = \sum_{p=0}^j \frac{(-1)^{j-p}(2j)!}{2^{\alpha+\frac{1}{2}}(j-p)!\Gamma(2j+\alpha+\frac{3}{2})} (2x)^{2j}.$$

In view of (10) we get

$$f(y) = \sqrt{\frac{2}{\pi}} \frac{1}{(\sqrt{2\pi})^n} \int_0^\infty e^{-\lambda^2\beta} \lambda^{n-1} \cdot$$

$$\cdot \int_{R^n} \sum_{j=0}^\infty \frac{(-1)^j \lambda^{2j}}{(2j)!} \beta^j H_{2j}^\alpha\left(\frac{|y-\eta|}{2\sqrt{\beta}}\right) f(\eta)\, d\eta d\lambda,$$

To simplify last formula, we change the order of integration and compute the inner integral with respect to λ. Then we get new analytical representation for $f(x)$

$$f(y) = \sum_{j=0}^\infty \frac{(-1)^j \Gamma(\frac{n}{2}+j)}{(2j)!} f_{2j}, \tag{11}$$

where

$$f_{2j} = \frac{1}{(\sqrt{2\pi})^{n+1} \beta^{\frac{n}{2}}} \int_{R^n} H_{2j}^\alpha\left(\frac{|y-\eta|}{2\sqrt{\beta}}\right) f(\eta)\, d\eta,$$

Γ-gamma function [20].

4 Vector Fourier transform with discontinuous coefficients

Let's develop the method of vector Fourier transform for the solution this problem. Let's consider Sturm–Liouville vector theory [7] about a design bounded on the set of nontrivial solution of separate simultaneous ordinary differential equations with constant matrix coefficients

$$\left(A_m^2 \frac{d^2}{dx^2} + \lambda^2 \mathrm{E}\right) y_m = 0, m = \overline{1, n+1}, \tag{12}$$

where E-unit matrix size $r \times r$. On the boundary conditions

$$\|y_1\|\, |_{x=-\infty} < \infty, \quad \|y_{n+1}\|\, |_{x=\infty} < \infty \tag{13}$$

and conditions of the contact in the points of conjugation of intervals

$$\left(\left(\alpha_{j1}^k + \lambda^2 \delta_{j1}^k\right) \frac{d}{dx} + \left(\beta_{j1}^k + \lambda^2 \gamma_{j1}^k\right)\right) y_k =$$

$$= \left(\left(\alpha_{j2}^k + \lambda^2 \delta_{j2}^k\right) \frac{d}{dx} + \left(\beta_{j2}^k + \lambda^2 \gamma_{j2}^k\right)\right) y_{k+1}, \tag{14}$$

$$x = l_k, \;\; k = \overline{1, n}, \;\; j = 1, 2.,$$

where

$$y_m(x,\lambda) = \begin{pmatrix} y_{1m}(x,\lambda) \\ \vdots \\ y_{rm}(x,\lambda) \end{pmatrix},$$

$$\|y_m\| = \sqrt{y_{1m}^2 + ... + y_{rm}^2}, \;\; m = \overline{1, n+1}.$$

For some λ, the boundary problem under consideration has a nontrivial solution

$$y(x, \lambda) = \sum_{k=1}^{n+1} \theta(x - l_{k-1})\, \theta(l_k - x)\, y_k(x, \lambda).$$

The number λ is called an Eigenvalue in this case, and the corresponding decision $y(x, \lambda)$ is called vector-valued Eigenfunction.

