Some iterative algorithms for k-strictly pseudo-contractive mappings in a $CAT(0)$ space

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Abstract: In this paper, we prove the Δ-convergence theorems of the cyclic algorithm and the new multi-step iteration for k-strictly pseudo-contractive mappings and give also the strong convergence theorem of the modified Halpern's iteration for these mappings in a $CAT(0)$ space. Our results extend and improve the corresponding recent results announced by many authors in the literature.

Key–Words: $CAT(0)$ space, fixed point, strong convergence, Δ-convergence, k-strictly pseudo-contractive mapping, iterative algorithm.

1 Introduction

Let C be a nonempty subset of a real Hilbert space X. Recall that a mapping $T : C \to C$ is said to be k-strictly pseudo-contractive if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k \|(I - T)x - (I - T)y\|^2$$

for all $x, y \in C$.

A point $x \in C$ is called a fixed point of T if $x = Tx$. We will denote the set of fixed points of T by $F(T)$. Note that the class of k-strictly pseudo-contractions includes the class of nonexpansive mappings which are mappings T on C such that

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

That is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive. The mapping T is also said to be pseudo-contractive if $k = 1$ and T is said to be strongly pseudo-contractive if there exists a constant $\lambda \in (0, 1)$ such that $T - \lambda I$ is pseudo-contractive. Clearly, the class of k-strictly pseudo-contractive mappings is the one between classes of nonexpansive mappings and pseudo-contractive mappings. Also we remark that the class of strongly pseudo-contractive mappings is independent from the class of k-strictly pseudo-contractive mappings (see, e.g., [1]-[3]). Recently, many authors have been devoted the studies on the problems of finding fixed points for k-strictly pseudo-contractive mappings (see, e.g., [4]-[10]).

We define the concept of k-strictly pseudo-contractive mapping in a $CAT(0)$ space as follows.

Let C be a nonempty subset of a $CAT(0)$ space X. A mapping $T : C \to C$ is said to be k-strictly pseudo-contractive if there exists a constant $k \in [0, 1)$ such that

$$d(Tx, Ty)^2 \leq d(x, y)^2 + k\left(d\left(x, Tx\right) + d(y, Ty)\right)^2 \tag{1}$$

for all $x, y \in C$.

Gürsoy, Karakaya and Rhoades [11] introduced a new multi-step iteration in a Banach space. Recently, Başarır and Şahin [12] modified this iteration in a $CAT(0)$ space as follows.

For an arbitrary fixed order $k \geq 2$,

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)y_n^1 \oplus \alpha_n T y_n^1, \\ y_n^1 = (1-\beta_n^1)y_n^2 \oplus \beta_n^1 T y_n^2, \\ y_n^2 = (1-\beta_n^2)y_n^3 \oplus \beta_n^2 T y_n^3, \\ \vdots \\ y_n^{k-2} = (1-\beta_n^{k-2})y_n^{k-1} \oplus \beta_n^{k-2} T y_n^{k-1}, \\ y_n^{k-1} = (1-\beta_n^{k-1})x_n \oplus \beta_n^{k-1} T x_n, \ \ \forall n \geq 0, \end{cases}$$

or, in short,

$$\begin{cases} x_0 \in C \\ x_{n+1} = (1-\alpha_n)y_n^1 \oplus \alpha_n T y_n^1, \\ y_n^i = (1-\beta_n^i)y_n^{i+1} \oplus \beta_n^i T y_n^{i+1}, \quad i = 1, 2, ..., k-2, \\ y_n^{k-1} = (1-\beta_n^{k-1})x_n \oplus \beta_n^{k-1} T x_n, \ \forall n \geq 0. \end{cases} \tag{2}$$

By taking $k = 3$ and $k = 2$ in (2), we obtain the SP-iteration of Phuengrattana and Suantai [13] and the two-step iteration of Thianwan [14], respectively.

Acedo and Xu [15] introduced a cyclic algorithm in a Hilbert space. We modify this algorithm in a $CAT(0)$ space.

