A characterization of some groups by their orders and degree patterns

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Abstract: Let G be a finite group. Moghaddamfar et al defined the prime graph $\Gamma(G)$ of group G as follows. The vertices of $\Gamma(G)$ are the primes dividing the order of G and two distinct vertices p, q are joined by an edge, denoted by $p \sim q$, if there is an element in G of order $p \cdot q$. Assume $|G| = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ with $P_1 < \cdots < p_k$ and nature numbers α_i with $i = 1, 2, \cdots, k$. For $p \in \pi(G)$, let the degree of p be $deg(p) = |\{q \in \pi(G) \mid q \sim p\}|$, and $D(G) = (deg(p_1), deg(p_2), \cdots, deg(p_k))$. In this note we give an example showing that S_{27} is 9-fold OD-characterizable, which gives a negative answer to an open Problem of Yan et al.

Key–Words: Order component, Element order, Symmetric group, Degree pattern, Prime graph, Simple group.

1 Introduction

In this paper, all groups under consideration are finite, and for a simple group, we mean a non-Abelian simple group. Let G be a group. Then $\omega(G)$ denotes the set of orders of its elements of G and $\pi(G)$ denotes the set of prime divisors of $|G|$. Associated to $\omega(G)$ a graph is called prime graph of G, which is denoted by $\Gamma(G)$. The vertex set of $\Gamma(G)$ is $\pi(G)$, and two distinct vertices p, q are joined by an edge if $p \cdot q \in \omega(G)$ which is denoted by $p \sim q$.

Through this paper, we also use the following symbols. For a finite group G, then socle of G is defined as the subgroup generated by the minimal normal subgroups of G, denoted by $Soc(G)$. $Syl_p(G)$ denotes the set of all Sylow p-subgroups of G, where $p \in \pi(G)$, P_r denotes the Sylow r-subgroup of G for $r \in \pi(G)$. S_n and A_n denotes the symmetric and alternating groups of degree n, respectively. Let p be a prime and we use $Exp(m, p)$ to denote the exponent of the largest power of a prime p in the factorization of a positive integer $m(> 1)$. The other symbols are standard (see [5], for instance).

Definition 1 *[12] Let G be a finite group and* $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$*, where* p_i*s are primes and* α_i*s are integers. For* $p \in \pi(G)$*, let* $deg(p) := |\{q \in \pi(G) | p \sim q\}|$*, which we call the degree of p. We also define* $D(G) := (deg(p_1), deg(p_2), \cdots, deg(p_k))$*, where* $p_1 < p_2 < \cdots < p_k$*. We call* $D(G)$ *the degree pattern of G.*

Given a finite group M, denote by $h_{OD}(M)$ the number of isomorphism classes of finite groups G such that (1) $|G| = |M|$ and (2) $D(G) = D(M)$.

Definition 2 *[12] A finite group* M *is called k-fold OD-characterizable if* $h_{OD}(M) = k$*. Moreover, a 1-fold* OD*-characterizable group is simply called an* OD*-characterizable group.*

A group G is called C_{pp}-group if $p \in \pi(G)$, and the centralizer of any nontrivial p-element in G is a p-group. A group G is called to be a K_n-group if G is nonabelian simple group and $|\pi(G)| = n$. Many finite groups are k-fold OD-characterizable.

Proposition 3 *A finite group G is OD-characterizable if G is one of the following groups:*

(1) The alternating groups A_p*,* A_{p+1} *and* A_{p+2}*, where p is a prime [10].*

(2) The alternating groups A_{p+3}*, where p is a prime and* $7 \neq p \in \pi(100!)$ *[6, 8].*

(3) All finite almost simple K_3*-groups except* $Aut(A_6)$ *and* $Aut(U_4(2))$ *[17].*

(4) The symmetric groups S_p *and* S_{p+1}*, where p is a prime [10].*

(5) All finite simple $C_{2,2}$*-groups [10].*

(6) All finite simple K_4*-groups except* A_{10} *[23].*

(7) The simple groups of the Lie type $L_3(q)$, $U_3(q)$, ${}^2B_2(q)$ and ${}^2G_2(q)$ for a certain prime power q [12].

(8) All sporadic simple groups and their automorphism groups except $Aut(J_2)$ and $Aut(M^cL)$ [10].

(9) The almost simple groups of $Aut(F_4(2))$ and $Aut(O^{\pm}_{10}(2))$ [14].

(10) $L_2(q)$ where q is a prime power of prime p [25].

(11) $L_7(3)$ [18]

(12) $U_3(5)$, $U_3(5).2$, $U_6(2)$, $U_6(2).2$, $L_2(49)$ and $L_2(49).2$ [27, 24, 22]

(13) $L_4(q)$ where q =4, 8, 9, 11, 13, 16, 17, 19, 23, 27, 29, 31, 32, 37 [1, 2]

(14) $L_n(2)$ for $n \geq 2$, $L_{10}(2)$, $L_{11}(2)$ and $Aut(L_p(2))$ with 2^p is a Mersenne prime [9].

(15) $C_p(2)$ with $2^p - 1 > 7$ Mersenne prime[3].

Proposition 4 *A finite group G is 2-fold OD-characterizable if G is one of the following groups:*

(1) $B_3(5)$ and $C_3(5)$ [4].

