Qualitative Analysis for A Lotka-Volterra Model with Time Delays

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Abstract: This paper considers a Lotka-Volterra model with time delays and delay dependent parameters. The linear stability conditions are obtained with characteristic root method. The Hopf bifurcation is demonstrated. Using normal form theory and center manifold theory, Some explicit formulae for determining the stability and the direction of the Hopf bifurcation periodic solutions are derived. Finally, numerical simulations are carried out to support the theoretical predictions.

Key–Words: Time delay; Lotka-Volterra model, Stability, Hopf bifurcation, Delay dependent parameter

1 Introduction

The past decades have witnessed an enormous interest in predator-prey models with stage structures. The dynamical phenomena of predator-prey models with stage structures have been extensively investigated because of the wide application in the field of biomathematics. For example, Ma et al. [1] investigated the permanence of a stage-structured predator-prey system with a class of functional responses. Xu et al. [2] considered the permanence and periodicity of a delayed ratio-dependent predator-prey model with stage structure. Bairagi and Jana [3] focused on the oscillations and control of an age-structured predator-prey model with habit complexity. Chakraborty et al. [3] analyzed the local stability, global stability and Hopf bifurcation of a stage structured prey-predator model incorporating cannibalism in competitive environment. For more research on the predator-prey models with stage structures, one can see [4-14]. In particular, the appearance of a cycle bifurcating from the equilibrium of an ordinary or a delayed predator-prey model with a single parameter, which is known as a Hopf bifurcation, has attracted much attention due to its theoretical and practical significance [15-19]. Here we shall point out that most of the research literatures on these models are only connected with parameters which are independent of time delay, thus the corresponding characteristic equations are easy to deal with. While in most applications of delay differential equations in population dynamics, the need of incorporation of a time delay is often the result of existence of some stage structure [20-25]. In fact, every population goes through some distinct life stages [26-28]. Since the through-stage survival rate is often a function of time delay, it is easy to conceive that these models will inevitably involve some delays dependent parameters. Thus the corresponding characteristic equations dependent on the delay τ become more complicated. It is often difficult to analytically study models with delay dependent parameters even if only a single discrete delay is present, we are resort to the help of computer programs.

In 2010, Xiang [29] focused on the following impulsive delay differential equation

$$\begin{cases} \left.\begin{aligned} \frac{dx(t)}{dt} &= rx(t)\left(1-\frac{x(t)}{k}\right)-\frac{\beta x(t)y_2(t)}{1+\alpha x(t)},\\ \frac{dy_1(t)}{dt} &= \frac{k\beta x(t)y_2(t)}{1+\alpha x(t)}\\ &\quad -e^{-\tilde{\omega}\tau}\frac{k\beta x(t-\tau)y_2(t-\tau)}{1+\alpha x(t-\tau)}-\tilde{\omega}y_1(t),\\ \frac{dy_2(t)}{dt} &= e^{-\tilde{\omega}\tau}\frac{\tilde{\lambda}k\beta x(t-\tau)y_2(t-\tau)}{1+\alpha x(t-\tau)}\\ &\quad -\tilde{\omega}y_2(t)-\mu y_2^2(t), \end{aligned}\right\} t\neq nT,\\ \left.\begin{aligned} &\triangle x(t)=-px(t),\\ &\triangle y_1(t)=0,\\ &\triangle y_2(t)=0, \end{aligned}\right\} t=nT, n=1,2\cdots,\\ (\varphi_1(s),\varphi_2(s),\varphi_3(s))\in C_+=C_+([-\tau,0],R_+^3),\\ \varphi_i(0)>0, i=1,2,3. \end{cases} \tag{1}$$

where $x(t), y_1(t), y_2(t)$ denote densities of the prey, the immature and mature predator, respectively. β is the predation rate of predator, $\tilde{\omega}$ is the death rate of the predator, $\tilde{\lambda}$ represents the conversion rate at which ingested prey in excess of what is needed for maintenance is translated into predator population increase. $p(0 \leq p < 1)$ represents partial impulsive harvest to preys by catching or pesticides, τ is the mean length of the juvenile period, the capacity rate k is concerned with the resources which maintain the evolution of the population, T is the period of the impulsive of the prey. Considering that the first and third equations of (1) do not contain $y_1(t)$, Xiang [29] simplified model (1) and only restricted their attention to the following model:

$$\begin{cases} \left.\begin{array}{l} \frac{dx(t)}{dt} = rx(t)\left(1-\frac{x(t)}{k}\right) - \frac{\beta x(t)y_2(t)}{1+\alpha x(t)}, \\ \frac{dy_2(t)}{dt} = e^{-\tilde{\omega}\tau}\frac{\tilde{\lambda}k\beta x(t-\tau)y_2(t-\tau)}{1+\alpha x(t-\tau)} \\ -\tilde{\omega}y_2(t) - \mu y_2^2(t), \end{array}\right\} t \neq nT, \\ \left.\begin{array}{l} \triangle x(t) = -px(t), \\ \triangle y_2(t) = 0, \end{array}\right\} t = nT, n = 1, 2 \cdots, \\ (\varphi_1(s), \varphi_3(s)) \in C_+ = C_+([-\tau, 0], R_+^2), \\ \varphi_i(0) > 0, i = 1, 2. \end{cases} \tag{2}$$

If there is no impulsive effects on the prey, then system (2) takes the form

$$\begin{cases} \frac{dx(t)}{dt} = rx(t)\left(1-\frac{x(t)}{k}\right) - \frac{\beta x(t)y_2(t)}{1+\alpha x(t)}, \\ \frac{dy_2(t)}{dt} = e^{-\tilde{\omega}\tau}\frac{\tilde{\lambda}k\beta x(t-\tau)y_2(t-\tau)}{1+\alpha x(t-\tau)} \\ \qquad - \tilde{\omega}y_2(t) - \mu y_2^2(t). \end{cases} \tag{3}$$

It is well known that time delays which occur in the interaction between predator-prey will affect the stability of a model by creating instability, oscillation and chaos phenomena. The main purpose of this paper is to investigate the stability and the properties of Hopf bifurcation of the model (3) which involves some delays dependent parameters. Recently, there are few papers on the topic that involves some delays dependent parameters, for example, Liu and Zhang [30] investigated the stability and Hopf bifurcation of the following SIS model with nonlinear birth rate:

$$\begin{cases} \dot{I}(t) = \beta(N(t) - I(t))\frac{I(t)}{N(t)} - (d + \varepsilon + \gamma)I(t), \\ \dot{N}(t) = \frac{PN(t-\tau)}{1+qN^3(t-\tau)}e^{-d_1\tau} - dN(t) - \varepsilon I(t). \end{cases} \tag{4}$$