$$\alpha_{11}^0, \beta_{11}^0, \gamma_{11}^0, \delta_{11}^0, \alpha_{ji}^k, \beta_{ji}^k, \gamma_{ji}^k, \delta_{ji}^k, A_j -$$

are matrices of the size $r \times r$. We shall require invertibility

$$\det \;\; M_{mk} \neq 0, \;\; \lambda \in (-\infty, \infty) \tag{15}$$

for matrixes

$$M_{mk} \equiv \begin{pmatrix} \beta_{1m}^k + \lambda^2 \gamma_{1m}^k & \alpha_{1m}^k + \lambda^2 \delta_{1m}^k \\ \beta_{2m}^k + \lambda^2 \gamma_{2m}^k & \alpha_{2m}^k + \lambda^2 \delta_{2m}^k \end{pmatrix},$$

$$m = 1, 2; \;\; k = \overline{1, n}.$$

Matrices A_m^2, $m = \overline{1, n+1}$, are positive-defined [21]. We denote

$$\Phi_{n+1}(x) = e^{q_{n+1} x i}; \;\; \Psi_{n+1}(x) = e^{-q_{n+1} x i}\;.$$

$$q_{n+1}^2 = \lambda^2 A_{n+1}^{-2}.$$

Define the induction relations the others n-pairs a matrix-importance functions (Φ_k, Ψ_k), $k = 1, n$:

$$\left[\left(\alpha_{j1}^k + \lambda^2 \delta_{j1}^k\right) \frac{d}{dx} + \left(\beta_{j1}^k + \lambda^2 \gamma_{j1}^k\right)\right] (\Phi_k, \Psi_k) =$$

$$\left[\left(\alpha_{j2}^k + \lambda^2 \delta_{j2}^k\right) \frac{d}{dx} + \left(\beta_{j2}^k + \lambda^2 \gamma_{j2}^k\right)\right] (\Phi_{k+1}, \Psi_{k+1}), \tag{16}$$

$$k = \overline{1, n}, \quad j = \overline{1, 2}$$

Let us introduce the following notation

$$\Omega_k = \begin{pmatrix} \Phi_k & \Psi_k \\ \Phi_k^{/} & \Psi_k^{/} \end{pmatrix}, \quad i = \overline{1, n+1}.$$

Theorem 7 *The spectrum of the problem (12), (13), (14) is a continuous and fills all axis $(-\infty, \infty)$. Sturm–Liouville theory r time is degenerate. To each Eigenvalue λ corresponds to exactly r linearly independent vector-valued functions. As the last it is possible to take r columns matrix-importance functions.*

$$v(x, \lambda) = \sum_{k=1}^{n+1} \theta(x - l_{k-1})\, \theta(l_k - x)\, v_k(x, \lambda),$$

$$v_j(x, \lambda) = \Psi_j(x, \lambda). \tag{17}$$

That is

$$y^m(x, \lambda) = \begin{pmatrix} v_{1m}(x, \lambda) \\ \vdots \\ v_{rm}(x, \lambda) \end{pmatrix}.$$

Dual Sturm–Liouville theory consists in a finding of the non-trivial solution of separate simultaneous ordinary differential equations with constant matrix coefficients.

$$\left(A_m^2 \frac{d^2}{dx^2} + \lambda^2 \mathrm{E}\right) y_m^* = 0, \;\; , m = \overline{1, n+1} \tag{18}$$

on the boundary conditions

$$\|y_1^*\| < \infty, \;\; \|y_{n+1}^*\| < \infty, \tag{19}$$

and conditions of the contact in the points of conjugation of intervals

$$\left(-\frac{d}{dx} y_k^*, y_k^*\right) \begin{pmatrix} \beta_{11}^k + \lambda^2 \gamma_{11}^k & \alpha_{11}^k + \lambda^2 \delta_{11}^k \\ \beta_{21}^k + \lambda^2 \gamma_{21}^k & \alpha_{21}^k + \lambda^2 \delta_{21}^k \end{pmatrix}^{-1}$$

$$= \left(-\frac{d}{dx} y_{k+1}^*, y_{k+1}^*\right) \cdot$$

$$\cdot \begin{pmatrix} \beta_{12}^k + \lambda^2 \gamma_{12}^k & \alpha_{12}^k + \lambda^2 \delta_{12}^k \\ \beta_{22}^k + \lambda^2 \gamma_{22}^k & \alpha_{22}^k + \lambda^2 \delta_{22}^k \end{pmatrix}^{-1}, \tag{20}$$

