The reduction method for approximative solution of systems of Singular Integro-Differential Equations in Lebesgue spaces(case $\gamma \neq 0$)

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Abstract: In this article we have elaborated the numerical schemes of reduction methods for approximate solution of system of singular integro-differential equations when the kernel has a weak singularity. The equations are defined on the arbitrary smooth closed contour of complex plane. We suggest the numerical schemes of the reduction method over the system of Faber-Laurent polynomials for the approximate solution of weakly singular integro- differential equations defined on smooth closed contours in the complex plane. We use the cut-off technique kernel to reduce the weakly singular integro- differential equation to the continuous one. Our approach is based on the Krykunov theory and Zolotarevski results. We have obtained the theoretical background for these methods in classical Lebesgue spaces.

Key–Words: Faber- Laurent Polynomials, systems of singular integro-differential equations, reduction methods

1 Introduction

Singular integro-differential equations (SIDE) model many problems in physics, elasticity theory, aerodynamics, mechanics, etc. [1]-[9].

The problem of approximate solution of systems of SIDE was studied in many scientific articles [10]-[18].

At the same time the reduction method applied to the approximative solution of systems of SIDE is not studied enough, particularly the case when the equations are defined on a close contour of the complex plain different from the standard one(unit circle:Γ_0).

It is known that the exact solution for SIDE can only be found in particular cases and even in these cases the exact solution is expressed by multiple singular integrals, their calculation presents many theoretical and practical difficulties. That is why the necessity exists to elaborate approximate methods for solving SIDE with the corresponding theoretical background. The first results in this direction have been obtained for SIDE on a standard contour: segment or unity circle. [10]-[18] Transition to another contour, different from the standard one, implies many difficulties. It should be noted that the conformal mapping of SIDE from the arbitrary contour to the standard one using some reflection function does not solve the problem, but only it makes more difficult: a nucleus with a weak singularity appears, so the method of mechanical quadratures cannot be applied.

We would like to mention only the scientific articles [24]-[25] which study the reduction method applied to the Singular Integral Equations(SIE), defined on the smooth closed contour (different from a standard one), where the theoretical background of reduction method has been obtained in Hölder spaces. In case when the SIE are defined on the unit circle the reduction method was studied in monograph [26] and scientific papers [27]-[28] where the theoretical background of SIE has been obtained in Lebesgue spaces $L_p(\Gamma_0)$ and Holder spaces $H_\beta(\Gamma_0)$(Γ_0 is unit circle). The case of SIDE of reduction methods has been studied in [29]. Theoretical background has been obtained in Holder spaces.

Theoretical background of the collocation for approximate solution of SIDE in Höder spaces and Lebesgue spaces was proved in [22]-[23]. The equations are given on a closed contour satisfying some conditions of smoothness. The stability of collocation methods was obtained in [20]. The numerical results can be found in [21].

In the present work the numerical schemes of reduction methods for approximative solution of systems of weakly SIDE have been obtained on the Faber-Laurent polynomial system, generated by an arbitrary smooth closed contour. The theoretical background has been obtained in Lebesgue spaces.

2 Definitions of Function Spaces and Notations

Let Γ be a smooth closed contour bounding a simply connected domain D^+ that contains the point $z = 0$, and let $D^- = C \setminus D^+ \cup \Gamma$.

Let function $w = \Phi(z)$ apply conformably D^- in the domain $|w| > 1$ so that
$\Phi(\infty) = \infty, \lim\limits_{z\to\infty} z^{-1}\Phi(z) = \alpha > 0$, and $z = \phi(w)$ is the inverse function of function $\Phi(z)$.

And let $w = F(z)$ apply conformably D^+ in the domain $|w| > 1$, so that
$F(0) = \infty, \lim\limits_{z\to 0} zF(z) = \beta > 0$, and $z = \varphi(w)$ is the inverse function.

In the area of the infinite point the function $\Phi(z)$ will have the decomposition:

$$\Phi(z) = \alpha z + \alpha_0 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + ...,$$

and the inverse function $z = \phi(w) = \gamma w + \gamma_0 + \frac{\gamma_1}{w} + \frac{\gamma_2}{w^2} + ..., |w| > 1$, where $\gamma = \frac{1}{\alpha} > 0$.