(2) $S_6(3)$ and $O_7(3)$ [12].

(3) A_{10} and $Aut(M^cL)$ [11, 23].

(4) $U_4(2)$ [26]

Proposition 5 *A finite group G is 3-fold OD-characterizable if G is one of the following groups:*

(1) $Aut(J_2)$ [11].

(2) S_{p+3} with ($p < 1000$) prime [6, 8, 15, 16].

(3) $GL_7(3)$ [18].

(4) $U_3(5).3$ and $U_6(2).3$ [27, 24].

Proposition 6 *[17, Main Theorem] $Aut(A_6)$ is 4-fold OD-characterizable. In particular, $Aut(U_4(2))$ is at least 4-fold OD-characterizable.*

Proposition 7 *[27] $U_3(5).S_3$ are 6-fold OD-characterizable.*

Proposition 8 *[22] $L_2(49).2^2$ are 9-fold OD-characterizable.*

Proposition 9 *[11] The group S_{10} is 8-fold OD-characterizable.*

2 Main results

Let p be a prime. By proposition 3, the symmetric groups A_p, A_{p+1}, A_{p+2} and A_{p+3} except A_{10} are OD-characterizable. But in general, we do not know if the alternating groups A_{p+4} are OD-characterization. So we put forward the following Conjecture:

Conjuecture 1. Let p be a prime with $p + 2$ and $p + 4$ composite. Then the alternating group A_{p+4} is OD-characterizable.

Not all alternating groups A_{p+4} are OD-characterizable since A_{10} is 2-fold OD-characterizable (see Proposition 4).

From Propositions 3 and 5, we have that S_p, S_{p+1}, S_{p+2} and S_{p+3} are OD-characterizable, and by Proposition 9, S_{10} are 8-fold OD-characterizable. Omitting the symmetric groups S_p, S_{p+1}, S_{p+2} and S_{p+3}, there remain the following groups: S_{27}, S_{28}, S_{35}, S_{36}, S_{51}, S_{52}, S_{57}, S_{58}, S_{65}, S_{66}, S_{77}, S_{78}, S_{87}, S_{93}, S_{94}, S_{95}, S_{96}, $\cdots$. We will prove that S_{27} is 9-fold OD-characterizable. So we put forward the following conjecture.

Conjecture 2. Let p be a prime with $p + 2$ and $p + 4$ composite. Then the symmetric group S_{p+4} except S_{27} is 9-fold OD-characterizable.

In fact, we will prove the following result.

Theorem 10 *Let $p \in \{23, 31, 47, 53, 61, 73, 83, 89\}$. Then*

(1) The alternating groups A_{p+4}, where p =23, 31, 47, 53, 61, 73, 83, 89, are OD-characterizable.

(2) The symmetric group S_{27} is 9-fold OD-characterizable.

(3) The symmetric groups S_{p+4}, where p =31, 47, 53, 61, 73, 83, 89 are 3-fold OD-characterizable.

Our results show that the symmetric group S_{27} is 9-fold OD-characterizable which gives a negative answer to an open problem of Yan et al in [16, 15].

Open Problem. [16, 15] Are symmetric groups $S_n(n \neq p, p+1)$, except S_{10}, 3-fold OD-characterizable?

3 Preliminary Results

In this section, we will give some results which will be used.

Lemma 11 *[19] Let $S = P_1 \times P_2 \times \cdots \times P_r$, where P_i's are isomorphic non-abelian simple group. Then $Aut(S) = (Aut(P_1) \times Aut(P_2) \times \cdots \times Aut(P_r)) \cdot S_r$.*

Lemma 12 *[20] The group S_n (or A_n) has an element of order $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, where $p_1, p_2, \cdots, p_s$ are distinct primes and $\alpha_1, \alpha_2, \cdots, \alpha_s$ are nature numbers, if and only if $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n$ (or $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n$ for m odd, and $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n-2$ for m even).*

As a corollary of Lemma 12, we have the following result.

Lemma 13 *Let A_n (or S_n) be an alternating (or symmetric group) of degree n. Then the following hold.*

(1) Let $p, q \in \pi(A_n)$ be odd primes. Then $p \sim q$ if and only if $p + q \leq n$.

(2) Let $p \in \pi(A_n)$ be odd prime. Then $2 \sim p$ if and only if $p + 4 \leq n$.

(3) Let $p, q \in \pi(S_n)$. Then $p \sim q$ if and only if $p + q \leq n$.

By [13], we know that A_{p+4} and S_{p+4} for $p \in \{23, 31, 47, 53, 61, 73, 83, 89\}$ have connected prime graphs. By [5], we have that $|A_n| = n!/2$ and $|S_n| = n!$.

Since the degree patterns of alternating groups A_{p+4} for $p = 23, 31, 47, 53, 61, 73, 83, 89$, are the same as those of their automorphism groups. So we only list the order and degree pattern of alternating groups A_{p+4} in Table 1.

Lemma 14 *Let A_{p+4} be an alternating group of degree $p+4$, where p is a prime, and assume that the numbers $p+2$ and $p+4$ are composite. Set $|\pi(A_{p+4})| = d$. Then the following hold.*

(1) $deg(2) = deg(3) = d$. In particular, $2 \sim r$ for all $r \in \pi(A_{p+4})$.