Jiang and Wei [31] studied the stability and Hopf bifurcation of the following SIR model:

$$\begin{cases} \dot{S}(t) = \mu - \mu S(t) - \frac{\phi I(t)S(t)}{1+I(t)} \\ \qquad + \gamma I(t-\tau)e^{-\mu\tau}, \\ \dot{I}(t) = \frac{\phi I(t)S(t)}{1+I(t)} - (\mu + \gamma)I(t). \end{cases} \tag{5}$$

Wang et al. [32] introduced and investigated the following predator-prey interaction model with time delay and delay dependent parameters:

$$\begin{cases} \dot{x}_1(t) = a_1(t)x_2(t) - r_1(t)x_1(t) \\ \qquad - a_1(t-\tau_1)e^{\int_{t-\tau_1}^{t} -r_1(s)-k_1(s)y_2(s)ds} \\ \qquad \times x_2(t-\tau_1) - k_1(t)x_1(t)y_2(t), \\ \dot{x}_2(t) = a_1(t-\tau_1)e^{\int_{t-\tau_1}^{t} -r_1(s)-k_1(s)y_2(s)ds} \\ \qquad \times x_2(t-\tau_1) - \beta_1(t)x_2^2(t), \\ \dot{y}_1(t) = a_2(t)x_1(t)y_2(t) - r_2(t)y_1(t) \\ \qquad - a_2(t-\tau_2)e^{\int_{t-\tau_2}^{t} -r_2(s)ds} \\ \qquad \times x_1(t-\tau_2)y_2(t-\tau_2), \\ \dot{y}_2(t) = a_2(t-\tau_2)e^{\int_{t-\tau_2}^{t} -r_2(s)ds} \\ \qquad \times x_1(t-\tau_2)y_2(t-\tau_2) - \beta_2(t)y_2^2(t). \end{cases} \tag{6}$$

The meaning of all the parameters in system (4), (5) and (6), one can see [30], [31] and [32], respectively. For more work on models with delay dependent parameters, on can see [33-43].

This paper is organized as follows. In Section 2, the stability of the equilibrium and the existence of Hopf bifurcation at the equilibrium are analyzed. In Section 3, the direction of Hopf bifurcation and the stability and periodic of bifurcating periodic solutions on the center manifold are determined. In Section 4, computer simulations are carried out to illustrate the validity of the main results. Some main conclusions are drawn in Section 5.

2 Stability of the equilibrium and local Hopf bifurcations

Throughout this paper, we assume that the following condition holds:

(H1): $x^* < k$,

where x^* is the positive root of the following equation

$$\rho_3 x^3 + \rho_2 x^2 + \rho_1 x_1 + \rho_0 = 0, \tag{7}$$

where $\rho_0 = -(k\tilde{\omega}\beta + kr\mu)$, $\rho_1 = ke^{-\tilde{\omega}\tau}\tilde{\lambda}\beta^2 - k\tilde{\omega}\alpha\beta + r\mu - 2\alpha kr\mu$, $\rho_2 = (2\alpha - k\alpha^2)r\mu$, $\rho_3 = \alpha^2 r\mu$.

The hypothesis (H1) implies that system (3) has a unique positive equilibrium $E(x^*, y_2^*)$, where

$$y_2^* = \frac{r\left(1-\frac{x^*}{k}\right)(1+\alpha x^*)}{\beta}.$$

The linearized system of (3) around $E(x^*, y_2^*)$ is given by

$$\begin{cases} \frac{dx(t)}{dt} = a_1 x(t) + a_2 y_2(t), \\ \frac{dy_2(t)}{dt} = b_1 y_2(t) + b_2 x(t-\tau) \\ \qquad + b_3 y_2(t-\tau), \end{cases} \tag{8}$$

where

$$\begin{aligned} a_1 &= r-\frac{2rx^*}{k}-\frac{\alpha\beta x^* y_2^*}{(1+\alpha x^*)^2}-\frac{\beta y_2^*}{1+\alpha x^*},\\ a_2 &= -\frac{\beta x^*}{1+\alpha x^*}, b_1=-(\tilde{\omega}+2y^*\mu),\\ b_2 &= \frac{e^{-\tilde{\omega}\tau}\tilde{\lambda}\beta y_2^*}{1+\alpha x^*}-\frac{e^{-\tilde{\omega}\tau}\tilde{\lambda}\beta y_2^*\alpha}{(1+\alpha x^*)^2},\\ b_3 &= -\frac{e^{-\tilde{\omega}\tau}\tilde{\lambda}\beta x^*}{1+\alpha x^*}. \end{aligned}$$

The associated characteristic equation of (8) reads as

$$P(\lambda,\tau)+Q(\lambda,\tau)e^{-\lambda\tau}=0, \quad (9)$$

where

$$\begin{aligned} P(\lambda,\tau)=\lambda^2+c_1\lambda+c_0,\\ Q(\lambda,\tau)=d_1\lambda+d_0, \end{aligned} \quad (10)$$

where $c_0=a_1b_1$,$c_1=-(a_1+b_1)$, $d_0=a_1b_3-a_2b_2$, $d_1=-b_3$.

When $\tau=0$, then Eq.(9) becomes

$$\lambda^2+(c_1+d_1)\lambda+c_0+d_0=0, \quad (11)$$

Then we have the following result:

Lemma 1 *If the condition*
(H2)*:* $c_1+d_1>0, c_0+d_0>0$
holds, then the positive equilibrium $E(x^*,y_2^*)$ of system (3) is asymptotically stable.

In the sequel, we discuss the existence of purely imaginary roots $\lambda=i\omega(\omega>0)$ of Eq.(9). Eq.(9) takes the form of a second-degree exponential polynomial in λ, which some of the coefficients of P and Q depend on τ. Beretta and Kuang [44] established a geometrical criterion which gives the existence of purely imaginary roots of a characteristic equation with delay dependent coefficients. In order to apply the criterion due to Beretta and Kuang [44], we need to verify the following properties for all $\tau\in[0,\tau_{\max})$, where $\tau_{\max}$ is the maximum value which $E(x^*,y_2^*)$ exists.

(a) $P(0,\tau)+Q(0,\tau)\neq 0$;

(b) $P(i\omega,\tau)+Q(i\omega,\tau)\neq 0$;

(c) $\limsup\{|\frac{Q(\lambda,\tau)}{P(\lambda,\tau)}|:|\lambda|\to\infty,\mathrm{Re}\lambda\geq 0\}<1$;

(d) $F(\omega,\tau)=|P(i\omega,\tau)|^2-|Q(i\omega,\tau)|^2$ has a finite number of zeros;

(e) Each positive root $\omega(\tau)$ of $F(\omega,\tau)=0$ is continuous and differentiable in τ whenever it exists.

Here, $P(\lambda,\tau)$ and $Q(\lambda,\tau)$ are defined as in (10).