$$x = l_k, \quad k = \overline{1, n}$$

The solution of the boundary value problem we write in the form of

$$y^*(\xi, \lambda) = \sum_{k=1}^{n+1} \theta(\xi - l_{k-1})\, \theta(l_k - \xi)\, y_k^*(\xi, \lambda),$$

$$y_m^*(\xi, \lambda) = \begin{pmatrix} y_{m1}^*(\xi, \lambda) & \cdots & y_{mr}^*(\xi, \lambda) \end{pmatrix},$$

$$\|y_m^*\| = \sqrt{(y_{1m}^*)^2 + ... + (y_{rm}^*)^2}, m = \overline{1, n+1}.$$

Theorem 8 *The spectrum of the problem (18), (19), (20) is a continuous and fills axis $(-\infty, \infty)$. Sturm–Liouville theory r time is degenerate. To each Eigenvalue λ corresponds to exactly r linearly independent vector-valued functions. As the last it is possible to take r rows matrix-importance functions.*

$$v^*(x, \lambda) = \sum_{k=1}^{n+1} \theta(x - l_{k-1})\, \theta(l_k - x)\, v_k^*(x, \lambda),$$

$$v_j^*(x, \beta) = (\mathrm{E}, 0)\, \Omega_j^{-1}(x, \beta) \begin{pmatrix} 0 \\ \mathrm{E} \end{pmatrix} A_j^{-2},$$

That is

$$y^{*j}(\xi, \lambda) = \begin{pmatrix} v_{j1}^*(\xi, \lambda) & \cdots & v_{jr}^*(\xi, \lambda) \end{pmatrix}, j = \overline{1, r}. \tag{21}$$

The existence of spectral functions $u(x,\lambda)$ and the conjugate spectral function $u^*(x,\lambda)$ allows to write the a vector decomposition theorem on the set of I_n^+.

Theorem 9 *Let the vector-valued function $f(x)$ be defined on I_n continuous, absolutely integrated and have the bounded total variation. Then for any $x \in I_n$ the following formula decomposition is true:*

$$f(x) = -\frac{1}{\pi i}\int_{-\infty}^{\infty} v(x,\lambda)\,(\int_{-\infty}^{\infty} v^*(\xi,\lambda)\, f(\xi)\, d\xi +$$

$$+\sum_{k=1}^{n} (\mathrm{E},0)\ \Omega_k^{-1}(l_k,\lambda)\ M_{k1}^{-1}(\lambda)\cdot(N_2 - N_1))\lambda d\lambda,$$

$$N_i = \begin{pmatrix} \gamma_{1+i1}^k & \delta_{1+i1}^k \\ \gamma_{1+i2}^k & \delta_{1+i2}^k \end{pmatrix}\begin{pmatrix} f_{k+i-1}(l_k) \\ f'_{k+i-1}(l_k) \end{pmatrix}. \quad (22)$$

The decomposition theorem allows to enter the direct and inverse matrix integral Fourier transform on the real axis with conjugation points:

$$F_n[f](\lambda) \equiv \tilde{f}(\lambda) = \int_{-\infty}^{\infty} v^*(\xi,\lambda)\, f(\xi)\, d\xi +$$

$$+\sum_{k=1}^{n} (\mathrm{E},0)\ \Omega_k^{-1}(l_k,\lambda)\ M_{k1}^{-1}(\lambda)\cdot(N_2 - N_1) \quad (23)$$

and

$$F_n^{-1}\left[\tilde{f}\right](x) = -\frac{1}{\pi i}\int_{-\infty}^{\infty} \lambda v(x,\lambda)\ \tilde{f}(\lambda)\, d\lambda, \quad (24)$$

when

$$f(x) = \sum_{k=1}^{n+1} \theta(l_k - x)\ \theta(x - l_{k-1})\ f_k(x).$$

Let's result the basic identity of integral transform of the differential operator

$$B = \sum_{j=1}^{n} \theta(x - l_{j-1})\ \theta(l_j - x)\, A_j^2 \frac{d^2}{dx^2}.$$