Let $x_0 \in C$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. The cyclic algorithm generates a sequence $\{x_n\}$ in the following way:

$$\begin{cases} x_1 = \alpha_0 x_0 \oplus (1-\alpha_0) T_0 x_0, \\ x_2 = \alpha_1 x_1 \oplus (1-\alpha_1) T_1 x_1, \\ \vdots \\ x_N = \alpha_{N-1} x_{N-1} \oplus (1-\alpha_{N-1}) T_{N-1} x_{N-1}, \\ x_{N+1} = \alpha_N x_N \oplus (1-\alpha_N) T_0 x_N, \\ \vdots \end{cases}$$

or, shortly,

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) T_{[n]} x_n, \quad \forall n \geq 0, \tag{3}$$

where $T_{[n]} = T_i$, with $i = n(modN)$, $0 \leq i \leq N-1$. By taking $T_{[n]} = T$ for all n in (3), we obtain the Mann iteration in [16].

In this paper, motivated by the above results, we prove the demiclosedness principle for k-strictly pseudo-contractive mappings in a $CAT(0)$ space. Also we present the Δ-convergence theorems of the cyclic algorithm and the new multi-step iteration and the strong convergence theorem of the modified Halpern's iteration which is introduced for Hilbert space by Hu [17] for these mappings in a $CAT(0)$ space.

2 Preliminaries on $CAT(0)$ space

A metric space X is a $CAT(0)$ space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a $CAT(0)$ space. Other examples include Pre-Hilbert spaces (see [18]), Euclidean buildings (see [19]), R-trees (see [20]), the complex Hilbert ball with a hyperbolic metric (see [21]) and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [18].

Fixed point theory in a $CAT(0)$ space has been first studied by Kirk (see [22], [23]). He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space always has a fixed point. Since then the fixed point theory in a $CAT(0)$ space has been rapidly developed and many papers have appeared (see, e.g., [24]-[32]). It is worth mentioning that fixed point theorems in a $CAT(0)$ space (specially in R-trees) can be applied to graph theory, biology and computer science (see, e.g., [20], [33]-[36]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset$ R to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic and X is said to be a *uniquely geodesic* if there is exactly one geodesic joining x to y for each $x, y \in X$.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consist of three points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) = \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane R^2 such that

$$d_{\mathrm{R}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$$

for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [18]).

A geodesic metric space is said to be a $CAT(0)$ space [18] if all geodesic triangles of appropriate size satisfy the following comparison axiom:

Let Δ be a geodesic triangle in X and $\overline{\Delta}$ be a comparison triangle for Δ. Then, Δ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x, y) \leq d_{\mathrm{R}^2}(\overline{x}, \overline{y}).$$

If x, y_1, y_2 are points in a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies that

$$d\,(x, y_0)^2 \leq \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2.$$

This is the (CN) inequality of Bruhat and Tits [37]. In fact (see [18, p.163]), a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality. It is worth mentioning that the results in a $CAT(0)$ space can be applied to any $CAT(k)$ space with $k \leq 0$ since any $CAT(k)$ space is a $CAT(k')$ space for every $k' \geq k$ (see [18, p.165]).

Let $x, y \in X$ and by Lemma 2.1(iv) of [27] for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = t d(x, y), \quad d(y, z) = (1-t) d(x, y). \tag{4}$$

From now on, we will use the notation $(1-t)\,x \oplus ty$ for the unique point z satisfying (4). We now collect some elementary facts about $CAT(0)$ spaces which will be used in sequel the proofs of our main results.

Lemma 1 *Let X be a $CAT(0)$ space. Then*

(i) (see [27, Lemma 2.4]) for each $x, y, z \in X$ and $t \in [0,1]$, one has

$$d\left((1-t)x \oplus ty, z\right) \leq (1-t)d(x,z) + td(y,z),$$

(ii) (see [27, Lemma 2.5]) for each $x, y, z \in X$ and $t \in [0,1]$, one has

$$\begin{aligned} & d\left((1-t)x \oplus ty, z\right)^2 \\ \leq\ & (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2. \end{aligned}$$

3 Demiclosedness principle for k-strictly pseudo-contractive mappings

In 1976 Lim [38] introduced a concept of convergence in a general metric space setting which is called Δ-convergence. Later, Kirk and Panyanak [39] used the concept of Δ-convergence introduced by Lim [38] to prove on the $CAT(0)$ space analogs of some Banach space results which involve weak convergence. Also, Dhompongsa and Panyanak [27] obtained the Δ-convergence theorems for the Picard, Mann and Ishikawa iterations in a $CAT(0)$ space for nonexpansive mappings under some appropriate conditions.

We now give the definition and collect some basic properties of the Δ-convergence.