Let $\tau\in[0,\tau_{\max})$. It is easy to see that

$$P(0,\tau)+Q(0,\tau)=c_0+d_0\neq 0,$$

which implies that (a) is satisfied, and (b)

$$\begin{aligned} &P(i\omega,\tau)+Q(i\omega,\tau)\\ &=-\omega^2+c_0+d_0+i\omega(c_1+d_1)\neq 0. \end{aligned} \quad (12)$$

From (2.3), we have

$$\lim_{|\lambda|\to+\infty}\Big|\frac{Q(\lambda,\tau)}{P(\lambda,\tau)}\Big|=\lim_{|\lambda|\to+\infty}\Big|\frac{d_1\lambda+d_0}{\lambda^2+c_1\lambda+c_0}\Big|=0.$$

Therefore (c) follows. Let F be defined as in (d). From

$$|P(i\omega,\tau)|^2=\omega^4+(c_1^2-2c_0)\omega^2+c_0^2$$

and

$$|Q(i\omega,\tau)|^2=d_1^2\omega^2+d_0^2,$$

we obtain

$$F(\omega,\tau)=\omega^4+(c_1^2-2c_0-d_1^2)\omega^2+c_0^2-d_0^2.$$

Obviously, property (d) is satisfied, and by implicit function theorem, (e) is fulfilled.

Let $\lambda=i\omega(\omega>0)$ be a root of Eq.(9) and substituting it into Eq.(9) and separating the real and imaginary parts yields

$$\begin{cases} d_0\cos\omega\tau+d_1\omega\sin\omega\tau=\omega^2-c_0,\\ d_1\omega\cos\omega\tau-d_0\sin\omega\tau=-c_1\omega. \end{cases} \quad (13)$$

It follows from (13) that

$$\sin\omega\tau = \frac{(\omega^2-c_0)d_1\omega+c_1d_1\omega^2}{d_1^2\omega^2+d_0^2}, \quad (14)$$

$$\cos\omega\tau = \frac{(\omega^2-c_0)d_0-c_1d_1\omega^2}{dd_1^2\omega^2+d_0^2}. \quad (15)$$

By the definitions of P and Q as in (10), respectively, and applying the property (a), then (14) and (15) can be written as

$$\begin{aligned} \sin\omega\tau &= \mathrm{Im}\left[\frac{P(i\omega,\tau)}{Q(i\omega,\tau)}\right],\\ \cos\omega\tau &= -\mathrm{Re}\left[\frac{P(i\omega,\tau)}{Q(i\omega,\tau)}\right], \end{aligned}$$

which leads to $|P(i\omega,\tau)|^2=|Q(i\omega,\tau)|^2$. Assume that $I\in R_{+0}$ is the set where $\omega(\tau)$ is a positive root of

$$F(\omega,\tau)=|P(i\omega,\tau)|^2-|Q(i\omega,\tau)|^2$$

and for $\tau\bar{\in}I,\omega(\tau)$ is not definite. Then for all τ in $I,\omega(\tau)$ satisfied $F(\omega,\tau)=0$. The polynomial function F can be written as

$$F(\omega,\tau)=h(\omega^2,\tau), \quad (16)$$

where h is a second degree polynomial, defined by

$$h(z,\tau) = z^2 + (c_1^2 - 2c_0 - d_1^2)z + c_0^2 - d_0^2. \quad (17)$$

It is easy to see that

$$h(z,\tau) = z^2 + (c_1^2 - 2c_0 - d_1^2)z + c_0^2 - d_0^2 = 0. \quad (18)$$

has only one positive real root if the following condition (H3) holds.

(H3): $c_0^2 < d_0^2$.

We denote this positive real root by z_+. Hence, Eq.(16) has only one positive real root $\omega = \sqrt{z_+}$. Since the critical value of τ and $\omega(\tau)$ are impossible to solve explicitly, so we shall use the procedure described in Beretta and Kuang [44]. According to this procedure, we define $\theta(\tau) \in [0, 2\pi)$ such that $\sin\theta(\tau)$ and $\cos\theta(\tau)$ are given by the right hand sides of (14) and (15), respectively, with $\theta(\tau)$ given by (18). This define $\theta(\tau)$ in a form suitable for numerical evaluation using standard software. And the relation between the argument θ and $\omega\tau$ in (17) for $\tau > 0$ must be $\omega\tau = \theta + 2n\pi, n = 1, 2, \cdots$

Hence we can define the maps: $\tau_n : I \to R_{+0}$ given by

$$\tau_n(\tau) := \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, \tau_n > 0, n = 0, 1, 2, \cdots, \quad (19)$$

where a positive root $\omega(\tau)$ of $(F(\omega, \tau) = 0$. exists in I. Let us introduce the functions $S_n(\tau) : I \to R$,

$$S_n(\tau) = \tau - \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, n = 0, 1, 2, \cdots, \quad (20)$$

which are continuous and differentiable in τ. Thus we give the following theorem which is borrowed from Beretta and Kuang [39].

Theorem 2 *Assume that $\omega(\tau)$ is a positive root of (9) defined for $\tau \in I, I \subseteq R_{+0}$, and at some $\tau_0 \in I, S_n(\tau_0) = 0$ for some $n \in N_0$. Then a pair of simple conjugate pure imaginary roots $\lambda = \pm i\omega$ exists at $\tau = \tau_0$ which crosses the imaginary axis from left to right if $\delta(\tau_0) > 0$ and crosses the imaginary axis from right to left if $\delta(\tau_0) < 0$, where $\delta(\tau_0) = sign\left[F'_\omega(\omega\tau_0, \tau_0)\right] sign\left[\frac{dS_n(\tau)}{d\tau}\Big|_{\tau=\tau_0}\right]$.*

Applying Lemma 1 and the Hopf bifurcation theorem for functional differential equation [45], we can conclude the existence of a Hopf bifurcation as stated in the following theorem.

Theorem 3 *For system (3), if (H1)-(H3) hold, then there exists s $\tau_0 \in I$ such that the equilibrium $E(x^*, y_2^*)$ is asymptotically stable for $0 \le \tau < \tau_0$, and becomes unstable for τ staying in some right neighborhood of τ_0, with a Hopf bifurcation occurring when $\tau = \tau_0$.*

3 Direction and stability of the Hopf bifurcation

In section 2, we have already derived some conditions which ensure that the Lotka-Volterra model with time delays and delay dependent parameters undergoes the Hopf bifurcation at some values of $\tau = \tau_0$. In this section, we shall obtain the explicit formulae for determining the direction, stability, and period of these periodic solutions bifurcating from the positive equilibrium $E(x^*, y_2^*)$ at these critical value of τ, by applying techniques from normal form and center manifold theory [45], Throughout this section, we always assume that system (3) undergoes Hopf bifurcation at the positive equilibrium $E(x^*, y_2^*)$ for $\tau = \tau_0$, and then $\pm i\omega_0$ is corresponding purely imaginary roots of the characteristic equation at the equilibrium $E(x^*, y_2^*)$.