(2) $deg(5) = d - 1$. In particular, $5 \sim r$ for all $r \in \pi(A_{p+4}) \backslash \{p\}$.

(3) $deg(p) = 2$. In particular, $p \sim r$, where $r \in \pi(A_{p+4})$, if and only if $r = 2, 3$.

(4) $Exp(|A_{p+4}|, 2) = \sum\limits_{i=1}^{\infty} [\frac{p+4}{2^i}] - 1$. In particular, $Exp(|A_{p+4}|, 2) \leq p + 3 < p + 4$.

(5) $Exp(|A_{p+4}|, r) = \sum\limits_{i=1}^{\infty} [\frac{p+4}{r^i}]$ for each $r \in \pi(A_{p+4}) \backslash \{2\}$. Furthermore, $Exp(|A_{p+4}|, r) < \frac{p-1}{2}$, where $3 \leq r \in \pi(A_{p+4})$. In particular, if $r > [\frac{p+4}{2}]$, then $Exp(|A_{p+4}|, r) = 1$.

Proof. (1) By Lemma 12, $r + 4 \leq p + 4$ for each $r \in \pi(A_{p+4})$. So we have $deg(2) = d$. For each $r \in \pi(A_{p+4})$, $r + 3 \leq p + 4$. Hence $deg(3) = d$.

(2) By Lemma 12, $r + 5 \leq p + 4$ for each $r \in \pi(A_{p+4}) \backslash \{p\}$. So we have $deg(5) = d - 1$.

(3) For $r \in \pi(A_{p+4})$, by Lemma 12, it is easy to get that $p \sim r$ if and only if $p + r \leq p + 4$. Thus $r \leq 4$ and so $r = 2, 3$. So we have $deg(p) = 2$.

(4) By definition of Gaussian integer function, we have that

$$
\begin{aligned}
Exp(|A_{p+4}|, 2) &= \sum_{i=1}^{\infty} [\frac{p+4}{2^i}] - 1 \\
&= ([\frac{p+4}{2}] + [\frac{p+4}{2^2}] + [\frac{p+4}{2^3}] + \cdots) - 1 \\
&\leq (\frac{p+4}{2} + \frac{p+4}{2^2} + \frac{p+4}{2^3} + \cdots) - 1 \\
&= (p+4)(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots) - 1 \\
&= p + 3.
\end{aligned}
$$

(5) Similarly as (4), we have that

$$
\begin{aligned}
Exp(|A_{p+4}|, r) &\leq (p+4)(\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \cdots) \\
&= \frac{p+4}{r-1} \leq \frac{p+4}{2}
\end{aligned}
$$

for an odd prime $r \in \pi(A_{p+4})$.

If $r > [\frac{p+4}{2}]$, $Exp(|A_{p+4}|, r) = 1$. The proof is complete. □

Similarly as the proof of Lemma 14, we can prove the following Lemma 15.

Lemma 15 *Let S_{p+4} be a symmetric group of degree $p+4$, where p is a prime, and assume that the numbers $p+2$ and $p+4$ are composite. Set $|\pi(S_{p+4})| = d$. Then the following hold.*

(1) $deg(2) = deg(3) = d$. In particular, $2 \sim r$ for all $r \in \pi(S_{p+4})$.

(2) $deg(p) = 2$. In particular, $p \sim r$, where $r \in \pi(S_{p+4})$, if and only if $r = 2, 3$.

(3) $Exp(|S_{p+4}|, 2) = \sum\limits_{i=1}^{\infty} [\frac{p+4}{2^i}] - 1$. In particular, $Exp(|S_{p+4}|, 2) \leq p + 3 < p + 4$.

(4) $Exp(|S_{p+4}|, r) = \sum\limits_{i=1}^{\infty} [\frac{p+4}{r^i}]$ for each $r \in \pi(S_{p+4}) \backslash \{2\}$. Furthermore, $Exp(|S_{p+4}|, r) < \frac{p-1}{2}$, where $3 \leq r \in \pi(S_{p+4})$. In particular, if $r > [\frac{p+4}{2}]$, then $Exp(|S_{p+4}|, r) = 1$.