Theorem 10 *If vector-valued function*

$$f(x) = \sum_{k=1}^{n+1} \theta(x - l_{k-1})\ \theta(l_k - x)\, f_k(x),$$

is three times continuously differentiable on the set, and the limit values together with its derivatives up to the third order inclusive satisfies to the boundary condition on infinity

$$\lim_{x\to\infty}\left(u^*(x,\lambda)\frac{d}{dx}f(x) - \frac{d}{dx}u^*(x,\lambda)\ f(x)\right) = 0$$

and homogeneous conditions of conjugation (14), that basic identity of integral transform of the differential operator B hold

$$F_n[B(f)](\lambda) = -\lambda^2\tilde{f}. \quad (25)$$

The proofs of Theorems 7–10 are spent by a method of the method of contour integration. Similarly presented to work of the author [7].

5 Piece-wise homogeneous analogues of Hermite polynomials and Hermite functions

Definition 11 *Right and left analogs of power function are defined by formulas*

$$\xi_n^{*k} = i^k D_\lambda^k v^*(\xi,0)\,, x_n^k = (-i)^k D_\lambda^k v(x,0)\,.$$

respectively.

The function $e^{\lambda^2\beta}v^*(\xi,\lambda)$ is a generating function for Hermite polynomials [6] with piece-wise constants coefficients, this means that

$$e^{\lambda^2\beta}v^*(\xi,\lambda) = \sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}H_{j,n}^*(\xi,\beta)\,. \quad (26)$$

Definition 12 *The Hermite piece-wise polynomials are called the the sequence of functions $H_{j,n}^*(z,\beta)$ from (26).*

In the homogeneous case

$$v^*(\xi,\lambda) = e^{-i\lambda\xi},$$

$$H_{j,0}^*(\xi,\beta) = \beta^{\frac{j}{2}} H_j\left(\frac{\xi}{2\sqrt{\beta}}\right),$$

where $H_j(z)$ is classical Hermite polynomial [6]. Expansion of piece-wise homogeneous analogues of Hermite polynomials on the right piece-wise analogues of the power function is followed from Definitions 11 and 12.

Theorem 13 *If*

$$H_{j,0}^*(\xi,\beta) = \sum_{k=0}^{j} h_{k,j}\xi^k$$

is an expansion of Hermite polynomial with respect to ξ, then for their piece-wise analogues $H^*_{j,n}(\xi,\beta)$, the representation

$$H^*_{j,n}(\xi,\beta) = \sum_{k=0}^{j} h_{k,j}\xi_n^{*k}$$

is true.

Definition 14 *For each fixed $j = 0,1,2,...$, we define a piece-wise analogue of Hermite function $H_{j,n}(x,\beta)$, $j = 0,1,2,...$ as follows:*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\lambda)^j e^{-\lambda^2\beta} v(x,\lambda)\,d\lambda = H_{j,n}(x,\beta). \tag{27}$$

In the homogeneous case, we have

$$v(x,\lambda) = e^{i\lambda x},$$

$$H_{j,0}(x,\beta) = \frac{e^{-\frac{x^2}{4(\beta)}}}{2\sqrt{\pi\beta}}\frac{1}{(2\sqrt{\beta})^j}H_j\left(\frac{x}{2\sqrt{\beta}}\right),$$

where $H_j(z)$ is classical Hermite polynomial [6]. Expansion of piece-wise analogues of Hermite polynomials on the left piece-wise analogues of the power function is followed From Definitions 11 and 12.

Theorem 15 *If*

$$H_{j,0}(x,\beta) = \sum_{k=0}^{\infty} h_{k,j}x^k$$

is the expansion of the Hermite function into Taylor series with respect to x, then for its piece-wise analogue $H_{j,n}(x,\beta)$, the representation

$$H_{j,n}(x,\beta) = \sum_{k=0}^{\infty} h_{k,j}x_n^k.$$

holds true.