For convenience, let $\tau = \tau_0 + \nu, \nu \in R$. Then $\nu = 0$ is the Hopf bifurcation value of (3). Thus, we shall study Hopf bifurcation of small amplitude periodic solutions of (3) from the positive equilibrium point $E_*(x^*, y_2^*)$ for ν close to 0.

Let $u_1(t) = x(t) - x^*, u_2(t) = y_2(t) - y_2^*, x(t) = u_1(\tau t), y_2(t) = u_2(\tau t), \tau = \tau_0 + \nu$,then system (3) is transform into an functional differential equation (FDE) in $(C = C(-1, 0], R^2)$ as

$$\frac{du}{dt} = L_\nu(u_t) + f(\nu, u_t), \quad (21)$$

where $u(t) = (x(t), y_2(t))^T \in R^2$ and $L_\nu : C \to R$, $f : R \times C \to R$ are given respectively by

$$L_\nu\phi = (\tau_0 + \nu)B\phi(0) + (\tau_0 + \nu)G\phi(-1), \quad (22)$$

where

$$B = \begin{pmatrix} a_1 & a_2 \\ 0 & b_1 \end{pmatrix}, G = \begin{pmatrix} 0 & 0 \\ b_2 & b_3 \end{pmatrix}$$

and

$$f(\nu, \phi) = \begin{pmatrix} f_1(\nu, \phi) \\ f_2(\nu, \phi) \end{pmatrix}, \quad (23)$$

where

$$\begin{aligned} f_1(\nu, \phi) &= (\tau_0 + \nu)[a_3\phi_1^2(0) + a_4\phi_1(0)\phi_2(0) \\ &\quad + a_5\phi_1^3(0) + a_6\phi_1^2(0)\phi_2(0) + \text{h.o.t.}], \\ f_2(\nu, \phi) &= (\tau_0 + \nu)[b_4\phi_2^2(0) + b_5\phi_1(-1)\phi_2(-1) \\ &\quad + b_6\phi_1^3(-1) + b_7\phi_1^2(-1)\phi_2(-1) + \text{h.o.t.}], \end{aligned}$$

where

$$a_3 = \frac{\alpha\beta y_2^*}{(1 + \alpha x^*)^2} - \frac{\alpha^2\beta x^* y_2^*}{(1 + \alpha x^*)^3} - \frac{r}{k},$$

$$a_4 = \frac{\alpha\beta x^*}{(1 + \alpha x^*)^2} - \frac{\beta}{1 + \alpha x^*},$$

$$a_5 = \frac{\alpha^2\beta x^* y_2^*}{(1+\alpha x^*)^4} - \frac{\alpha^2\beta y_2^*}{(1+\alpha x^*)^3},$$
$$a_6 = \frac{\alpha\beta}{(1+\alpha x^*)^2} - \frac{\alpha^2\beta x^*}{(1+\alpha x^*)^3},$$
$$b_4 = \frac{e^{-\tilde{\omega}\tau}\alpha\beta\lambda y_2^*}{(1+\alpha x^*)^2} - \frac{e^{-\tilde{\omega}\tau}\alpha^2\beta\tilde{\lambda}x^* y_2^*}{(1+\alpha x^*)^3},$$
$$b_5 = \frac{e^{-\tilde{\omega}\tau}\alpha\beta\tilde{\lambda}x^*}{(1+\alpha x^*)^2} - \frac{e^{-\tilde{\omega}\tau}\beta\tilde{\lambda}}{1+\alpha x^*},$$
$$b_6 = \frac{e^{-\tilde{\omega}\tau}\alpha^2\beta\tilde{\lambda}x^* y_2^*}{(1+\alpha x^*)^4} - \frac{e^{-\tilde{\omega}\tau}\alpha^2\beta\tilde{\lambda}y_2^*}{(1+\alpha x^*)^3},$$
$$b_7 = \frac{e^{-\tilde{\omega}\tau}\alpha\beta\tilde{\lambda}}{(1+\alpha x^*)^2} - \frac{e^{-\tilde{\omega}\tau}\alpha^2\beta\tilde{\lambda}x^*}{(1+\alpha x^*)^3}.$$

Clearly, L_ν is a linear continuous operator from C to R^2. By the Riesz representation theorem, there exists a matrix function with bounded variation components $\eta(\theta,\nu), \theta \in [-1,0]$ such that

$$L_\nu\phi = \int_{-1}^{0} d\eta(\theta,\nu)\phi(\theta), \qquad \text{for } \phi \in C. \tag{24}$$

In fact, we can choose

$$\eta(\theta,\nu) = (\tau_0+\nu)\begin{pmatrix} a_1 & a_2 \\ 0 & b_1 \end{pmatrix}\delta(\theta) - (\tau_0+\nu)\begin{pmatrix} 0 & 0 \\ b_2 & b_3 \end{pmatrix}\delta(\theta+1), \tag{25}$$

where δ is the Dirac delta function.

For $\phi \in C([-1,0],R^2)$, define

$$A(\nu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \int_{-1}^{0} d\eta(s,\nu)\phi(s), & \theta = 0 \end{cases} \tag{26}$$

and

$$R(\nu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ f(\nu,\phi), & \theta = 0. \end{cases} \tag{27}$$

Then (3) is equivalent to the abstract differential equation

$$\dot{u}_t = A(\nu)u_t + R(\nu)u_t, \tag{28}$$

where $u = (x, y_2)^T, u_t(\theta) = u(t+\theta), \theta \in [-1,0]$.

For $\psi \in C([0,1],(R^2)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0. \end{cases} \tag{29}$$

For $\phi \in C([-1,0],R^2)$ and $\psi \in C([0,1],(R^2)^*)$, define the bilinear form

$$<\psi,\phi> = \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi, \tag{30}$$

where $\eta(\theta) = \eta(\theta,0)$. We have the following result on the relation between the operators $A = A(0)$ and A^*.

Lemma 4 *$A = A(0)$ and A^* are adjoint operators.*

The proof is easy from (30), so we omit it.

By the discussion in Section 2, we know that $\pm i\omega_0\tau_0$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_0\tau_0$ and $-i\omega_0\tau_0$, respectively. We have the following result.