Table 1: Order of some alternating with their degree patterns

G	$\lvert G\rvert$	$D(G)$
A_{27}	$2^{22} \cdot 3^{13} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23$	(8,8,7,7,5,5,4,4,2)
A_{35}	$2^{31} \cdot 3^{15} \cdot 5^8 \cdot 7^5 \cdot 11^3 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 23 \cdot 29 \cdot 31$	(10,10,9,8,8,7,6,5,5,3,2)
A_{51}	$2^{46} \cdot 3^{23} \cdot 5^{12} \cdot 7^8 \cdot 11^4 \cdot 13 \cdot 17^3 \cdot 19^2 \cdot 23^2 \cdot 29$	(14,14,13,13,11,11,10,
	$\cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47$	8,8,8,8,6,4,4,2)
A_{57}	$2^{52} \cdot 3^{27} \cdot 5^{12} \cdot 7^9 \cdot 11^5 \cdot 13^4 \cdot 17^3 \cdot 19^3 \cdot 23^2 \cdot 29$	(15,15,14,14,13,13,11,
	$\cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53$	11,10,9,9,8,6,4,2)
A_{65}	$2^{62} \cdot 3^{30} \cdot 5^{15} \cdot 7^{10} \cdot 11^5 \cdot 13^5 \cdot 17^3 \cdot 19^3 \cdot 23^2 \cdot 29^2$	(17,17,16,15,13,13,11,11,10,9,
	$\cdot 31^2 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61$	9,8,6,4,2)
A_{77}	$2^{72} \cdot 3^{35} \cdot 5^{17} \cdot 7^{12} \cdot 11^7 \cdot 13^5 \cdot 17^4 \cdot 19^4 \cdot 23^3 \cdot 29^2$	(20,20,19,19,17,17,16,15,15,14,
	$\cdot 31^2 \cdot 37^2 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73$	13,11,11,11,10,9,7,6,4,3,2)
A_{87}	$2^{81} \cdot 3^{42} \cdot 5^{19} \cdot 7^{13} \cdot 11^7 \cdot 13^6 \cdot 17^5 \cdot 19^4 \cdot 23^3 \cdot 29^3$	(22,22,21,21,20,20,18,18,17,15,
	$\cdot 31^2 \cdot 37^2 \cdot 41^2 \cdot 43^2 \cdot 47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot 83$	15,14,13,13,12,11,9,9,8,6,6,4,2)
A_{93}	$2^{87} \cdot 3^{45} \cdot 5^{20} \cdot 7^{14} \cdot 11^8 \cdot 13^7 \cdot 17^5 \cdot 19^4 \cdot 23^4 \cdot 29^3$	(23,23,21,21,20,20,18,18,17,15,
	$\cdot 31^3 \cdot 37^2 \cdot 41^2 \cdot 43^2 \cdot 47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot 83 \cdot 89$	15,14,13,13,12,11,9,9,8,6,6,4,2,2)

Lemma 16 *Let G be a finite non-abelian simple group with $p \in \pi(G) \subsetneqq \{2, 3, 5, 7, \cdots, p\}$, where $p = 23, 31, 47, 53, 61, 73, 83, 89$. Then G is isomorphic to one of the groups as listed in Table 1.*

Proof. From [21], we have the possible groups and their orders. By [5], we have the order of the outer automorphism groups by computations.

In the proof, we also need the following information of p-groups of order p^3 with odd p.

Lemma 17 *Let P be a p-group of order p^3 and x be the largest order elements of P. Then the following hold.*

- *P is abelian. If $|x| = 27$, then $P \cong Z_{p^3}$. If $|x| = 9$, then $P \cong Z_{p^2} \times Z_p$. If $|x| = 3$, then $P \cong Z_p \times Z_p \times Z_p$.*
- *P is nonabelian. If $|x| = 9$, then $P \cong Z_{p^2} \rtimes Z_p$. If $|x| = 3$, then $P \cong (Z_p \times Z_p) \rtimes Z_p$.*

From Table 2, we have the following Lemma.

Lemma 18 *Let S be a simple group as Lemma 16. If $|Out(S)| \neq 1$, then $\pi(Out(S)) \subsetneqq \{2, 3, 5\}$. Moreover, when $\{p\} \subset \pi(S) \subseteq \pi(p!)$,*

(1) if $p = 23$, then $\{2, 3, 11, 23\} \subset \pi(S)$;

(2) if $p = 31$ and $S \ncong L_2(32)$, then $\{2, 3, 5, 31\} \subset \pi(S)$;

(3) if $p = 47$, 61, or 73, then $\{2, 3, p\} \subset \pi(S)$;

(4) if $p = 53$ and $S \ncong L_2(53)$, then $\{2, 3, 5, 11, 23, 53\} \subset \pi(S)$.

4 Proof of the main theorem and its applications

In this section, we will give the proof of Theorem 10. We divide the proof of the following sections.

4.1 Proof for the alternating groups

From Proposition 3, we know that the alternating groups A_p, A_{p+1}, A_{p+2} and A_{p+3}, where p is a prime, are OD-characterizable, and from Proposition 4, alternating group A_{10} is 2-fold OD-characterizable. As a development of this topics, we prove the following theorem.

Theorem 19 *The alternating groups A_{p+4}, where $p \in \{23, 31, 47, 53, 61, 73, 83, 89\}$, are OD-characterizable.*

Proof. Let $M \cong A_{p+4}$. Assume that G is a finite group such that $|G| = |M|$ and $D(G) = D(M)$. From Lemma 14, we have that the prime graph $\Gamma(G)$ is connected, in particular, $\Gamma(G) = \Gamma(M)$.

In the following, we only consider the case "$p = 23$".