Let X be a $CAT(0)$ space and $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set

$$r\left(x, \{x_n\}\right) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r\left(\{x_n\}\right)$ of $\{x_n\}$ is given by

$$r\left(\{x_n\}\right) = \inf\left\{r\left(x, \{x_n\}\right) : x \in X\right\}.$$

The *asymptotic center* $A\left(\{x_n\}\right)$ of $\{x_n\}$ is the set

$$A\left(\{x_n\}\right) = \left\{x \in X : r\left(x, \{x_n\}\right) = r\left(\{x_n\}\right)\right\}.$$

It is known that in a complete $CAT(0)$ space, $A\left(\{x_n\}\right)$ consists of exactly one point (see [40, Proposition 7]).

Definition 2 *([38], [39]) A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and x is called the Δ-limit of $\{x_n\}$.*

Lemma 3 *(i) Every bounded sequence in a complete $CAT(0)$ space always has a Δ-convergent subsequence. (see [39, p.3690])*

(ii) Let C be a nonempty closed convex subset of a complete $CAT(0)$ space and let $\{x_n\}$ be a bounded sequence in C. Then the asymptotic center of $\{x_n\}$ is in C. (see [41, Proposition 2.1])

Lemma 4 *([27, Lemma 2.8]) If $\{x_n\}$ is a bounded sequence in a complete $CAT(0)$ space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ is convergent then $x = u$.*

Let C be a closed convex subset of a $CAT(0)$ space X and $\{x_n\}$ be a bounded sequence in C. We denote the notation

$$\{x_n\} \rightharpoonup w \Leftrightarrow \Phi(w) = \inf_{x\in C} \Phi(x) \tag{5}$$

where $\Phi(x) = \limsup_{n\to\infty} d(x_n, x)$.

Nanjaras and Panyanak [42] gave a connection between the "$\rightharpoonup$" convergence and Δ-convergence.

Proposition 5 *([42, Proposition 3.12]) Let C be a closed convex subset of a $CAT(0)$ space X and $\{x_n\}$ be a bounded sequence in C. Then $\Delta\text{-}\lim_{n\to\infty} x_n = p$ implies that $\{x_n\} \rightharpoonup p$.*

The purpose of this section is to prove demiclosedness principle for k -strictly pseudo-contractive mappings in a $CAT(0)$ space by using the convergence defined in (5).

Theorem 6 *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be a k-strictly pseudo-contractive mapping such that $k \in \left[0, \frac{1}{2}\right)$ and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a bounded sequence in C such that $\Delta\text{-}\lim_{n\to\infty} x_n = w$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $Tw = w$.*

Proof: By the hypothesis, $\Delta\text{-}\lim_{n\to\infty} x_n = w$. From Proposition 5, we get $\{x_n\} \rightharpoonup w$. Then we obtain $A(\{x_n\}) = \{w\}$ by Lemma 3 (ii) (see [42]). Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then we get

$$\Phi(x) = \limsup_{n\to\infty} d(x_n, x) = \limsup_{n\to\infty} d(Tx_n, x) \tag{6}$$

for all $x \in C$. In (6) by taking $x = Tw$, we have

$$\begin{aligned} \Phi(Tw)^2 &= \limsup_{n\to\infty} d(Tx_n, Tw)^2 \\ &\leq \limsup_{n\to\infty} \{d(x_n, w)^2 \\ &\quad + k(d(x_n, Tx_n) + d(w, Tw))^2\} \\ &\leq \limsup_{n\to\infty} d(x_n, w)^2 \\ &\quad + k \limsup_{n\to\infty} \left(d(x_n, Tx_n) + d(w, Tw)\right)^2 \\ &= \Phi(w)^2 + kd(w, Tw)^2 \end{aligned} \tag{7}$$

The (CN) inequality implies that

$$\begin{aligned} & d\left(x_n, \frac{w \oplus Tw}{2}\right)^2 \\ \leq \ & \frac{1}{2}d(x_n, w)^2 + \frac{1}{2}d(x_n, Tw)^2 - \frac{1}{4}d(w, Tw)^2. \end{aligned}$$

Letting $n \to \infty$ and taking superior limit on the both sides of the above inequality, we get

$$\Phi\left(\frac{w \oplus Tw}{2}\right)^2 \leq \frac{1}{2}\Phi(w)^2 + \frac{1}{2}\Phi(Tw)^2 - \frac{1}{4}d(w, Tw)^2.$$