Lemma 5 *The vector*

$$q(\theta) = (1,\gamma)^T e^{i\omega_0\tau_0\theta}, \quad \theta \in [-1,0],$$

where

$$\gamma = \frac{i\omega_0 - a_1}{a_2}$$

is the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0\tau_0$, and

$$q^*(s) = D(1,\gamma^*)e^{i\omega_0\tau_0 s}, \quad s \in [0,1],$$

where

$$\gamma^* = -\frac{i\omega_0 + a_1}{b_2 e^{-i\omega\tau_0}}$$

is the eigenvector of A^* corresponding to the eigenvalue $-i\omega_0\tau_0$, moreover, $< q^*(s), q(\theta) >= 1$, where

$$D = 1 + \bar{\gamma}\bar{\gamma^*} + b_2\gamma^* e^{i\omega_0\tau_0} + b_3\bar{\gamma}\gamma^* e^{i\omega_0\tau_0}.$$

Proof. Let $q(\theta)$ be the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0\tau_0$ and $q^*(s)$ be the eigenvector of A^* corresponding to the eigenvalue $-i\omega_0\tau_0$, namely, $A(0)q(\theta) = i\omega_0\tau_0 q(\theta)$ and $A^* q^{*T}(s) = -i\omega_0\tau_0 q^{*T}(s)$. From the definitions of $A(0)$ and A^*, we have $A(0)q(\theta) = dq(\theta)/d\theta$ and $A^* q^{*T}(s) = -dq^{*T}(s)/ds$. Thus, $q(\theta) = q(0)e^{i\omega_0\tau_0\theta}$ and $q^*(s) = q^*(0)e^{i\omega_0\tau_0 s}$. In addition,

$$\begin{aligned}\int_{-1}^{0} d\eta(\theta)q(\theta) &= \tau_0 Bq(0) + \tau_0 Gq(-1) \\ &= \tau_0 A(0)q(0) = i\omega_0\tau_0 q(0).\end{aligned}$$

Namely

$$\begin{pmatrix} i\omega_0 - a_1 & -a_2 \\ -b_2 e^{-i\omega_0\tau_0} & i\omega_0 - b_1 - b_3 e^{-i\omega_0\tau_0} \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{31}$$

Therefore we can easily obtain

$$\gamma = \frac{i\omega_0 - a_1}{a_2}.$$

So

$$q(0) = \left(1, \frac{i\omega_0 - a_1}{a_2}\right)^T$$

and hence

$$q(\theta) = \left(1, \frac{i\omega_0 - a_1}{a_2}\right)^T e^{i\omega_0\tau_0\theta}.$$

On the other hand,

$$\int_{-1}^{0} q^*(-t)d\eta(t) = \tau_0 B^T q^{*T}(0) + \tau_0 G^T q^{*T}(-1)$$
$$= \tau_0 A^* q^{*T}(0) = -i\omega_0\tau_0 q^{*T}(0). \quad (32)$$

Namely,

$$\begin{pmatrix} -i\omega_0 - a_1 & -b_2 e^{-i\omega_0\tau_0} \\ -a_2 & -i\omega_0 - b_1 - b_3 e^{-i\omega_0\tau_0} \end{pmatrix} q^{*T}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (33)$$

Therefore we can easily get

$$\gamma^* = -\frac{i\omega_0 + a_1}{b_2 e^{-i\omega_0\tau_0}},$$

and so

$$q^*(0) = \left(1, -\frac{i\omega_0 + a_1}{b_2 e^{-i\omega_0\tau_0}}\right)$$

and hence

$$q^*(s) = \left(1, -\frac{i\omega_0 + a_1}{b_2 e^{-i\omega_0\tau_0}}\right) e^{i\omega_0\tau_0 s}.$$

In what follows, we shall verify that $< q^*(s), q(\theta) >= 1$. In fact, from (30), we have

$$< q^*(s), q(\theta) >= \bar{D}(1, \bar{\gamma^*})(1, \gamma)^T$$
$$- \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}(1\bar{\gamma^*})e^{-i\omega_0(\xi-\theta)}d\eta(\theta)(1,\gamma)^T e^{i\omega_0\xi}d\xi$$
$$= \bar{D}\left[1 + \gamma\bar{\gamma^*} - \int_{-1}^{0}(1, \bar{\gamma^*})\theta e^{i\omega_0\theta}d\eta(\theta)(1,\gamma)^T\right]$$
$$= \bar{D}\left\{1 + \gamma\bar{\gamma^*} - (1, \bar{\gamma^*})\left[-\tau_0 G e^{-i\omega_0\tau_0}\right](1,\gamma)^T\right\}$$
$$= \bar{D}\left[1 + \gamma\bar{\gamma^*} + b_2\bar{\gamma}^* e^{-i\omega_0\tau_0} + b_3\gamma\bar{\gamma^*}e^{-i\omega_0\tau_0}\right]$$
$$= 1.$$

Next, we use the same notations as those in Hassard, Kazarinoff and Wan [45], and we first compute the coordinates to describe the center manifold C_0 at $\nu = 0$. Let u_t be the solution of Eq.(3) when $\nu = 0$.

Define

$$z(t) =< q^*, u_t >,$$
$$W(t,\theta) = u_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\}. \quad (34)$$

on the center manifold C_0, and we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta), \quad (35)$$

where

$$W(z(t), \bar{z}(t), \theta) = W(z, \bar{z}) = W_{20}\frac{z^2}{2}$$
$$+ W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots \quad (36)$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Noting that W is also real if u_t is real, we consider only real solutions. For solutions $u_t \in C_0$ of (3),

$$\dot{z}(t) =< q^*(s), \dot{u}_t >$$
$$=< q^*(s), A(0)u_t + R(0)u_t >$$
$$=< q^*(s), A(0)u_t > + < q^*(s), R(0)u_t >$$
$$=< A^*q^*(s), u_t > + \bar{q^*}(0)R(0)u_t$$
$$- \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{q^*}(\xi - \theta)d\eta(\theta)A(0)R(0)u_t(\xi)d\xi$$
$$=< i\omega_0\tau_0 q^*(s), u_t > + \bar{q^*}(0)f(0, u_t(\theta)$$
$$\stackrel{\text{def}}{=} i\omega_0\tau_0 z(t) + \bar{q^*}(0)f_0(z(t), \bar{z}(t)). \quad (37)$$

That is

$$\dot{z}(t) = i\omega_0 z + g(z, \bar{z}), \quad (38)$$

where

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \quad (39)$$

Hence, we have

$$g(z,\bar{z}) = \bar{q}^*(0)f_0(z,\bar{z}) = f(0, u_t)$$
$$= \bar{D}(1, \gamma^*)(f_1(0, u_t), f_2(0, u_t))^T, \quad (40)$$

where

$$f_1(0, u_t) = \tau_0[a_3x_t^2(0) + a_4x_t(0)y_{2t}(0) + a_5x_t^3(0) + a_6x_t^2(0)y_{2t}(0) + \text{h.o.t.}],$$
$$f_2(0, u_t) = \tau_0[b_4y_{2t}^2(0) + b_5x_t(-1)y_{2t}(-1) + b_6x_t^3(-1) + b_7x_t^2(-1)y_{2t}(-1) + \text{h.o.t.}],$$