We know that

$$|G| = 2^{22} \cdot 3^{13} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23$$

and

$$D(G) = (8, 8, 7, 7, 5, 4, 4, 4, 2).$$

Lemma 20 *Let K be a maximal normal soluble subgroup of G. Then K is a $\{2, 3, 11\}$-group. In particular, G is insoluble.*

Table 2: $p \in \pi(G) \subseteqq \{2, 3, 5, 7, \cdots, p\}$ where $p = 23, 31, 47, 53, 61, 73, 83, 89$

G	$\|G\|$	$\|O_{ut}(G)\|$
$L_2(23)$	$2^3 \cdot 3 \cdot 11 \cdot 23$	2
$U_3(23)$	$2^7 \cdot 3^2 \cdot 11 \cdot 13^2 \cdot 23^2$	4
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	1
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	1
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	1
Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	1
Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	1
Fi_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	1
$L_2(31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$	2
$L_3(5)$	$2^5 \cdot 3 \cdot 5^3 \cdot 31$	2
$L_2(32)$	$2^5 \cdot 3 \cdot 11 \cdot 31$	2
$L_2(5^3)$	$2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 31$	12
$G_2(5)$	$2^6 \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 31$	1
$L_5(2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$	2
$L_6(2)$	$2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31$	2
$L_4(5)$	$2^7 \cdot 3^2 \cdot 5^6 \cdot 13 \cdot 31$	8
$L_3(25)$	$2^7 \cdot 3^2 \cdot 5^6 \cdot 7 \cdot 13 \cdot 23$	12
$O_7(5)$	$2^9 \cdot 3^4 \cdot 5^9 \cdot 7 \cdot 13 \cdot 23$	2
$S_6(5)$	$2^9 \cdot 3^4 \cdot 5^9 \cdot 13 \cdot 31$	2
$O_8^{\pm}(5)$	$2^{12} \cdot 3^5 \cdot 5^{12} \cdot 7 \cdot 13^2 \cdot 31$	24
$O_{10}^{\pm}(2)$	$2^{20} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$	2
$U_3(31)$	$2^{11} \cdot 3 \cdot 5 \cdot 7^2 \cdot 19 \cdot 31^3$	2
$L_5(4)$	$2^{20} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 31$	4
$S_{10}(2)$	$2^{25} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 31$	1
$O_{12}^{\pm}(2)$	$2^{30} \cdot 3^8 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 17 \cdot 31$	2
ON	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	2
TH	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	1
$O_{12}^{-}(2)$	$2^{30} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 31$	2
$L_6(4)$	$2^{30} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 31$	12
$S_{12}(2)$	$2^{36} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 31$	1
$L_2(47)$	$2^4 \cdot 3 \cdot 23 \cdot 47$	2
$L_2(47^2)$	$2^5 \cdot 3 \cdot 5 \cdot 13 \cdot 17 \cdot 23 \cdot 47^2$	4
$S_4(47)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 23^2 \cdot 47^4$	2
B	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$ $\cdot 19 \cdot 23 \cdot 31 \cdot 47$	1
$L_2(53)$	$2^2 \cdot 3^3 \cdot 13 \cdot 53$	2
$L_2(23^2)$	$2^4 \cdot 3 \cdot 5 \cdot 11 \cdot 23^2 \cdot 53$	4
$S_4(23)$	$2^8 \cdot 3^2 \cdot 5 \cdot 11^2 \cdot 23^4 \cdot 53$	2
$U_4(23)$	$2^{10} \cdot 3^4 \cdot 5 \cdot 11^2 \cdot 13^2 \cdot 23^6 \cdot 53$	4
$L_2(3^5)$	$2^2 \cdot 3^5 \cdot 11^2 \cdot 61$	10
$U_5(3)$	$2^{11} \cdot 3^{10} \cdot 5 \cdot 7 \cdot 61$	2
$L_2(11^2)$	$2^3 \cdot 3 \cdot 5 \cdot 11^2 \cdot 61$	4
$S_4(11)$	$2^6 \cdot 3^2 \cdot 5^2 \cdot 11^4 \cdot 61$	2
$L_2(61)$	$2^2 \cdot 3 \cdot 5 \cdot 31 \cdot 61$	2
$L_3(13)$	$2^5 \cdot 3^2 \cdot 7 \cdot 13^3 \cdot 61$	6

Table 2: $p \in \pi(G) \subseteqq \{2, 3, 5, 7, \cdots, p\}$, where $p = 23, 31, 47, 53, 61, 73, 83, 89$