The function $F(z)$ in the area of the point zero will have the decomposing:

$$F(z) = \beta z^{-1} + \beta_0 + \beta_1 z + \beta_2 z^2 + ...,$$

and the inverse function will have a decomposing that is obtained in the same way.

Suppose in the future that $\alpha = 1, \beta = 1$. As is proved in the monograph [30], this supposition doesn't limit the general case.

With $\Phi_k(z)(k = 0, 1, 2, ...)$ mark the polynomial that represents the totality of members with non-negative power of z in the Laurent decomposition of the function $[\Phi(z)]^k$, and with $F_k(1/z)(k = 1, 2, ...)-$ the polynomial that represents the totality of members with negative power of z in Laurent expansion of the function $[F(z)]^k$. Polynomials $\Phi_k(z)(k = 0, 1, 2, \ldots,)$ and $F_k(1/z)(k = 1, 2, \ldots,)$ are called Faber polynomials for the contour Γ for the powers of z and $1/z, z \in \Gamma$, respectively. Let S_n be the operator that takes each continuous function $g(t)$ on Γ to the nth partial sum of its Faber-Laurent polynomials (see [30]):

$$g(t) = \sum_{k=0}^{\infty} g_k \Phi_k(t) + \sum_{k=1}^{\infty} g_{-k} F_k(1/t), \qquad (1)$$

where $g_k(k = 0, \pm 1, \ldots)$ is calculated from:

$$g_k = \frac{1}{2\pi i} \int\limits_{|\tau|=\rho} g(\phi(\tau))\tau^{-k-1} d\tau, (k = 0, 1, 2, \ldots), \quad (2)$$

$$g_{-k} = \frac{1}{2\pi i} \int\limits_{|\tau|=\rho} g(\varphi(\tau))\tau^{-k-1} d\tau, (k = 1, 2, \ldots). \quad (3)$$

Mark with S_n the operator that puts in correspondence to each function $g(t)$ defined on Γ partial sum of order n of Faber-Laurent series:

$$(S_n g)(t) = \sum_{k=0}^{n} g_k \Phi_k(t) + \sum_{k=1}^{n} g_{-k} F_k(1/t). \quad (4)$$

We assume that the function $z = \psi(w)$ has a second derivative, satisfying on unit circle the Hölder condition with some parameter q $(0 < \nu < 1)$; the class of such contours is denoted by $C(2;\nu)$. [19]

In the complex space $[L_p(\Gamma)]_m (1 < p < \infty)$ of vector functions (v.f.)

$$g(t) = (g_1(t), \dots, g_m(t));$$

$g_j(t) \in L_p(\Gamma)$, $j = \overline{1,m}$ summarized on Γ on power p, $1 < p < \infty$ and with the norm

$$||g|| = \sum_{k=1}^{m} ||g_k||_p; \quad ||g_k||_p = \left(\frac{1}{l} \int_{\Gamma} |g_k|^p(\tau)|d\tau| \right)^{\frac{1}{p}},$$

where l is the length of Γ,

We denote by $[H_\beta^{(q)}]_m$, $0 < \beta \le 1$, the Banach space of $m-dimensional$ vector functions, satisfying on Γ the Holder condition with degree β.

The norm is defined as $\forall g(t) = \{g_1(t), \dots, g_m(t)\}$

$$||g||_\beta = \sum_{1}^{m} (||g||_C + H(g_k, \beta)),$$

$$||g||_C = \max_C |g(t)|,$$

$$H(g,\beta) = \sup_{t' \neq t''} \{|t' - t''|^{-\beta} |g(t' - g(t''|\},\ t', t'' \in \Gamma.$$

Definition 1 *A factorization of non-singular matrix $G(t)$ relative to the contour Γ is a representation of $G(t)$ in the form*

$$G(t) = G^+ \Delta(t) G^-$$

where $G^\pm$ are matrix functions analytic and non-singular in $D^\pm$, satisfying $detG^\pm \neq 0$, respectively, $\Delta(t) = diag\{t^{\kappa_1}, t^{\kappa_2}, \dots, t^{\kappa_m}\}$, and $\kappa_1, \kappa_2, \dots, \kappa_m$ are integer. The numbers $\kappa_1, \kappa_2, \dots, \kappa_m$ are called left partial indexes [36]