Noticing $u_t(\theta) = (x_t(\theta), y_{2t}(\theta))^T = W(t,\theta) + zq(\theta) + \bar{z}q(\theta)$ and $q(\theta) = (1,\gamma)^T e^{i\omega_0\tau_0\theta}$, we have

$$\begin{aligned}
x_t(0) &= z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} \\
&\quad +W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots, \\
y_t(0) &= \gamma z + \bar{\gamma}\bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} \\
&\quad +W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots, \\
x_t(-1) &= e^{-i\omega_0\tau_0}z + e^{i\omega_0\tau_0}\bar{z} + W_{20}^{(1)}(-1)\frac{z^2}{2} \\
&\quad +W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + \cdots, \\
y_{2t}(-1) &= \gamma e^{-i\omega_0\tau_0}z + \bar{\gamma}e^{i\omega_0\tau_0}\bar{z} + W_{20}^{(2)}(-1)\frac{z^2}{2} \\
&\quad +W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + \cdots.
\end{aligned}$$

From (39) and (40), we have

$$\begin{aligned}
&g(z,\bar{z}) = \bar{q}^*(0)f_0(z,\bar{z}) \\
&= \bar{D}\left[f_1(0,x_t) + \bar{\gamma}^* f_2(0,x_t)\right] \\
&= \bar{D}\tau_0\left[(a_3 + a_4\gamma) + \bar{\gamma}*(b_4 + b_5e^{-2i\omega_0\tau_0})\gamma\right]z^2 \\
&+\bar{D}\tau_0\Big[2a_3 + 2a_4\mathrm{Re}\{\gamma\} + \bar{\gamma}*(2b_4|\gamma|^2 \\
&+2b_5\mathrm{Re}\{\gamma\})\Big]z\bar{z} + \bar{D}\tau_0\Big[a_3 + a_4\bar{\gamma}^2 \\
&+\bar{\gamma}*(b_4\bar{\gamma}^2 + b_5\bar{\gamma}e^{2i\omega_0\tau_0})\Big]\bar{z}^2 \\
&+\bar{D}\tau_0\Big\{a_3\left(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\right) \\
&+a_4\left(2W_{11}^{(2)}(0) + 2\gamma W_{11}^{(1)}(0)\right) \\
&+3a_5 + a_6(2\bar{\gamma} + \gamma) \\
&+\bar{\gamma}*\Big[b_4\left(2\gamma W_{11}^{(2)}(0) + W_{20}^{(2)}(0)\right) \\
&+b_5\Big(\gamma e^{-i\omega_0\tau_0}W_{11}^{(1)}(-1) + e^{-i\omega_0\tau_0}W_{11}^{(2)}(-1) \\
&+\bar{\gamma}_1 e^{i\omega_0\tau_0}W_{11}^{(1)}(-1) + e^{i\omega_0\tau_0}W_{11}^{(2)}(-1)\Big) \\
&+3b_6e^{-i\omega_0\tau_0} + b_7\left(2\gamma + \bar{\gamma}e^{-i\omega_0\tau_0}\right)\Big]\Big\}z^2\bar{z} \\
&+\text{h.o.t.}.
\end{aligned}$$

Then

$$\begin{aligned}
&g_{20} = 2\bar{D}\tau_0\left[(a_3 + a_4\gamma) + \bar{\gamma}*(b_4 + b_5e^{-2i\omega_0\tau_0})\gamma\right], \\
&g_{11} = \bar{D}\tau_0\Big[2a_3 + 2a_4\mathrm{Re}\{\gamma\} + \bar{\gamma}*(2b_4|\gamma|^2 \\
&\quad + 2b_5\mathrm{Re}\{\gamma\})\Big], \\
&g_{02} = 2\bar{D}\tau_0\left[a_3 + a_4\bar{\gamma}^2 + \bar{\gamma}*(b_4\bar{\gamma}^2 + b_5\bar{\gamma}e^{2i\omega_0\tau_0})\right],
\end{aligned}$$

$$\begin{aligned}
&g_{21} = 2\bar{D}\tau_0\Big\{a_3\left(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\right) \\
&\quad + a_4\left(2W_{11}^{(2)}(0) + 2\gamma W_{11}^{(1)}(0)\right) \\
&\quad + 3a_5 + a_6(2\bar{\gamma} + \gamma) \\
&\quad + \bar{\gamma}*\Big[b_4\left(2\gamma W_{11}^{(2)}(0) + W_{20}^{(2)}(0)\right) \\
&\quad + b_5\Big(\gamma e^{-i\omega_0\tau_0}W_{11}^{(1)}(-1) + e^{-i\omega_0\tau_0}W_{11}^{(2)}(-1) \\
&\quad + \bar{\gamma}_1 e^{i\omega_0\tau_0}W_{11}^{(1)}(-1) + e^{i\omega_0\tau_0}W_{11}^{(2)}(-1)\Big) \\
&\quad +3b_6e^{-i\omega_0\tau_0} + b_7\left(2\gamma + \bar{\gamma}e^{-i\omega_0\tau_0}\right)\Big]\Big\}.
\end{aligned}$$

In the sequel, we will compute the following value: $W_{20}^{(1)}(0), W_{20}^{(1)}(-1), W_{11}^{(1)}(0), W_{11}^{(2)}(-1), W_{11}^{(2)}(0), W_{11}^{(2)}(-1)$. It follows from (28) and (34) that

$$W' = \begin{cases} AW - 2\mathrm{Re}\{\bar{q}^*(0)f_0q(\theta)\}, -1 \le \theta < 0, \\ AW - 2\mathrm{Re}\{\bar{q}^*(0)f_0q(\theta)\} + f_0, \theta = 0. \end{cases}$$

$$\stackrel{\text{def}}{=} AW + H(z,\bar{z},\theta), \tag{41}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{42}$$

Comparing the coefficients, we obtain

$$(A - 2i\omega_0\tau_0)W_{20} = -H_{20}(\theta) \tag{43}$$

$$AW_{11}(\theta) = -H_{11}(\theta), \tag{44}$$

and we know that for $\theta \in [-1, 0)$,

$$\begin{aligned}
H(z,\bar{z},\theta) &= -\bar{q}^*(0)f_0q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) \\
&= -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta). \quad (45)
\end{aligned}$$

Comparing the coefficients of (42) with (45) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tag{46}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{47}$$

From (43),(46) and the definition of A , we get

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_0W_{20}(\theta) + g_{20}q(\theta) + \bar{g_{02}}\bar{q}(\theta). \tag{48}$$

Considering that $q(\theta) = q(0)e^{i\omega_0\tau_0\theta}$, we have

$$\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} \\
&\quad +E_1e^{2i\omega_0\tau_0\theta}, \quad (49)
\end{aligned}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)})^T$ is a constant vector.