G	$\|G\|$	$\|O_{ut}(G)\|$
$U_6(3)$	$2^{13} \cdot 3^{15} \cdot 5 \cdot 7^2 \cdot 13 \cdot 61$	4
$U_4(11)$	$2^7 \cdot 3^4 \cdot 5^2 \cdot 11^6 \cdot 37 \cdot 61$	8
$L_3(47)$	$2^6 \cdot 3 \cdot 23^2 \cdot 37 \cdot 47^3 \cdot 61$	2
$L_4(11)$	$2^7 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11^6 \cdot 19 \cdot 61$	4
$L_4(13)$	$2^7 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13^6 \cdot 17 \cdot 61$	8
$O_{10}^{-}(3)$	$2^{15} \cdot 3^{20} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41 \cdot 41 \cdot 61$	8
$L_5(9)$	$2^{15} \cdot 3^{20} \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 41 \cdot 61$	4
$S_{10}(3)$	$2^{17} \cdot 3^{25} \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 41 \cdot 61$	2
$O_{11}(3)$	$2^{17} \cdot 3^{25} \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 41 \cdot 61$	2
$O_{12}^{+}(3)$	$2^{19} \cdot 3^{30} \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^2$ $\cdot 41 \cdot 61$	8
$L_3(11^2)$	$2^7 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11^6 \cdot 19 \cdot 37 \cdot 61$	12
$S_6(11)$	$2^9 \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11^9 \cdot 19 \cdot 37 \cdot 61$	2
$O_7(11)$	$2^9 \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 11^9 \cdot 19 \cdot 37 \cdot 61$	2
$O_8^{+}(11)$	$2^{12} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 11^{12} \cdot 19$ $\cdot 37 \cdot 61^2$	24
$L_4(47)$	$2^{11} \cdot 3^2 \cdot 5 \cdot 13 \cdot 17 \cdot 23^3 \cdot 37 \cdot$ $\cdot 47^6 \cdot 61$	4
$U_3(9)$	$2^5 \cdot 3^6 \cdot 5^2 \cdot 73$	4
$L_3(8)$	$2^9 \cdot 3^2 \cdot 7^2 \cdot 73$	6
$L_2(73)$	$2^3 \cdot 3^2 \cdot 37 \cdot 73$	24
$U_4(9)$	$2^9 \cdot 3^{12} \cdot 5^3 \cdot 41 \cdot 73$	8
${}^3D_4(3)$	$2^6 \cdot 3^{12} \cdot 7^2 \cdot 13^2 \cdot 73$	3
$L_2(2^9)$	$2^9 \cdot 3^3 \cdot 7 \cdot 19 \cdot 73$	9
$G_2(8)$	$2^{18} \cdot 3^5 \cdot 7^2 \cdot 19 \cdot 73$	3
$L_2(3^6)$	$2^3 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13 \cdot 73$	12
$S_4(27)$	$2^6 \cdot 3^{12} \cdot 5 \cdot 7^2 \cdot 13^2 \cdot 73$	6
$E_6(2)$	$2^{36} \cdot 3^6 \cdot 5^2 \cdot 7^3 \cdot 13 \cdot 17 \cdot 31 \cdot 73$	2
$U_4(27)$	$2^7 \cdot 3^{18} \cdot 5 \cdot 7^3 \cdot 13^2 \cdot 19 \cdot 37 \cdot 73$	6
$O_{12}^{-}(3)$	$2^{18} \cdot 3^{30} \cdot 5^3 \cdot 7 \cdot 11^2 \cdot 13 \cdot 41$ $\cdot 61 \cdot 73$	2
$L_6(9)$	$2^{18} \cdot 3^{30} \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2$ $\cdot 41 \cdot 61 \cdot 73$	4
$O_{13}(3)$	$2^{21} \cdot 3^{36} \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2$ $\cdot 41 \cdot 61 \cdot 73$	2
$S_{12}(3)$	$2^{21} \cdot 3^{36} \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2$ $\cdot 41 \cdot 61 \cdot 73$	2
${}^2E_6(2)$	$2^{19} \cdot 3^{30} \cdot 5^2 \cdot 7^3 \cdot 13^2 \cdot 19 \cdot 37$ $\cdot 41 \cdot 61 \cdot 73$	2
$L_2(83)$	$2^2 \cdot 3 \cdot 7 \cdot 41 \cdot 83$	2
$L_2(83^2)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 41 \cdot 53 \cdot 83^2$	2
$S_4(83)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 41^2 \cdot 53 \cdot 83^4$	2
$L_2(89)$	$2^3 \cdot 3^2 \cdot 5 \cdot 11 \cdot 89$	2
$L_2(97)$	$2^5 \cdot 3 \cdot 7^2 \cdot 97$	2
$L_3(61)$	$2^5 \cdot 3^2 \cdot 5^2 \cdot 13 \cdot 31 \cdot 61^3 \cdot 97$	2
A_n	$n!$ with $23 \leq n \leq 100$	2

Proof. First show that K is a $23'$-group. Otherwise, K contains an element x of order 23. Set $C = C_G(x)$ and $N = N_G(x)$. From $D(G)$, we have that C is a $\{2, 23\}$-group. By N/C theorem, N/C is isomorphic to a subgroup of automorphism group $Aut(< x >) \cong Z_{22}$, where Z_{22} is a cyclic group of order 22. Hence, N is a $\{2, 11, 23\}$-group. By Frattini arguments, $G = KN_G(< x >)$, which means that $\{3, 5, 7, 13, 17, 19\} \subseteq \pi(K)$. Since K is soluble, K contains a Hall $\{19, 23\}$-subgroup H of order $19 \cdot 23$. Obviously, H is nilpotent, then $19 \cdot 23 \in \omega(G)$, a contradiction.

Second prove that K is a p'-group, where $p = 5, 7, 13, 17, 19$. Let $p \in \pi(K)$ and P be a Syl_p-subgroup of K. Then by Frattini arguments, $G = KN_G(P)$. Considering the order of G, $23 \mid |N_G(P)|$. Obviously, the Sylow 23-subgroup of G acts fixed point freely on the set of elements of order p, which means that $23 \cdot p \in \omega(G)$, a contradiction.

So we have K is a $\{2, 3, 11\}$-group. Since $K \neq G$, G is insoluble.

Lemma 21 *The quotient group G/K is an almost simple group. More precisely, there is a normal series such that $S \leq G/H \leq Aut(S)$, where $S \cong A_{26}$ or A_{27}.*

Proof. Let $H = G/K$ and $S = Soc(H)$. Then $S = B_1 \times B_2 \times \cdots B_n$, where B_i's are nonabelian simple groups and $S \leq H \leq Aut(S)$. In what follows, we will prove that $n = 1$ and $S \cong A_{26}$ or A_{27}.