Theorem 16 *System of functions $H_{j,n}(x,\beta)$, $H^*_{k,n}(x,\beta)$ is biorthogonal, i.e.,*

$$\int_{-\infty}^{\infty} H_{j,n}(x,\beta)H^*_{k,n}(x,\beta)dx = \delta_{j,k}.$$

Proof. Consider the integral

$$\int_{-\infty}^{\infty} H_{j,n}(x,\beta)e^{s^2\beta}v^*(\xi,s)\,dx$$
$$= \int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\lambda)^j e^{-\lambda^2\beta}\cdot$$
$$\cdot v(x,\lambda)\,d\lambda e^{s^2\beta}v^*(x,s)\,dx.$$

Changing the order of integration and applying the decomposition theorem, we obtain the equality

$$\int_{-\infty}^{\infty} H_{j,n}(x,\beta)e^{s^2\beta}v^*(\xi,s)\,dx = (-is)^j.$$

To complete the proof we use equation (27) and the uniqueness of Taylor's expansion. The theorem is proved. □

6 New expansion theorems

We use the well-known expansion theorem for function $f(x)$ in Fourier integral form [7]:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} v(x,\lambda)\cdot$$
$$\cdot\left(\int_{-\infty}^{\infty} v^*(\xi,\lambda)\,f(\xi)\,d\xi\right)d\lambda. \tag{28}$$

Write the last equality in the form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\lambda^2\beta}v(x,\lambda)\cdot$$
$$\cdot\left(\int_{-\infty}^{\infty} e^{\lambda^2\beta}v^*(\xi,\lambda)\,f(\xi)d\xi\right)d\lambda, \tag{29}$$

where $\beta > 0$.

In accordance with (26), formula (29) takes the form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\lambda^2\beta}v(x,\lambda)\cdot$$
$$\cdot\sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}\int_{-\infty}^{\infty} H^*_{j,n}(\xi,\beta)\,f(\xi)\,d\xi d\lambda.$$

Then we use definition 12 and finally get new analytical representation at the point x

$$f(x) = \sum_{j=0}^{\infty} H_{j,n}(x,\beta)\frac{f_j}{j!}, \tag{30}$$

where

$$f_j = \int_{-\infty}^{\infty} H^*_{j,n}(\xi,\beta)\,f(\xi)\,d\xi.$$

We get a new expansion theorem. Classical expansion theorem takes the form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda^2\tau}v(x,\lambda)\cdot$$
$$\cdot\left(\int_{-\infty}^{\infty} v^*(\xi,\lambda)\,u(\tau,\xi)\,d\xi\right)d\lambda.$$

If $\beta > 0$, then the last formula takes the form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\lambda^2\beta}e^{\lambda^2\beta}v(x,\lambda)\cdot$$

$$\cdot \left(\int_{-\infty}^{\infty} v^* (\xi, \lambda) \, u (\tau, \xi) \, d\xi \right) d\lambda. \tag{31}$$

Because of formula (26) we get

$$f (x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda^2 \beta} v (x, \lambda) \cdot$$

$$\cdot \sum_{j=0}^{\infty} \frac{(-i\lambda)^j}{j!} \int_{-\infty}^{\infty} H_{j,n}^* (\xi, \beta) f (\xi) \, d\xi d\lambda.$$

We shall change the order of integration and calculate the inner integral with respect to λ. On the basis of (27) we can write

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\lambda)^j e^{-\lambda^2 \beta} v (x, \lambda) \, d\lambda = H_{j,n} (x, \beta) .$$

Finally, second new expansion theorem takes the form

$$f(x) = \sum_{j=0}^{\infty} H_{j,n} (x, \beta) \frac{f_j}{j!}, \tag{32}$$

where

$$f_j = \int_{-\infty}^{\infty} H_{j,n}^* (\xi, \beta) f (\xi) \, d\xi.$$

Now we get third new formula. To do this, formula (28) can be written in the form

$$f (x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda^2 \beta} \cdot$$

$$\cdot \left(\int_{-\infty}^{\infty} e^{\lambda^2 \beta} v (x, \lambda) \, v^* (\xi, \lambda) \, f (\xi) \, d\xi \right) d\lambda,$$

where $\beta > 0$.