Since $A(\{x_n\}) = \{w\}$, we have

$$\begin{aligned} \Phi(w)^2 & \leq \Phi\left(\frac{w \oplus Tw}{2}\right)^2 \\ & \leq \frac{1}{2}\Phi(w)^2 + \frac{1}{2}\Phi(Tw)^2 - \frac{1}{4}d(w, Tw)^2. \end{aligned}$$

which implies that

$$d(w, Tw)^2 \leq 2\Phi(Tw)^2 - 2\Phi(w)^2. \tag{8}$$

By (7) and (8), we get $(1 - 2k)d(w, Tw)^2 \leq 0$. Since $k \in \left[0, \frac{1}{2}\right)$, then we have $Tw = w$ as desired. □

Now, we prove the Δ-convergence of the new multi-step iteration for k-strictly pseudo-contractive mappings in a $CAT(0)$ space.

Theorem 7 *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be a k-strictly pseudo-contractive mapping such that $k \in \left[0, \frac{1}{2}\right)$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n^i\}$, $i = 1, 2, ..., k - 2$ be sequences in $[a, b]$ for some $a, b \in (0, 1)$ and $k < 1 - b$. Let $\{x_n\}$ be a sequence defined by (2). Then the sequence $\{x_n\}$ is Δ convergent to a fixed point of T.*

Proof: Let $p \in F(T)$. From (1), (2) and Lemma 1, we have

$$\begin{aligned} & d(x_{n+1}, p)^2 = d((1 - \alpha_n)y_n^1 \oplus \alpha_n Ty_n^1, p)^2 \\ \leq \ & (1 - \alpha_n)d(y_n^1, p)^2 + \alpha_n d(Ty_n^1, p)^2 \\ & -\alpha_n(1 - \alpha_n)d(y_n^1, Ty_n^1)^2 \\ \leq \ & (1 - \alpha_n)d(y_n^1, p)^2 \\ & +\alpha_n \left\{d(y_n^1, p)^2 + kd(y_n^1, Ty_n^1)^2\right\} \\ & -\alpha_n(1 - \alpha_n)d(y_n^1, Ty_n^1)^2 \\ = \ & d(y_n^1, p)^2 - \alpha_n((1 - \alpha_n) - k)d(y_n^1, Ty_n^1)^2 \\ \leq \ & d(y_n^1, p)^2. \end{aligned}$$

Also, we obtain

$$\begin{aligned} & d(y_n^1, p)^2 = d((1 - \beta_n^1)y_n^2 \oplus \beta_n^1 Ty_n^2, p)^2 \\ \leq \ & (1 - \beta_n^1)d(y_n^2, p)^2 + \beta_n^1 d(Ty_n^2, p)^2 \\ \leq \ & (1 - \beta_n^1)d(y_n^2, p)^2 \\ & +\beta_n^1 \left\{d(y_n^2, p)^2 + kd(y_n^2, Ty_n^2)^2\right\} \\ & -\beta_n^1(1 - \beta_n^1)d(y_n^2, Ty_n^2)^2 \\ = \ & d(y_n^2, p)^2 - \beta_n^1((1 - \beta_n^1) - k)d(y_n^2, Ty_n^2)^2 \\ \leq \ & d(y_n^2, p)^2. \end{aligned}$$

Continuing the above process we have

$$d(x_{n+1}, p) \leq d(y_n^2, p) \leq ... \leq d(y_n^{k-1}, p) \leq d(x_n, p). \tag{9}$$

This inequality guarantees that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. Let $\lim_{n\to\infty} d(x_n, p) = r$. By using (9), we get

$$\lim_{n\to\infty} d(y_n^{k-1}, p) = r.$$

By Lemma 1, we also have

$$\begin{aligned} & d(y_n^{k-1}, p)^2 = d((1 - \beta_n^{k-1})x_n \oplus \beta_n^{k-1}Tx_n, p)^2 \\ \leq \ & (1 - \beta_n^{k-1})d(x_n, p)^2 + \beta_n^{k-1}d(Tx_n, p)^2 \\ & -\beta_n^{k-1}(1 - \beta_n^{k-1})d(x_n, Tx_n)^2 \\ \leq \ & (1 - \beta_n^{k-1})d(x_n, p)^2 \\ & +\beta_n^{k-1}\left\{d(x_n, p)^2 + kd(x_n, Tx_n)^2\right\} \\ & -\beta_n^{k-1}(1 - \beta_n^{k-1})d(x_n, Tx_n)^2 \\ = \ & d(x_n, p)^2 - \beta_n^{k-1}((1 - \beta_n^{k-1}) - k)d(x_n, Tx_n)^2, \end{aligned}$$

which implies that

$$\begin{aligned} & d(x_n, Tx_n)^2 \\ \leq \ & \frac{1}{a((1 - b) - k)}\left[d(x_n, p)^2 - d(y_n^{k-1}, p)^2\right]. \end{aligned}$$