3 Problem formulation

In the complex space $[L_p(\Gamma)]_m$ we consider the singular integro-differential equations (SIDE)

$$(Mx \equiv) \sum_{r=0}^{q} [\tilde{A}_r(t) x^{(r)}(t) + \tilde{B}_r(t) \frac{1}{\pi i} \int_{\Gamma} \frac{x^{(r)}(\tau)}{\tau - t} d\tau +$$

$$+ \frac{1}{2\pi i} \int_{\Gamma} \frac{K_r(t,\tau)}{|t-\tau|^{\gamma_r}} \cdot x^{(r)}(\tau) d\tau]$$

$$= f(t), \quad t \in \Gamma, \qquad (5)$$

where $0 < \gamma_r < 1$, $\tilde{A}_r(t), \tilde{B}_r(t)$ are given matrix functions(m.f.), $f(t)$ is given v.f. which belong to $[H_\beta(\Gamma)]_m$;and $K_r(t,\tau)$ $(r = \overline{0,q})$ is m.f. which belong to $[H_\beta(\Gamma)]_m$ by both variables. $x^{(0)}(t) = x(t)$ is the required v.f.; $x^{(r)}(t) = \frac{d^r x(t)}{dt^r}$ $(r = \overline{1,q})$; (q is a natural number) which belong to $[H_\beta(\Gamma)]_m$.

Using the Riesz operators $P = \frac{1}{2}(I + S)$, $Q = I - P$, (where I is the identity operator, and S is the singular operator (with Cauchy kernel)), we rewrite the system of Eq. (5) in the following form convenient for consideration:

$$(Mx \equiv) \sum_{r=0}^{q} [A_r(t)(Px^{(r)})(t) + B_r(t)(Qx^{(r)})(t) +$$

$$+ \frac{1}{2\pi i} \int_{\Gamma} \frac{K_r(t,\tau)}{|t-\tau|^{\gamma}} \cdot x^{(r)}(\tau) d\tau]$$

$$= f(t), \quad t \in \Gamma, \qquad (6)$$

where $A_r(t) = \tilde{A}_r(t) + \tilde{B}_r(t)$, $B_r(t) = \tilde{A}_r(t) - \tilde{B}_r(t)$, $\quad r = 0, \dots, q$.

We need that the function $x^{(q)}(t)$ belongs to $[H_\beta(\Gamma)]_m$. From this condition follows that

$$x^{(k)} \in [H_\beta(\Gamma)]_m, \qquad k = 0, \dots, q-1.$$

We search for the solution of equation (5) in the class of functions, satisfying the condition

$$\frac{1}{2\pi i} \int_{\Gamma} x(\tau) \tau^{-k-1} d\tau = 0, \quad k = \overline{0, q-1}. \qquad (7)$$

We introduce the denotation "the problem (5)-(7)" for the SIDE (5) together with the conditions (7).

In order to reduce the schemes of reduction methods we introduce a new integro- differential equation from the initial equation. The kernel with the weak singular peculiarities is substituted by continue kernel. Consequently we solve the approximative equation:

$$(M_\rho x)\equiv)(M_0x)(t)+\frac{1}{2\pi i}\sum_{r=0}^{q}\int_\Gamma K_{r,\rho}(t,\tau)x^{(r)}(\tau)d\tau \quad \text{F}$$

$$= f(t),\ t\in\Gamma. \tag{8}$$

where $$K_{r,\rho}(t,\tau)=\begin{cases}\dfrac{K_r(t,\tau)}{|t-\tau|^{\gamma_r}} & \text{when} \quad |t-\tau|\geq\rho;\\ \dfrac{K_r(t,\tau)}{\rho^{\gamma_r}} & \text{when} \quad |t-\tau|<\rho;\end{cases} \tag{9}$$

ρ is a arbitrary positive number, M_0 is a characteristic part of S which corresponds system (5). And so the equation (5) is changed on the new equation (8).