Suppose that $n \geq 2$. In this case, it is easy to have that 23 does not divide the order of S, since, otherwise, $5 \sim 23$, a contradiction. Hence, for every i, we have that $B_i \in \mathcal{F}_{19}$. On the other hand, by Lemma 20, K is a $\{2, 3, 11\}$-group. Therefore, $23 \in \pi(H) \subseteq \pi(Aut(S))$ and so 23 divides the order of $Out(S)$. But by Lemma 11,

$$Out(S) = Out(P_1) \times Out(P_2) \times \cdots Out(P_r),$$

where the group P_i's such that $S \cong P_1 \times P_2 \times \cdots P_r$. Therefore, for some j, 23 divides the order of an outer automorphism group of a direct P_j of t isomorphic simple groups B_i. Since $B_i \in \mathcal{F}_{19}$, we have that $|Out(B_i)|$ is not divisible by 23 (see Table 2). Now by Lemma 11, $|Aut(P_j)| = |Aut(P_j)|^t \cdot t!$. Therefore $t \geq 23$, Now 2^{46} must divide the order of G, a contradiction. Thus $n = 1$ and $S = B_1$.

Now by Lemmas 14 and 20, it is evident that

$$|S| = 2^a \cdot 3^b \cdot 5^6 \cdot 7^3 \cdot 11^c \cdot 13^2 \cdot 17 \cdot 19 \cdot 23$$

where $2 \leq a \leq 22$, $0 \leq b \leq 13$ and $0 \leq c \leq 2$. By Lemma 16, the only possible group which is isomorphic to S is A_{26} or A_{27}.

Lemma 22 *G is isomorphic to A_{27}.*

Proof. By Lemma 21, we have that $S \cong A_{26}$ or A_{27}.

- If $S \cong A_{27}$, then

  $$A_{27} \leq G/K \leq Aut(A_{27}) \cong S_{27}.$$

  Therefore, $G/K \cong A_{27}$ or $G/K \cong S_{27}$. If the latter, then $K = 1$ and $G \cong S_{27}$, which contradicts the hypotheses. So $G/K \cong A_{27}$ and $K = 1$ by considering the order of G. Therefore $G \cong A_{27}$.

- If $S \cong A_{26}$, then

  $$A_{26} \leq G/K \leq Aut(A_{26}).$$

  Therefore, $G/K \cong A_{26}$ or S_{26}.

  If $G/K \cong S_{26}$, order consideration can rule out this case.

  If $G/K \cong A_{26}$, then $|K| = 3^3$. In this case, $2 \nsim 23$, a contradiction.

This completes the proof of Theorem 19.

Let K be a maximal normal soluble subgroup of G. Similarly as the proof of the case "$p = 23$", we have that, for $p = 31, 47, 53, 61, 73, 83, 89$, K is a $\{2, 3, 5\}$, $\{2, 3\}$, $\{2, 3, 5, 11, 23\}$, $\{2, 3\}$, $\{2, 3\}$, $\{2, 3, 7\}$, $\{2, 3\}$-group, respectively. We also have that G/K is an almost simple group. In particular, $S \leq G/K \leq Aut(S)$, where $S \cong A_{35}$, A_{51}, A_{65}, A_{77}, A_{87} or A_{93} respectively. Order consideration, $G/K \cong A_{p+4}$. It is easy to see that $K = 1$ and so $G \cong A_{p+4}$ for $p = 31, 47, 53, 61, 73, 83, 89$.

The proof of theorem is completed. □

4.2 Proof for symmetric groups

From Proposition 5, we have that the symmetric groups S_p, S_{p+1} and S_{p+2} are OD-characterizable. Also by Proposition 9, S_{10} are 8-fold OD-characterizable. Some authors proved that the symmetric groups S_{p+3} except S_{10} are 3-fold OD-characterizable. We prove the following theorem.

Theorem 23 *(1) If $D(G) = D(S_{27})$, then G is isomorphic to $(Z_3 \times Z_3 \times Z_3) \times S_{26}$, $((Z_3 \times Z_3) \rtimes Z_3) \times S_{26}$, $(Z_3 \times Z_3 \times Z_3).S_{26}$, $((Z_3 \times Z_3) \rtimes Z_3).S_{26}$, $(Z_2.((Z_3 \times Z_3) \rtimes Z_3)) \times A_{26}$, $(Z_2.(Z_3 \times Z_3 \times Z_3)) \times A_{26}$, S_{27}, $Z_2.A_{27}$ and $Z_2 \times A_{27}$. In particular, S_{27} is 9-fold OD-characterizable.*

(2) The symmetric groups S_{p+4}, where $p \in \{31, 47, 53, 61, 73, 83, 89\}$, are 3-fold OD-characterizable.

Proof. Let $M \cong S_{p+4}$. Assume that G is a finite group such that $|G| = |M|$ and $D(G) = D(M)$. From Lemma 15, we have that the prime graph $\Gamma(G)$ is connected, in particular, $\Gamma(G) = \Gamma(M)$.

In the following, we only consider the case "$p = 23$".