Thus $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. To show that the sequence $\{x_n\}$ is Δ-convergent to a fixed point of T, we prove that

$$W_\Delta(x_n) = \bigcup_{\{u_n\}\subseteq\{x_n\}} A(\{u_n\}) \subseteq F(T)$$

and $W_\Delta(x_n)$ consists of exactly one point. Let $u \in W_\Delta(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 3, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_{n\to\infty} v_n = v \in K$. By Theorem 6, we have $v \in F(T)$ and by Lemma 4, we have $u = v \in F(T)$. This shows that $W_\Delta(x_n) \subseteq F(T)$. Now, we prove that $W_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$

be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F(T)$. Finally, since $\{d(x_n, v)\}$ is convergent, we have $x = v \in F(T)$ by Lemma 4. This shows $W_\Delta(x_n) = \{x\}$. This completes the proof. □

Also, we prove the Δ-convergence of the cyclic algorithm for k-strictly pseudo-contractive mappings in a $CAT(0)$ space.

Theorem 8 *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $N \geq 1$ be an integer. Let, for each $0 \leq i \leq N-1$, $T_i : C \to C$ be k_i-strictly pseudo-contractive mappings for some $0 \leq k_i < \frac{1}{2}$. Let $k = \max\{k_i; 0 \leq i \leq N-1\}$, $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$ and $k < a$. Let $F = \bigcap_{i=0}^{N-1} F(T_i) \neq \emptyset$. For $x_0 \in C$, let $\{x_n\}$ be a sequence defined by (3). Then the sequence $\{x_n\}$ is Δ-convergent to a common fixed point of the family $\{T_i\}_{i=0}^{N-1}$.*

Proof: Let $p \in F$. Using (1), (3) and Lemma 1, we have

$$\begin{aligned} & d(x_{n+1}, p)^2 \\ = \; & d(\alpha_n x_n \oplus (1-\alpha_n) T_{[n]} x_n, p)^2 \\ \leq \; & \alpha_n d(x_n, p)^2 + (1-\alpha_n) d(T_{[n]} x_n, p)^2 \\ & -\alpha_n (1-\alpha_n) d(x_n, T_{[n]} x_n)^2 \\ \leq \; & \alpha_n d(x_n, p)^2 \\ & +(1-\alpha_n) \left\{ d(x_n, p)^2 + k d(x_n, T_{[n]} x_n)^2 \right\} \\ & -\alpha_n (1-\alpha_n) d(x_n, T_{[n]} x_n)^2 \\ = \; & d(x_n, p)^2 \\ & -(1-\alpha_n)(\alpha_n - k) d(x_n, T_{[n]} x_n)^2 \qquad (10) \\ \leq \; & d(x_n, p)^2. \end{aligned}$$

This inequality guarantees that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$. By (10), we also have

$$\begin{aligned} & d(x_n, T_{[n]} x_n)^2 \\ \leq \; & \frac{1}{(1-\alpha_n)(\alpha_n - k)} \left[d(x_n, p)^2 - d(x_{n+1}, p)^2 \right] \\ \leq \; & \frac{1}{(1-b)(a-k)} \left[d(x_n, p)^2 - d(x_{n+1}, p)^2 \right]. \end{aligned}$$

Since $\lim_{n\to\infty} d(x_n, p)$ exists, we obtain $\lim_{n\to\infty} d(x_n, T_{[n]} x_n) = 0$. The rest of the proof closely follows the proof of Theorem 7 and is therefore omitted. □

4 The strong convergence theorem for the modified Halpern's iteration

In [17], Hu introduced a modified Halpern's iteration. We modify this iteration in a $CAT(0)$ space as follows.

For an arbitrary initial value $x_0 \in C$ and a fixed anchor $u \in C$, the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = \alpha_n u \oplus (1-\alpha_n) y_n, \\ y_n = \frac{\beta_n}{1-\alpha_n} x_n \oplus \frac{\gamma_n}{1-\alpha_n} T x_n, \;\; \forall n \geq 0, \end{cases} \qquad (11)$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three real sequences in $(0, 1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$.