4 Auxiliary results

We formulate one result from [31],[32] establishing the equivalence (in sense of solvability) of problem (5)-(7) and system of SIE. We use this result for proving Theorem 7. V.F. $\frac{d^q(Px)(t)}{dt^q}$ and $\frac{d^q(Qx)(t)}{dt^q}$ can be represented by integrals of Cauchy type with the same density $v(t)$:

$$\left.\begin{aligned}\frac{d^q(Px)(t)}{dt^q}&=\frac{1}{2\pi i}\int_\Gamma\frac{v(\tau)}{\tau-t}d\tau, \quad t\in F^+,\\ \frac{d^q(Qx)(t)}{dt^q}&=\frac{t^{-q}}{2\pi i}\int_\Gamma\frac{v(\tau)}{\tau-t}d\tau, \quad t\in F^-.\end{aligned}\right\} \tag{10}$$

Using the integral representation (10) we reduce the problem (5)-(7) to the equivalent (in sense of solvability) system of SIE

$$(\Upsilon v\equiv)C(t)v(t)+\frac{D(t)}{\pi i}\int_\Gamma\frac{v(\tau)}{\tau-t}d\tau+$$

$$\frac{1}{2\pi i}\int_\Gamma\frac{h(t,\tau)}{|\tau-t|^\gamma}v(\tau)d\tau=f(t), \quad t\in\Gamma, \tag{11}$$

for unknown v.f. $v(t)$ where

$$C(t)=\frac{1}{2}[A_q(t)+t^{-q}B_q(t)],$$

$$D(t)=\frac{1}{2}[A_q(t)-t^{-q}B_q(t)], \tag{12}$$

$$h(t,\tau)=\frac{1}{2}\left[K_q(t,\tau)+\right.$$

$$\left.K_q(t,\tau)\tau^{-n}\right]-\frac{1}{2\pi i}\int_\Gamma$$

$$\left[K_q(t,t_1)-K_q(t,t_1)t_1^{-n}\right]\frac{dt_1}{t_1-\tau}$$

$$+\sum_{j=0}^{q-1}\left[A_j(t)\tilde{M}_j(t,\tau)+\right.$$

$$\left.\int_\Gamma K_j(t,t_1)\tilde{M}_j(t_1,\tau)dt_1\right]$$

$$-\sum_{j=0}^{q-1}\left[B_j(t)\tilde{N}_j(t,\tau)+\right.$$

$$\left.\int_\Gamma K_j(t,t_1)\tilde{N}_j(t_1,\tau)dt_1\right], \tag{13}$$

where $\tilde{M}_j(t,\tau),\tilde{N}_j(t,\tau)$ $j=0,\ldots,$ are Hölder m.f. An obvious form for these functions is given in [32]. By virtue of the proprieties of the M.F. $\tilde{M}_j(t,\tau),\tilde{N}_j(t,\tau),\ K_j(t,\tau),A_j(t),B_j(t),\ j=0,\ldots,q$ the function $h(t,\tau)$ is a continuous function in both variables.

Lemma 2 *The system of SIE (11) and problem (5)-(7) are equivalent in the sense of solvability. That is, for each solution v.f. $v(t)$ of system of SIE (11) there is a solution of problem (5)-(7), determined by formulae*

$$(Px)(t)=\frac{(-1)^q}{2\pi i(q-1)!}\int_\Gamma v(\tau)[(\tau-t)^{q-1}\times$$

$$\log\left(1-\frac{t}{\tau}\right)+\sum_{k=1}^{q-1}\tilde{\alpha}_k\tau^{q-k-1}t^k]d\tau,$$

$$(Qx)(t) = \frac{(-1)^q}{2\pi i(q-1)!}\int_\Gamma v(\tau)\tau^{-q}[(\tau-t)^{q-1}\times$$

$$\log\left(1-\frac{\tau}{t}\right)+\sum_{k=1}^{q-2}\tilde{\beta}_k\tau^{q-k-1}t^k]d\tau, \qquad (14)$$

where($\tilde{\alpha}_k = \sum_{j=0}^{k-1}\frac{(-1)^j C_{q-1}^j}{k-j}, k = 1,\ldots,q-1$, $\tilde{\beta}_k = \sum_{j=k+1}^{q-1}\frac{(-1)^j C_{q-1}^j}{j-k}$, $k = 1,\ldots,q-2$ *and* C_{q-1}^j *are the binomial coefficients.) On the other hand, for each solution v.f.* $x(t)$ *of the problem (5)-(7) there is a solution v.f.* $v(t)$

$$v(t) = \frac{d^q(Px)(t)}{dt^q} + t^q\frac{d^q(Qx)(t)}{dt^q},$$

to the system of SIE (11). Furthermore, for linearly-independent solutions of (11), there are corresponding linearly-independent solutions of the problem (5)-(7) from (14) and vice versa.