Use a Taylor series expansion with respect to λ

$$e^{\lambda^2 \beta} v (x, \lambda) \, v^* (\xi, \lambda) = \sum_{j=0}^{\infty} \frac{(-i\lambda)^j}{j!} H_{j,n} (x, \xi, \beta) . \tag{33}$$

In the homogeneous case we have

$$v(x, \lambda) v^* (\xi, \lambda) = e^{-i\lambda(\xi - x)},$$

$$H_{j,n}(x, \xi, \beta) = \beta^{\frac{j}{2}} H_j \left(\frac{\xi - x}{2\sqrt{\beta}} \right),$$

where $H_j(z)$ is the classical Hermite polynomial.

Let

$$H_{j,0}(\xi - x, \beta) = \sum_{k=0}^{j} h_{k,j} \sum_{\alpha+\beta=k} (-1)^\beta \frac{(\alpha + \beta)!}{\alpha! \beta!} \xi^\alpha x^\beta$$

be Hermite polynomial expansion on powers ξ, x. Then for their piece-wise homogeneous analogues $H_{j,n}(\xi, x, \beta)$, so called Hermite-type polynomials, we have the representation

$$H_{j,n}(\xi, x, \beta) = \sum_{k=0}^{j} h_{k,j} \sum_{\alpha+\beta=k} (-1)^\beta \frac{(\alpha + \beta)!}{\alpha! \beta!} \xi_n^{*\alpha} x_n^\beta.$$

In view of (10) we get

$$f (x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda^2 \beta} \sum_{j=0}^{\infty} \frac{(-i\lambda)^j}{j!} \cdot$$

$$\cdot \int_{-\infty}^{\infty} H_{j,n} (x, \xi, \beta) \, f (\xi) \, d\xi d\lambda.$$

To simplify last formula, we change the order of integration and compute the inner integral with respect to λ, substitute $x = 0$ in (33). Then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\lambda)^j e^{-\lambda^2 \beta} d\lambda = \frac{1}{(2\sqrt{\beta})^{j+1}} H_j (0) . \tag{34}$$

Taking into account the known formula from [6]

$$H_{2j} (0) = \frac{(-1)^n (2n)!}{2^n n!}, H_{2j+1} (0) = 0. \tag{35}$$

we get finally new analytical representation for $f(x)$

$$f(x) = \sum_{j=0}^{\infty} \frac{1}{(2\sqrt{\beta})^{2j+1}} \frac{(-1)^j f_{2j}}{2^j j!}, \tag{36}$$

where

$$f_{2j} = \int_{-\infty}^{\infty} H_{2j,n} (x, \xi, \beta) \, f (\xi) \, d\xi.$$

7 Cauchy problem for the heat equation.

For solution $u (\tau, x)$ of Cauchy problem [7],[8],[10] with the initial thermal field $f (x)$ for an piece-wise homogeneous infinite bar, we shall get as Hermite-type polynomial series. In order to get this result, we use the well-known analytic solution $u (\tau, x)$ in Fourier integral form [7]:

$$u (\tau, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\lambda^2 \tau} v (x, \lambda) \cdot$$

$$\cdot \left(\int_{-\infty}^{\infty} v^* (\xi, \lambda) \, f (\xi) \, d\xi \right) d\lambda, \tag{37}$$

where $f (x)$ is the initial thermal field, $u (\tau, x)$ is thermal field at time τ and in the point x.