Clearly, the iterative sequence (11) is a natural generalization of the well known iterations.

(i) If we take $\beta_n = 0$ for all n in (11), then the sequence (11) reduces to the Halpern's iteration in [43].

(ii) If we take $\alpha_n = 0$ for all n in (11), then the sequence (11) reduces to the Mann iteration in [16].

In this section, we prove the strong convergence of the modified Halpern's iteration in a $CAT(0)$ space.

Recall that a continuous linear functional μ on ℓ_∞, the Banach space of bounded real sequences, is called a Banach limit if $\|\mu\| = \mu(1, 1, ...) = 1$ and $\mu(a_n) = \mu(a_{n+1})$ for all $\{a_n\}_{n=1}^{\infty} \subset \ell_\infty$.

Lemma 9 *(see [44, Proposition 2]) Let $\{a_n\} \in \ell_\infty$ be such that $\mu(a_n) \leq 0$ for all Banach limits μ and $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$. Then, $\limsup_{n\to\infty} a_n \leq 0$.*

Lemma 10 *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X, $T : C \to C$ be a k-strictly pseudo-contractive mapping with $k \in [0, 1)$ and $S : C \to C$ be a mapping defined by $Sz = kz \oplus (1-k) Tz$, for $z \in C$. Let $u \in C$ be fixed. For each $t \in [0, 1]$, the mapping $S_t : C \to C$ defined by*

$$S_t z = tu \oplus (1-t) Sz = tu \oplus (1-t)(kz \oplus (1-k) Tz)$$

for $z \in C$, has a unique fixed point $z_t \in C$, that is,

$$z_t = S_t(z_t) = tu \oplus (1-t) S(z_t). \qquad (12)$$

Proof: As it has been proven in [45], if T is a k-strictly pseudo-contractive mapping with $k \in [0, 1)$, S is a nonexpansive mapping such that $F(S) = F(T)$. Then, from Lemma 2.1 in [29], the mapping S_t has a unique fixed point $z_t \in C$. □

Lemma 11 *Let X, C, T and S be as in Lemma 10. Then, $F(T) \neq \emptyset$ if and only if $\{z_t\}$ given by (12) remains bounded as $t \to 0$. In this case, the following statements hold:*

(1) $\{z_t\}$ converges to the unique fixed point z of T which is nearest to u;

(2) $d(u,z)^2 \leq \mu d(u,x_n)^2$ for all Banach limits μ and all bounded sequences $\{x_n\}$ with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof: If $F(T) \neq \emptyset$, then we have $F(S) = F(T) \neq \emptyset$. Also, if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we obtain that

$$\begin{aligned} d(x_n, Sx_n) &= d(x_n, kx_n \oplus (1-k)\, Tx_n) \\ &\leq (1-k) d(x_n, Tx_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, from Lemma 2.2 in [29], the rest of the proof of this lemma can be seen. □

The following lemma can be found in [46].

Lemma 12 *(see [46, Lemma 2.1]) Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the condition*

$$a_{n+1} \leq (1-\gamma_n) a_n + \gamma_n \sigma_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}$ and $\{\sigma_n\}$ are sequences of real numbers such that

(1) $\{\gamma_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(2) either $\limsup_{n\to\infty} \sigma_n \leq 0$ or $\sum_{n=1}^{\infty} |\gamma_n \sigma_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

We are now ready to prove our main result.

Theorem 13 *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be a k-strictly pseudo-contractive mapping such that $0 \leq k < \frac{\beta_n}{1-\alpha_n} < 1$ and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (11). Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C3) $\lim_{n\to\infty} \beta_n \neq k$ and $\lim_{n\to\infty} \gamma_n \neq 0$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof: We divide the proof into three steps. In the first step we show that $\{x_n\}, \{y_n\}$ and $\{Tx_n\}$ are bounded sequences. In the second step we show that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Finally, we show that $\{x_n\}$ converges to a fixed point $z \in F(T)$ which is nearest to u.