In formulas (14) by $\log(1-t/\tau)$ we understand the branch which vanishes as $t = 0$ and by $\log(1-\tau/t)$ the branch which vanishes as $t = \infty$.

Define $[\overset{\circ}{W}{}_p^{(q)}]_m$ as

$$[\overset{\circ}{W}{}_p^{(q)}]_m = \Big\{g \in [L_p(\Gamma)]_m : g^{(q)} \in [L_p(\Gamma)]_m\ ,$$

$$\frac{1}{2\pi i}\int_\Gamma g(\tau)\tau^{-k-1}d\tau = 0,$$

$$k = 0,\ldots,q-1\Big\}.$$

The norm in $[\overset{\circ}{W}{}_p^{(q)}]_m$ is determined by the equality

$$[||g||_{p,q}]_m = [||g^{(q)}||_{L_p}]_m.$$

We denote by $[L_{p,q}]_m$ the image of the space $[L_p]_m$ with respect to the map $P+t^{-q}Q$ equipped with the norm of $[L_p]_m$. We formulate Lemma 3 and Lemma 4 from [33]. We use these lemmas to prove the convergence theorems.

Lemma 3 *The differential operator* $D^q : [\overset{\circ}{W}{}_p^{(q)}]_m \to [L_{p,q}]_m$, $(D^q g)(t) = g^{(q)}(t)$ *is continuously invertible and its inverse operator* $D^{-q} : [L_{p,q}]_m \to [\overset{\circ}{W}{}_p^{(q)}]_m$ *is determined by the equality*

$$(D^{-q}g)(t) = (N^+g)(t) + (N^-g)(t),$$

$$(N^+g)(t) = \frac{(-1)^q}{2\pi i(q-1)!}\times$$

$$\int_\Gamma (Pg)(\tau)(\tau-t)^{q-1}\log(1-\frac{t}{\tau})d\tau,$$

$$(N^-g)(t) = \frac{(-1)^{q-1}}{2\pi i(q-1)!}\times$$

$$\int_\Gamma (Qg)(\tau)(\tau-t)^{q-1}\log(1-\frac{\tau}{t})d\tau.$$

From Lemma 3 it follows

Lemma 4 *The operator* $B : [\overset{\circ}{W}{}_p^{(q)}]_m \to [L_p]_m, B = (P+t^qQ)D^q$ *is invertible and*

$$B^{-1} = D^{-q}(P+t^{-q}Q).$$

4.1 Estimates for weakly singular integral operators

Lemma 5 *Let* $h(t,\tau) \in C(\Gamma\times\Gamma)$, *and* $\psi(t) \in L_p(\Gamma)$, $1 < p < \infty$. *Then the function* $H(t) = \frac{1}{2\pi i}\int_\Gamma \frac{h(t,\tau)}{|\tau-t|^\gamma}\psi(\tau)d\tau$, *satisfies the inequality*

$$||H||_p \le d_1{}^1||\psi||_p,\ \frac{1}{p}+\frac{1}{q} = 1,$$

$$||(\cdot)||_p = \left|\frac{1}{l}\int_\Gamma |(\cdot)(\tau)|^p d\tau\right|^{1/p}. \qquad (15)$$

The proof can be found in [34].

Lemma 6 *Let the assumptions of Lemma 5 be satisfied; then* $||\chi_\rho||_p \le d_2\rho^{(1-\gamma)/q}||\psi||_p$, *where* $\chi_\rho = \frac{1}{2\pi i}\int_\Gamma\left[\frac{h(t,\tau)}{|\tau-t|^\gamma} - h_\rho(t,\tau)\right]\psi(\tau)d\tau$, $\frac{1}{p}+\frac{1}{q} = 1$.

The proof of this lemma can be found in [34].

[1]By $d_1, d_2, \ldots,$ we denote the constants.