Let G be a group with

$$|G| = 2^{23} \cdot 3^{13} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23$$

and

$$D(G) = (8, 8, 7, 7, 5, 4, 4, 4, 2).$$

Since $\{23 \cdot 19, 23 \cdot 11, 19 \cdot 11\} \cap \omega(G) = \phi$, then by Lemma 8 of [7], G is insoluble. Let K be a maximal normal soluble subgroup of G and $H = G/K$.

Similarly as the proof of Theorem 19, by Table 2, S is A_{26} or A_{27} and $S \le G/K \le Aut(S)$.

- If $S \cong A_{27}$, then

$$A_{27} \le G/K \le Aut(A_{27}) \cong S_{27}.$$

  Therefore, $G/K \cong A_{27}$ or $G/K \cong S_{27}$. If the latter, then $K = 1$ and $G \cong S_{27}$. So $G/K \cong A_{27}$ and $|K| = 2$ by considering the order of G. If G is a central extension of Z_2 by A_{27}, then $G \cong Z_2 \times A_{27}$. If G is a non-split of Z_2 by A_{27}, then $G \cong Z_2.A_{27}$.

- If $S \cong A_{26}$, then

$$A_{26} \le G/K \le Aut(A_{26}).$$

  Therefore, $G/K \cong A_{26}$ or S_{26}.

  If $G/K \cong S_{26}$, then $|K|$=27. Let K be abelian.

  - If $K \le Z(G)$, then since K is a maximal normal soluble subgroup of G, $K = Z(G) \cong Z_{27}$, and so G is a central extension of Z_{27} by S_{26}. It follows that there is an element of order $3^3 \cdot 23$, a contradiction.
  - If $|K \cap Z(G)| = 9$, then there is an element of order $3^3 \cdot 23$, a contradiction.
  - if $|K \cap Z(G)| = 3$, then $K \cong Z_3 \times Z_3 \times Z_3$. If G splits over K, then clearly, $G \cong K \times S_{26}$.
    If G is non-split extension of K by S_{26}, we have that $G \cong K.S_{26}$.

  Let K is nonabelian. Obviously, the order of the center of K is order 3 and the highest order element x of K is 9 or 3.

  - Let $|x| = 9$. Then there exists an element of order $3^2 \cdot 23$, a contradiction.
  - Let $|x| = 3$. Then $K = (Z_3 \times Z_3) \rtimes Z_3$. If G splits over K, then clearly, $G \cong K \times S_{26}$. If G is non-split extension of K by S_{26}, we have that $G \cong K.S_{26}$.

  If $G/K \cong A_{26}$, then $|K| = 2 \cdot 3^3$. We know that $K = Z_2 \times P$ or $K = Z_2.P$, where P is a p-group of order 27. In the following, we consider two cases: P is abelian and nonabelian.

  Let P be abelian.

  - If $K \cap P \le Z(G) \cap P \cong Z_{27}$, then since K is a maximal normal soluble subgroup of G, G is a central extension of K by A_{26}. It follows that there is an element of order $3^3 \cdot 23$, a contradiction.
  - If $Z(G) \cap P \cong Z_9$, then there is an element of order $3^2 \cdot 23$, a contradiction.
  - if $Z(G) \cap P \cong Z_3$, then $P \cong Z_3 \times Z_3 \times Z_3$. If G splits over K, then clearly, $G \cong K \times A_{26}$.
    If G is non-split extension of K by A_{26}, we have that $G \cong K.A_{26}$. On the other hand, the order of K divides by the Schur multiplier of A_{26}, a contradiction.

  Let P be nonabelian. Obviously, the order of the center of K is order 3 and the highest order element x of K is 9 or 3.

  - Let $|x| = 9$. Then there exists an element of order $3^2 \cdot 23$, a contradiction.
  - Let $|x| = 3$. Then $K = (Z_3 \times Z_3) \rtimes Z_3$. If G splits over K, then clearly, $G \cong K \times A_{26}$. If G is non-split extension of K by A_{26}, we have that $G \cong K.A_{26}$. On the other hand, the order of K divides the Schur multiplier of A_{26}, a contradiction.

Therefore S_{27} is 9-fold OD-characterizable.

We avoid the details for S_{p+4}, where $p \in \{31, 47, 53, 61, 73, 83, 89\}$, because the arguments are quite similar to those for S_{27}. We only mention that the non-isomorphic groups $Z_2.A_{p+4}$ and $Z_2 \times A_{p+4}$, where $p \in \{31, 47, 53, 61, 73, 83, 89\}$, have the same order and degree patterns as S_{p+4}, where $p \in \{31, 47, 53, 61, 73, 83, 89\}$, respectively. Hence S_{p+4}, for where $p \in \{31, 47, 53, 61, 73, 83, 89\}$, is 3-fold OD-characterizable, and the proof of the theorem is complete. □

5 Conclusion

The alternating groups A_{p+4}, where $p \in$ {23, 31, 47, 53, 61, 73, 83, 89}, are OD-characterizable.

The symmetric groups S_{p+4}, where $p \in \{31, 47, 53, 61, 73, 83, 89\}$, are 3-fold OD-characterizable.

The symmetric group S_{27} is 9-fold OD-characterizable.

Corollary 24 *The alternating groups A_{p+4}, where p is a odd prime and $p < 100$, are OD-characterizable.*

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