First step: Take any $p \in F(T)$, then, from Lemma 1 and (11), we have

$$\begin{aligned} & d(y_n,p)^2 \\ \leq\ & \frac{\beta_n}{1-\alpha_n} d(x_n,p)^2 + \frac{\gamma_n}{1-\alpha_n} d(Tx_n,p)^2 \\ & -\frac{\beta_n\gamma_n}{(1-\alpha_n)^2} d(x_n,Tx_n)^2 \\ \leq\ & \frac{\beta_n}{1-\alpha_n} d(x_n,p)^2 \\ & +\frac{\gamma_n}{1-\alpha_n}\left(d(x_n,p)^2 + kd(x_n,Tx_n)^2\right) \\ & -\frac{\beta_n\gamma_n}{(1-\alpha_n)^2} d(x_n,Tx_n)^2 \\ =\ & d(x_n,p)^2 - \frac{\gamma_n}{1-\alpha_n}\left(\frac{\beta_n}{1-\alpha_n} - k\right) d(x_n,Tx_n)^2 \\ \leq\ & d(x_n,p)^2. \end{aligned}$$

Also, we obtain

$$\begin{aligned} & d(x_{n+1},p)^2 \\ \leq\ & \alpha_n d(u,p)^2 + (1-\alpha_n)\, d(y_n,p)^2 \\ & -\alpha_n (1-\alpha_n)\, d(u,y_n)^2 \\ \leq\ & \alpha_n d(u,p)^2 + (1-\alpha_n)\Big\{d(x_n,p)^2 \\ & -\frac{\gamma_n}{1-\alpha_n}\left(\frac{\beta_n}{1-\alpha_n} - k\right) d(x_n,Tx_n)^2\Big\} \\ & -\alpha_n (1-\alpha_n)\, d(u,y_n)^2 \\ =\ & \alpha_n d(u,p)^2 + (1-\alpha_n)\, d(x_n,p)^2 \\ & -\gamma_n\left(\frac{\beta_n}{1-\alpha_n} - k\right) d(x_n,Tx_n)^2 \\ & -\alpha_n (1-\alpha_n)\, d(u,y_n)^2 \qquad (13) \\ \leq\ & \alpha_n d(u,p)^2 + (1-\alpha_n)\, d(x_n,p)^2 \\ \leq\ & \max\left\{d(u,p)^2, d(x_n,p)^2\right\}. \end{aligned}$$

By induction,

$$d(x_{n+1},p)^2 \leq \max\left\{d(u,p)^2, d(x_0,p)^2\right\}.$$

This proves the boundedness of the sequence $\{x_n\}$, which leads to the boundedness of $\{Tx_n\}$ and $\{y_n\}$.

Second step: In fact, we have from (13) (for some appropriate constant $M > 0$) that

$$\begin{aligned} & d(x_{n+1},p)^2 \\ \leq\ & \alpha_n d(u,p)^2 + (1-\alpha_n)\, d(x_n,p)^2 \\ & -\gamma_n\left(\frac{\beta_n}{1-\alpha_n} - k\right) d(x_n,Tx_n)^2 \\ =\ & \alpha_n (d(u,p)^2 - d(x_n,p)^2) + d(x_n,p)^2 \\ & -\gamma_n\left(\frac{\beta_n}{1-\alpha_n} - k\right) d(x_n,Tx_n)^2 \end{aligned}$$

$$\begin{aligned} &\le \alpha_n M + d(x_n,p)^2 \\ &\quad -\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right) d(x_n,Tx_n)^2, \end{aligned}$$

which implies that

$$\begin{aligned} &\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right) d(x_n,Tx_n)^2 - \alpha_n M \\ \le\ & d(x_n,p)^2 - d(x_{n+1},p)^2. \end{aligned} \tag{14}$$

If $\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right) d(x_n,Tx_n)^2 - \alpha_n M \le 0$, then

$$d(x_n,Tx_n)^2 \le \frac{\alpha_n}{\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)} M,$$

and hence the desired result is obtained by the conditions (C1) and (C3).