5 Convergence Theorem

We seek an approximate solution of problem (5)-(7) in the form of a polynomial

$$x_{n,\rho}(t) = t^q \sum_{k=0}^{n} \alpha_{k,\rho}\Phi_k(t)+$$

$$\sum_{k=1}^{n} \alpha_{-k,\rho}F_k(\frac{1}{t}),\ t \in \Gamma, \quad (16)$$

with unknown numerical v.f. $\alpha_{k,\rho} = k = -n, \dots, n$. The numerical v.f. $\alpha_{k,\rho}$ are found from the condition:

$$S_n[Mx_{n,\rho} - f] = 0,$$

$$S_n[Mx_{n,\rho}] = S_n f, \quad (17)$$

for the unknown v.f. $x_{n,\rho}(t)$ of the form (16). Note that Eq. (17) is a system of $m(2n+1)$ linear algebraic equations(SLAE) with $m(2n+1)$ unknowns $\alpha_{k,\rho}\ k = -n, \dots, n.$

Note that the functions of this system is determined by the Faber- Laurent coefficients of the M.f $A_r(t)$ and $B_r(t)$:

$$\frac{1}{2\pi i}\int_{\Gamma} K_{r,\rho}(t,\tau)\Phi_k(\tau)d\tau,\ k = 0, \dots, n,$$

$$\frac{1}{2\pi i}\int_{\Gamma} K_{r,\rho}(t,\tau)F_k(\frac{1}{\tau})d\tau,$$

$$k = 1, \dots, n, r = 0, \dots, q$$

In what follows, we give a theoretical background of the reduction method, i.e., derive conditions providing the solvability (starting from some indices n) of (5) and the convergence of the approximate solutions (‘16) to the exact solution

Theorem 7 *Let the following conditions be satisfied:*

1. *m.f.* $A_r(t)$, $B_r(t)$ *and* $K_r, (t,\tau)$, *r* = $0, \dots, q$, *belong to the space* $[H(\alpha)]_m$*;*
2. $det[A_q(t)]det[B_q(t)] \neq 0$;
3. *the left partial indexes* $A_q(t)t^qB_q(t)$ *of m.f. are equal to zero;*

5) *m.f.* $[K_r(t,\tau)]\ \ (r = 0, \dots, q) \in [H_\beta]_m(\Gamma \times \Gamma),\ 0 < \beta \leq 1,$ *function* $f(t) \in [C(\Gamma)]_m$, $\Gamma \in C(2;\nu)$*;*

6) *the operator* $M : [\overset{\circ}{W}{}_p^{(q)}]_m \to [L_p(\Gamma)]_m$ *is linear and invertible;*

Then starting from indices $n \geq n_1$ *and* ρ *small enough the SLAE (17) of reduction method is uniquely solvable. The approximate solutions* $x_{n,\rho}(t)$ *given by formula (16) converge in the norm of space* $[\overset{\circ}{W}{}_p^{(q)}]_m$ *to the exact solution* $(x(t))$ *of problem (5)-(7) in sense of:*

$$\lim_{n\to\infty\, \rho to 0} ||x - x_{n,\rho}||^m_{p,q} = 0. \quad (18)$$

Proof Using the conditions of Theorem 7 we have that the operator $M : [\overset{o}{W}_{p,q}] \to [L_p(\Gamma)]_m$ is invertible. We estimate the perturbation of M depending on ρ. Using Lemma 6 and the relation $M_\rho = M_0 + K_\rho$ we obtain

$$||M - M_\rho|| = O(\rho^{(1-\gamma)/q}), \quad (19)$$

Let us show that the operator M_ρ is invertible for sufficiently small values ρ. Using the representation $M_\rho = M[I - M^{-1}(M - M_\rho)]$ and (19), we obtain from Banach Theorem that the inverse operator $M_\rho^{-1} = [I - M^{-1}(M - M_\rho)]^{-1}M^{-1}$ exists. The following inequalities hold:

$$||M_\rho^{-1}|| \leq \frac{||M^{-1}||}{1-q},$$

$$||M^{-1} - M_\rho^{-1}|| \leq d_{11}\rho^{(1-\gamma)/q}||M^{-1}||. \quad (20)$$