Write the last equality in the form

$$u(\tau,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2(\tau+\beta)}v(x,\lambda)\cdot$$

$$\cdot\left(\int_{-\infty}^{\infty}e^{\lambda^2\beta}v^*(\xi,\lambda)\,f(\xi)d\xi\right)d\lambda, \tag{38}$$

where $\beta>0$.

In accordance with (26), formula (38) takes the form

$$u(\tau,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2(\tau+\beta)}v(x,\lambda)\cdot$$

$$\cdot\sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}\int_{-\infty}^{\infty}H^*_{j,n}(\xi,\beta)\,f(\xi)\,d\xi d\lambda.$$

Then we use definition 12 and finally get new analytical representation for the thermal field in time τ and at the point x

$$u(\tau,x)=\sum_{j=0}^{\infty}H_{j,n}(x,\tau+\beta)\,\frac{f_j}{j!}, \tag{39}$$

where

$$f_j=\int_{-\infty}^{\infty}H^*_{j,n}(\xi,\beta)\,f(\xi)\,d\xi.$$

Now we get the second new formula. To do this, formula (38) can be written in the form

$$u(\tau,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2(\tau+\beta)}\cdot$$

$$\cdot\left(\int_{-\infty}^{\infty}e^{\lambda^2\beta}v(x,\lambda)\,v^*(\xi,\lambda)\,f(\xi)\,d\xi\right)d\lambda,$$

where $\beta>0$.

Use a Taylor series expansion with respect to λ

$$e^{\lambda^2\beta}v(x,\lambda)\,v^*(\xi,\lambda)\;=\sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}H_{j,n}(x,\xi,\beta)\,. \tag{40}$$

In the homogeneous case, we have

$$v(x,\lambda)v^*(\xi,\lambda)=e^{-i\lambda(\xi-x)},$$

$$H_{j,n}(x,\xi,\beta)=\beta^{\frac{j}{2}}H_j\left(\frac{\xi-x}{2\sqrt{\beta}}\right),$$

where $H_j(z)$ is the classical Hermite polynomial.

Let

$$H_{j,0}(\xi-x,\beta)=\sum_{k=0}^{j}h_{k,j}\sum_{\alpha+\beta=k}(-1)^\beta\frac{(\alpha+\beta)!}{\alpha!\beta!}\xi^\alpha x^\beta$$

be Hermite polynomial expansion on powers ξ,x. Then for their piece-wise homogeneous analogues $H_{j,n}(\xi,x,\beta)$, so called Hermite-type polynomials, we have the representation

$$H_{j,n}(\xi,x,\beta)=\sum_{k=0}^{j}h_{k,j}\sum_{\alpha+\beta=k}(-1)^\beta\frac{(\alpha+\beta)!}{\alpha!\beta!}\xi_n^{*\alpha}x_n^\beta.$$

In view of (10) we get

$$u(\tau,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\lambda^2(\tau+\beta)}\cdot$$

$$\cdot\sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}\int_{-\infty}^{\infty}H_{j,n}(x,\xi,\beta)\,f(\xi)\,d\xi d\lambda.$$

To simplify last formula, we change the order of integration and compute the inner integral with respect to λ, substitute $x=0$. Then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\lambda)^je^{-\lambda^2(\tau+\beta)}d\lambda=\frac{1}{(2\sqrt{\tau+\beta})^{j+1}}H_j(0)\,. \tag{41}$$

Taking into account the well-known formula from [6]

$$H_{2j}(0)=\frac{(-1)^n(2n)!}{2^nn!},\;\;H_{2j+1}(0)=0, \tag{42}$$

we get new analytical representation for thermal field in piecewise-homogeneous bar

$$u(\tau,x)=\sum_{j=0}^{\infty}\frac{1}{(2\sqrt{\tau+\beta})^{2j+1}}\frac{(-1)^jf_{2j}}{2^jj!}, \tag{43}$$

where

$$f_{2j}=\int_{-\infty}^{\infty}H_{2j,n}(x,\xi,\beta)\,f(\xi)\,d\xi.$$

8 Inverse Cauchy problem for the heat equation

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The inverse problem [10]-[13] for the heat equation of an infinite bar is to find the unknown initial distribution $f(x)$ of thermal field by the known thermal field $u(\tau,x)$. This problem leads to the solving of first type Fredholm integral equation [23]:

$$\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi\tau}}\exp\left(-\frac{(x-\xi)^2}{4\tau}\right)f(\xi)\,d\xi=u(\tau,x)\,. \tag{44}$$