If $\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right) d(x_n,Tx_n)^2 - \alpha_n M > 0$, then following (14), we have

$$\begin{aligned} &\sum_{n=0}^{m}\left[\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right) d(x_n,Tx_n)^2 - \alpha_n M\right] \\ \le\ & d(x_0,p)^2 - d(x_{m+1},p)^2 \\ \le\ & d(x_0,p)^2. \end{aligned}$$

That is

$$\sum_{n=0}^{\infty}\left[\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right) d(x_n,Tx_n)^2 - \alpha_n M\right] < \infty.$$

Thus

$$\lim_{n\to\infty}\left[\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right) d(x_n,Tx_n)^2 - \alpha_n M\right] = 0.$$

Then we get

$$\lim_{n\to\infty} d(x_n,Tx_n) = 0. \tag{15}$$

Third step: Using the condition (C1) and (15), we obtain

$$\begin{aligned} & d(x_{n+1},x_n) \\ \le\ & d(x_{n+1},Tx_n) + d(Tx_n,x_n) \\ \le\ & \alpha_n d(u,Tx_n) + (1-\alpha_n) d(y_n,Tx_n) \\ & +d(Tx_n,x_n) \\ \le\ & \alpha_n d(u,Tx_n) + (1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n} d(x_n,Tx_n)\right) \\ & +d(Tx_n,x_n) \\ =\ & \alpha_n d(u,Tx_n) + (\beta_n+1)\, d(x_n,Tx_n) \\ \to\ & 0, \quad \text{as } n\to\infty. \end{aligned}$$

Also, from (15), we have

$$\begin{aligned} d(x_n,y_n) &\le \frac{\gamma_n}{1-\alpha_n} d(x_n,Tx_n) \\ &\to 0, \quad \text{as } n\to\infty. \end{aligned} \tag{16}$$

Let $z = \lim_{t\to 0} z_t$, where z_t is given by (12) in Lemma 10. Then, z is the point of $F(T)$ which is nearest to u. By Lemma 11 (2), we have $\mu\left(d(u,z)^2 - d(u,x_n)^2\right) \le 0$ for all Banach limits μ. Let $a_n = d(u,z)^2 - d(u,x_n)^2$. Moreover, since $\lim_{n\to\infty} d(x_{n+1},x_n) = 0$, we get

$$\limsup_{n\to\infty}\left(a_{n+1}-a_n\right) = 0.$$

By Lemma 9, we obtain

$$\limsup_{n\to\infty}\left(d(u,z)^2 - d(u,x_n)^2\right) \le 0. \tag{17}$$

It follows from the condition (C1) and (16) that

$$\begin{aligned} & \limsup_{n\to\infty}\left(d(u,z)^2 - (1-\alpha_n)\, d(u,y_n)^2\right) \\ =\ & \limsup_{n\to\infty}\left(d(u,z)^2 - d(u,x_n)^2\right). \end{aligned} \tag{18}$$

By (17) and (18), we have

$$\limsup_{n\to\infty}\left(d(u,z)^2 - (1-\alpha_n)\, d(u,y_n)^2\right) \le 0. \tag{19}$$

We observe that

$$\begin{aligned} & d(x_{n+1},z)^2 \\ \le\ & \alpha_n d(u,z)^2 + (1-\alpha_n)\, d(y_n,z)^2 \\ & -\alpha_n(1-\alpha_n)\, d(u,y_n)^2 \\ \le\ & \alpha_n d(u,z)^2 + (1-\alpha_n)\, d(x_n,z)^2 \\ & -\alpha_n(1-\alpha_n)\, d(u,y_n)^2 \\ =\ & (1-\alpha_n)\, d(x_n,z)^2 \\ & +\alpha_n\left[d(u,z)^2 - (1-\alpha_n)\, d(u,y_n)^2\right]. \end{aligned}$$

It follows from the condition (C2) and (19), using Lemma 12, that $\lim_{n\to\infty} d(x_n,z) = 0$. This completes the proof of Theorem 13. □

We obtain the following corollary as a direct consequence of Theorem 13.

Corollary 14 *Let X, C and T be as Theorem 13. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ satisfying the conditions (C1) and (C2). For a constant $\delta \in (k,1)$, an arbitrary initial value $x_0 \in C$ and a fixed anchor $u \in C$, let the sequence $\{x_n\}$ be defined by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)\left(\delta x_n \oplus (1-\delta) T x_n\right), \tag{20}$$

for all $n \ge 0$. Then the sequence $\{x_n\}$ is strongly convergent to a fixed point of T

Proof: If, in the proof of Theorem 13, we take $\beta_n = (1-\alpha_n)\delta$ and $\gamma_n = (1-\alpha_n)(1-\delta)$, then we get the desired conclusion. □

Remark 15 *The results in this section contain the strong convergence theorems of the iterative sequences (11) and (20) for nonexpansive mappings in a $CAT(0)$ space. Also, our results contain the corresponding theorems proved for these iterative sequences in a Hilbert space.*

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