Similarly, from (44), (47) and the definition of A, we have

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g_{11}}\bar{q}(\theta), \tag{50}$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\theta} + E_2. \tag{51}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)})^T$ is a constant vector.

In what follows, we shall seek appropriate E_1,E_2 in (49), (51), respectively. It follows from the definition of A, (46) and (47) that

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(0) - H_{20}(0) \tag{52}$$

and

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{53}$$

where $\eta(\theta) = \eta(0, \theta)$.

From (43), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g_{02}}\bar{q}(0) + \tau_0(M_1, M_2)^T, \tag{54}$$

where

$$M_1 = a_3 + a_4\gamma, M_2 = (b_4 + b_5e^{-2i\omega_0\tau_0})\gamma.$$

In view of (44), we have

$$H_{11}(0) = -g_{11}q(0) - \bar{g_{11}}(0)\bar{q}(0) + \tau_0(N_1, N_2)^T, \tag{55}$$

where

$$N_1 = 2a_3 + 2a_4\text{Re}\{\gamma\}, N_2 = 2b_4|\gamma|^2 + 2b_5\text{Re}\{\gamma\}.$$

Noting that

$$\left(i\omega_0\tau_0 I - \int_{-1}^{0} e^{i\omega_0\tau_0\theta}d\eta(\theta)\right)q(0) = 0, \tag{56}$$

$$\left(-i\omega_0\tau_0 I - \int_{-1}^{0} e^{-i\omega_0\tau_0\theta}d\eta(\theta)\right)\bar{q}(0) = 0 \tag{57}$$

and substituting (49) and (54) into (52), we have

$$\left(2i\omega_0\tau_0 I - \int_{-1}^{0} e^{2i\omega_0\tau_0\theta}d\eta(\theta)\right)E_1 = \tau_0(M_1, M_2)^T. \tag{58}$$

That is

$$(2i\omega_0\tau_0 I - \tau_0 B - \tau_0 G e^{-2i\omega_0\tau_0})E_1 = \tau_0(M_1, M_2)^T, \tag{59}$$

then

$$\begin{pmatrix} 2i\omega_0 - a_1 & -a_2 \\ -b_2e^{-2i\omega_0\tau_0} & 2i\omega_0 - b_1 - b_3e^{-2i\omega_0\tau_0} \end{pmatrix} \times \begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \end{pmatrix} = \begin{pmatrix} a_3 + a_4\gamma \\ (b_4 + b_5e^{-2i\omega_0\tau_0})\gamma \end{pmatrix}. \tag{60}$$

Hence $E_1^{(1)} = \frac{e_1}{\chi}, E_1^{(2)} = \frac{e_2}{\chi}$, where

$$\begin{aligned} e_1 &= (a_3 + a_4\gamma)(2i\omega_0 - b_1 - b_3e^{-2i\omega_0\tau_0}) \\ &\quad + a_2[b_4 + b_5e^{-2i\omega_0\tau_0})\gamma] \\ e_2 &= (2i\omega_0 - a_1)[b_4 + b_5e^{-2i\omega_0\tau_0})\gamma] \\ &\quad + (a_3 + a_4\gamma)b_2e^{-2i\omega_0\tau_0} \\ \chi &= (2i\omega_0 - a_1)(2i\omega_0 - b_1 - b_3e^{-2i\omega_0\tau_0}) \\ &\quad - a_2b_2e^{-2i\omega_0\tau_0}. \end{aligned}$$

Similarly, substituting (51) and (55) into (53), we have

$$\left(\int_{-1}^{0} d\eta(\theta)\right)E_2 = \tau_0(N_1, N_2)^T. \tag{61}$$

Then

$$(B + G)E_2 = (-N_1, -N_2)^T. \tag{62}$$

That is

$$\begin{pmatrix} a_1 & a_2 \\ b_2 & b_1 + b_3 \end{pmatrix}\begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \end{pmatrix} = \begin{pmatrix} -(2a_3 + 2a_4\text{Re}\{\gamma\}) \\ -(2b_4|\gamma|^2 + 2b_5\text{Re}\{\gamma\}) \end{pmatrix}. \tag{63}$$

Hence

$$\begin{aligned} E_2^{(1)} &= \frac{a_2((2b_4|\gamma|^2 + 2b_5\text{Re}\{\gamma\})}{a_1(b_1 + b_3) - a_2b_2}, \\ E_2^{(2)} &= \frac{\sigma}{a_1(b_1 + b_3) - a_2b_2}, \end{aligned}$$

where $\sigma = b_2(2a_3 + 2a_4\text{Re}\{\gamma\}) - a_1(2b_4|\gamma|^2 + 2b_5\text{Re}\{\gamma\})$. From (49),(51), we can calculate g_{21} and derive the following values:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega_0\tau_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\text{Re}\{c_1(0)\}}{\text{Re}\{\lambda'(\tau_0)\}}, \\ \beta_2 &= 2\text{Re}(c_1(0)), \\ T_2 &= -\frac{\text{Im}\{c_1(0)\} + \mu_2\text{Im}\{\lambda'(\tau_0)\}}{\omega_0\tau_0}. \end{aligned}$$

These formulaes give a description of the Hopf bifurcation periodic solutions of (3) at $\tau = \tau_0$ on the center manifold. From the discussion above, we have the following result.

Theorem 6 *The periodic solution is supercritical (subcritical) if* $\mu_2 > 0$ $(\mu_2 < 0)$*; The bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if* $\beta_2 < 0$ $(\beta_2 > 0)$*; The periods of the bifurcating periodic solutions increase (decrease) if* $T_2 > 0$ $(T_2 < 0)$.

4 Numerical Examples

In this section, we carry out some numerical simulations to verify the analytical predictions derived in the previous section. Let $r = 1, k = 2, \alpha = 3, \beta = 2, \tilde{\omega} = 0.2, \mu = 0.2, \tilde{\lambda} = 0.5$. Then system (3) becomes

$$\begin{cases} \frac{dx(t)}{dt} = x(t)\left(1 - \frac{x(t)}{2}\right) - \frac{2x(t)y_2(t)}{1+3x(t)}, \\ \frac{dy_2(t)}{dt} = e^{-0.2\tau}\frac{2x(t-\tau)y_2(t-\tau)}{1+3x(t-\tau)} \\ \qquad\qquad - 0.2y_2(t) - 0.2y_2^2(t). \end{cases} \tag{64}$$

Obviously, system (64) has a positive equilibrium $E(x^*, y_2^*)$.