The left side of equation (44) is the Poisson integral, [23]. As it is shown in [23] the solution of equation (44) is:

$$f(x)=\frac{1}{\sqrt{\pi}}\sum_{j=0}^{\infty}\frac{u^{(j)}(0)}{(2\sqrt{\tau})^{n+1}j!}H_j\left(\frac{x}{2\sqrt{\tau}}\right), \tag{45}$$

where $H_j(z)$ is the Hermite polynomials [22].

Formula (45) contains a derivatives of an arbitrarily high order so formula (45) can't serve as a basis for the regularizing computational algorithm. Consequently it is actual to find new formulas without derivatives for solution of equation (44).

We obtain two new formulas. We get a solution of equation (44) by Fourier integral transform method, see [6]-[7],

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda^2\tau} v(x,\lambda)\cdot$$

$$\cdot\left(\int_{-\infty}^{\infty} v^*(\xi,\lambda)\, u(\tau,\xi)\, d\xi\right) d\lambda.$$

If $\beta > 0$, then the last formula takes the form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\lambda^2\beta} e^{\lambda^2(\tau+\beta)} v(x,\lambda)\cdot$$

$$\cdot\left(\int_{-\infty}^{\infty} v^*(\xi,\lambda)\, u(\tau,\xi)\, d\xi\right) d\lambda. \tag{46}$$

Because of formula (26) we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\lambda^2\beta} v(x,\lambda)\cdot$$

$$\cdot\sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}\int_{-\infty}^{\infty} H^*_{j,n}(\xi,\tau+\beta)\, u(\tau,\xi)\, d\xi d\lambda.$$

We shall change the order of integration and calculate the inner integral with respect to λ. We can write

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\lambda)^j e^{-\lambda^2\beta} v(x,\lambda)\, d\lambda = H_{j,n}(x,\beta).$$

Finally, first new formula for the initial thermal field takes the form

$$f(x) = \sum_{j=0}^{\infty} H_{j,n}(x,\beta)\,\frac{u_j}{j!}, \tag{47}$$

where

$$u_j = \int_{-\infty}^{\infty} H^*_{j,n}(\xi,\tau+\beta)\, u(\tau,\xi)\, d\xi.$$

Finally, we shall prove the second new formula for solution of inverse Cauchy problem.

We use (45) which can be written as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\lambda^2\beta} e^{\lambda^2(\tau+\beta)}\cdot$$

$$\cdot\left(\int_{-\infty}^{\infty} v(x,\lambda)\, v^*(\xi,\lambda)\, u(\tau,\xi)\, d\xi\right) d\lambda,$$

where $\beta > 0$.

Thank to (26), we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\lambda^2\beta}\cdot$$

$$\cdot\sum_{j=0}^{\infty}\frac{(-i\lambda)^j}{j!}\int_{-\infty}^{\infty} H_{j,n}(x,\xi,\tau+\beta)\, u(\tau,\xi)\, d\xi d\lambda.$$

If we use formula (34), then the initial distribution of thermal field takes the form

$$f(x) = \sum_{j=0}^{\infty}\frac{1}{(2\sqrt{\beta})^{2j+1}}\frac{(-1)^j\, u_{2j}}{2^j j!}, \tag{48}$$

where

$$u_j = \int_{-\infty}^{\infty} H_{j,n}(x,\xi,\tau+\beta)\, u(\tau,\xi)\, d\xi.$$

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