The SLAE (17) of the reduction method for SIDE (5) for $\gamma \in (0;1)$ is equivalent to the operator equation

$$S_nM_\rho S_nx_{n,\rho} \equiv S_nM_0S_nx_{n,\rho}+$$

$$S_n\sum_{r=0}^{q}\left\{\frac{1}{2\pi i}\int_{\Gamma} K_{r,\rho}(t,\tau)x^{(r)}_{n,\rho}(\tau)d\tau\right\}$$

$$= S_n f, \qquad (21)$$

where $K_{r,\rho}(t,\tau)$, $(r = 0,\ldots,q)$ is defined by formula (9). Using the integral presentation (10), the equation (21) is equivalent to the operator equation

$$U_n \Upsilon_\rho U_n v_{n,\rho} = U_n f, \qquad (22)$$

where operator Υ_ρ is defined in (11), substituting Υ by Υ_ρ and $\dfrac{h(t,\tau)}{|\tau - t|^\gamma}$ by $h_\rho(t,\tau)$ (where $h_\rho(t,\tau)$ is calculated by formula (13)). The equation (22), represents the reduction method for the system of SIE

$$\Upsilon_\rho v_\rho = f, v_\rho(t) \in L_p(\Gamma). \qquad (23)$$

We should show that if n $(\geq n_1)$ is large enough and ρ small enough the operator $U_n M_\rho U_n$ is invertible. The operator acts from the subspace $[\overset{\circ}{X}_n]_m = \left\{ t^q \sum\limits_{k=0}^{n} \xi_{k,\rho} t^k + \sum\limits_{k=-n}^{-1} \xi_{k,\rho} t^k \right\}$ (the norm as in $[\overset{\circ}{W}{}_p^{(q)}]_m$) to the subspace

$$[X_n]_m = \sum_{k=-n}^{n} r_k t^k, \quad t \in \Gamma.$$

(the norm as in $[L_p(\Gamma)]_m$.)

Using formulas (10) the $\dfrac{d^q(Px_{n,\rho})(t)}{dt^q}$ and $\dfrac{d^q(Qx_{n,\rho})(t)}{dt^q}$ can be represented by Cauchy type integrals with the same density $v_{n,\rho}(t)$:

$$\left. \begin{aligned} \frac{d^q(Px_{n,\rho})(t)}{dt^q} &= \frac{1}{2\pi i} \int\limits_\Gamma \frac{v_{n,\rho}(\tau)}{\tau - t} d\tau, \quad t \in F^+ \\ \frac{d^q(Qx_{n,\rho})(t)}{dt^q} &= \frac{t^{-q}}{2\pi i} \int\limits_\Gamma \frac{v_{n,\rho}(\tau)}{\tau - t} d\tau, \quad t \in F^-. \end{aligned} \right\} \qquad (24)$$

Using the formulas

$$(Px)^{(r)}(t) = P(x^{(r)})(t), \; (Qx)^{(r)}(t) = Q(x^{(r)})(t),$$

and relations (10) we obtain from (24)

$$v_{n,\rho}(t) = \sum_{k=0}^{n} \frac{(k+q)!}{k!} t^k \xi_{k,\rho} +$$

$$(-1)^q \sum_{k=1}^{n} \frac{(k+q-1)!}{(k-1)!} t^{-k} \xi_{-k,\rho}.$$

We obtain from previous relation that $v_{n,\rho}(t) \in X_n$, $t \in \Gamma$. The reduction method for system of SIE was considered in [35] where sufficient conditions for solvability and convergence of this method were obtained. From (24), Lemma 2 and $v_{n,\rho}(t) \in X_n$ we conclude that if function $v_{n,\rho}(t)$ is the solution of the equation (22) then the function $x_{n,\rho}(t)$ is the discrete solution for the system $U_n M U_n x_{n,\rho} = U_n f$ and vice versa. We can determine the function $v_{n,\rho}(t)$ from relations (14):

$$(Px_{n,\rho})(t) = \frac{(-1)^q}{2\pi i (q-1)!} \int\limits_\Gamma v_{n,\rho}(\tau) [(\tau - t)^{q-1} \times$$

$$\log(1 - \frac{t}{\tau}) + \sum_{k=1}^{q-1} \tilde{\alpha}_k \tau^{q-k-1} t^k] d\tau;$$

$$(Qx_{n,\rho})(t) = \frac{(-1)^q}{2\pi i (q-1)!} \times$$

$$\int\limits_\Gamma v_{n,\rho}(\tau) \tau^{-q} [(\tau - t)^{q-1} \log(1 - \frac{\tau}{t}) +$$