By Matlab 7.0, we get only one critical value of the delay $\tau_0 \approx 0.9201, \omega_0 \approx 1.7135, \lambda'(\tau_0) \approx 0.4053 - 0.5172i$. Thus we derive $c_1(0) \approx -2.4451 - 4.0566i, \mu_2 \approx 6.0328, \beta_2 \approx -4.8902, T_2 \approx 4.5526$. The conditions in Theorem 3 hold true. It follows that $\mu_2 > 0$ and $\beta_2 < 0$ that the positive equilibrium $E(x^*, y^*)$ is stable when $\tau < \tau_0$ which is illustrated by the computer simulations (see Figs.1-4).

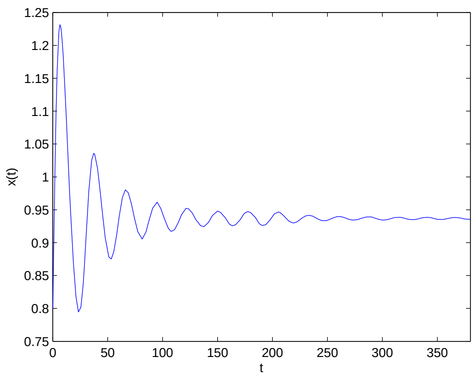

Figure 1

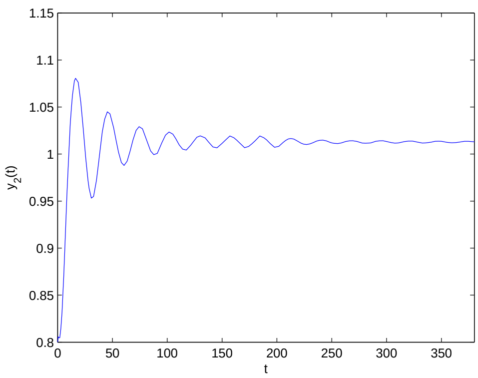

Figure 2

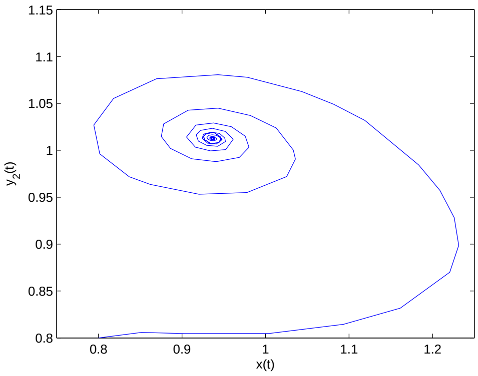

Figure 2

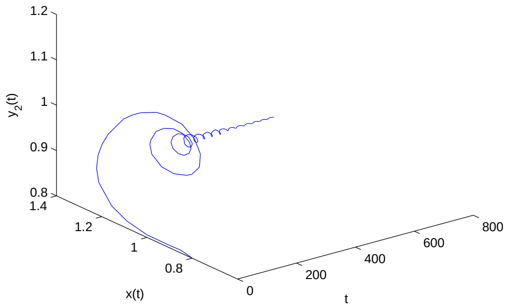

Figure 4

When τ passes through the critical value τ_0, the positive equilibrium $E(x^*, y_2^*)$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from the positive equilibrium $E_*(x^*, y^*)$. Due to $\mu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is $\tau > \tau_0$ and these bifurcating periodic solutions from $E(x^*, y_2^*)$ at τ_0 are stable, which are shown in Figs.5-8.

Figs.1-4. The time histories and phase portrait of system (64) with $\tau = 0.8 < \tau_0 \approx 0.9201$ and the initial value is (0.8,0.8). The positive equilibrium $E(x^*, y_2^*)$ is asymptotically stable.

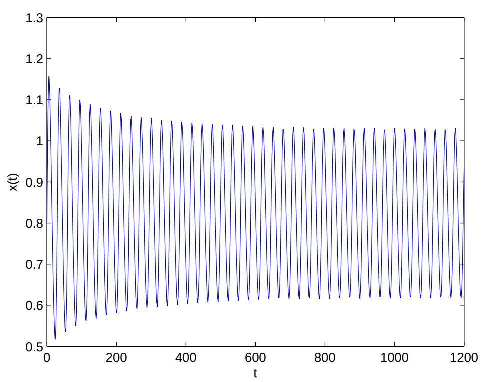

Figure 5

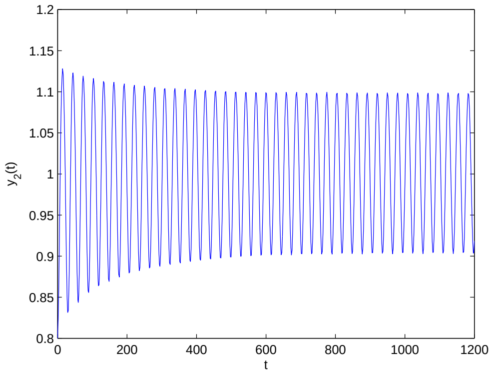

Figure 6

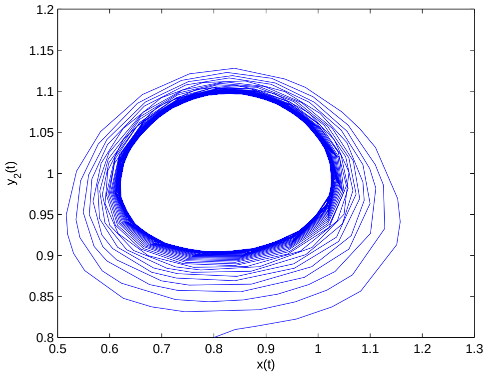

Figure 7

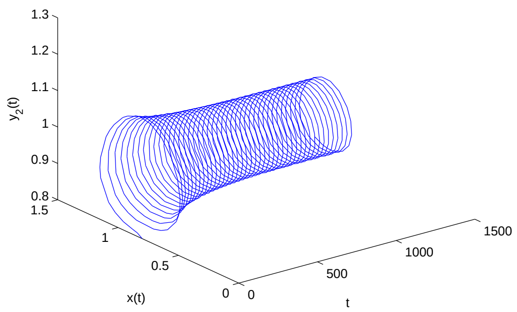

Figure 8

Figs.5-8. The time histories and phase portrait of system (64) with $\tau = 1 > \tau_0 \approx 0.9201$ and the initial value is (0.8,0.8). Hopf bifurcation occurs from the positive equilibrium $E(x^*, y_2^*)$.

5 Conclusions

In the present paper, we obtain the conditions to ensure that the positive equilibrium of a Lotka-Volterra model with time delays and delay dependent parameters is asymptotically stable by employing the method due to Beretta and Kuang [41] and analyzing the distributed of the eigenvalues. By regarding the time delay as bifurcation parameter, we find that Hopf bifurcation occurs when the delay passes through some critical values. Some formulae for determining the stability and the direction of Hopf bifurcation for a Lotka-Volterra model with time delays and delay dependent parameters are given by using the normal form theory and the center manifold theorem. Finally, numerical simulations are carried out to validate the theoretical findings.

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