$$\sum_{k=1}^{q-1} \tilde{\beta}_k \tau^{q-k-1} t^k] d\tau; \qquad (25)$$

From the conditions 3),4),6) of Theorem 7, Lemma 3 and Lemma 4, the invertibility of operator $\Upsilon : [L_p(\Gamma)]_m \to [L_p(\Gamma)]_m$ follows. From Banach Theorem and Lemma 6 for small numbers ρ, we have that the operator $\Upsilon_\rho : [L_p(\Gamma)]_m \to [L_p(\Gamma)]_m$ is invertible. We should show that for (22) all conditions of the Theorem 1 are satisfied from [35]. Theorem 1[35] gives the convergence of the reduction method for system of SIE in spaces $[L_p(\Gamma)]_m$. From condition 3 of Theorem 1[35] and from (12) we obtain the condition 3 of Theorem 7. From the equality

$$[C(t) - D(t)]^{-1} [C(t) + D(t)] = t^q B_q^{-1} A_q(t),$$

we conclude that the index of the function $[C(t) - D(t)]^{-1}[C(t) + D(t)]$ are equal to zero, which coincides with condition 4 of Theorem 7. Other conditions of Theorem 7 coincide with conditions of Theorem 1[35]. Conditions 1)- 6) in Theorem 7 provide the validity of all conditions of

Theorem 1[35]. Therefore beginning with numbers $n \geq n_1$ (22) is uniquely solvable and for numbers ρ small enough. The approximate solutions $v_{n,\rho}(t)$ of (22) converge to the exact solution of equation (11) in the norm of the space $[L_p(\Gamma)]_m$ as $n \to \infty$. Therefore the equations (21) and the SLAE (17) have the unique solutions for $(n \geq n_1)$. From Theorem 1[35] the following estimation holds:

$$||v_\rho - v_{n,\rho}||_p^m \leq O\left(\frac{1}{n^\alpha}\right) +$$

$$O(\omega(f;\frac{1}{n})) + O(\omega^t(h;\frac{1}{n})), \qquad (26)$$

where $O(\omega^t(h;\frac{1}{n})$ and $O(\omega(f;\frac{1}{n})$ are modulus of continuity. From (10) and (25) we obtain

$$(Px_\rho)^{(q)}(t) = (Pv_\rho)(t), \quad (Qx_\rho)^{(q)}(t) = t^{-q}(Qv_\rho)(t).$$

Therefore we have

$$(Px_{n,\rho})^{(q)}(t) = (Pv_{n,\rho})(t),$$

$$(Qx_{n,\rho})^{(q)}(t) = t^{-q}(Qv_{n,\rho})(t).$$

We proceed to get an error estimate

$$||x_\rho - x_{n,\rho}||_{p,q} = ||x_\rho^{(q)} - x_{n,\rho}^{(q)}||_{[L_p]} \leq$$

$$||P(v_\rho - v_{n,\rho})||_{[L_p]} + ||t^{-q}Q(v_\rho - v_{n,\rho})||_{[L_p]} \leq$$

$$||P||\cdot||v_\rho - v_{n,\rho}||_{[L_p]} + ||t^{-q}||||Q||\cdot||v_\rho - v_{n,\rho}||_{[L_p]} \leq$$

$$(||P|| + ||t^{-q}||||Q||)||v_\rho - v_{n,\rho}||_{[L_p]}. \qquad (27)$$

Using the inequality

$$||t^{-q}||_{L_p} = \left(\frac{1}{l}\int_\Gamma |t^{-q}|^p dt\right)^{\frac{1}{p}} = \left(\frac{1}{l}\int_\Gamma |t^{-qp}| dt\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{1}{l}\frac{1}{\min\limits_{t\in\Gamma}|t|^{pq}} l\right)^{\frac{1}{p}} = \left(\frac{1}{\min\limits_{t\in\Gamma}|t|^{pq}}\right)^{\frac{1}{p}} = c_1,$$

From (26),(27), (20) and from the inequality

$$||x - x_{n,\rho}||_{p,q}^m \leq ||M^{-1}f - M_\rho^{-1}||_{p,q}^m$$

$$+||x_\rho - x_{n,\rho}||_{p,q}^m. \qquad (28)$$

we obtain the relation (18). Thus Theorem 7 is proved.

6 Conclusion

In this article we studied more general case when the kernel in SIDE (5) contains a weak singularity ($\gamma \neq 0$.) Theoretical background was proved in classical Lebesgue spaces. In future we are going to prove theoretical background in for the others functional spaces.

Acknowledgements: The research of Iurie Caraus was supported by Fulbright Grant.

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