The Periodic Solution of Hypersonic Functionally Graded Plate Subjected to Aero-thermal Load

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Abstract: In this paper, the periodic solution of hypersonic functionally graded plate subjected to aero-thermal load with two degrees of freedom is investigated. Firstly, the average equations in 1:2 internal resonances are obtained by using multi-scale method. Based on the Poincare mapping and the Melnikov function, we get the sufficient condition for the existence of periodic solution about the system. Then we obtain the phase diagrams of the solution which were under different parameters by using Matlab. The simulation results demonstrate the sufficient condition which we have got is correct. Finally, by using the integral manifold theory and average method, we analyze the stability of the periodic orbit.

Key–Words: Poincare mapping, Periodic Solution, Stability, Numerical simulations

1 Introduction

The concept of Functional material was proposed by Dr. Morton J A in 1965. The development of contemporary aerospace and other high-technologies demand higher performance for the materials. Traditional materials can not satisfy the increasingly stringent environmental. So in 1987, Japanese scholars put forward a kind of advanced composite material, which is functionally graded material (FGM). Now this kind of material is widely used in many fields, such as aerospace engineering, nuclear engineering and so on. The nonlinear properties of FGMs can lead to a different behavior of the system from that predicted by a linear model. So it is worthwhile studying its nonlinear dynamical behavior.

It is a hot issue for the research on the nonlinear dynamic systems. In 2006, Zhou and Xu [1] analyzed a heteronomy strong nonlinear dynamics system by using the good properties of Chebyshev polynomials. The method used in this paper could be convenience for the analysis of systems with periodical varying coefficients or high dimensional problems. Finally, they compared the analytical results of Duffing equation with those obtained through using a Runge-Kutta integration algorithm and the standard Harmonic Balance Method. In 2007, Li et al [2] presented the bifurcations of the multiple limit cycles for a parametrically and externally excited mechanical system. The original mechanical system was first transformed to the average equation in the Cartesian form, which was in the form of a Z_2-symmetric perturbed polynomial Hamiltonian system of degree 5. Then, by using the bifurcation theory of planar dynamical system and the method of detection function, the bifurcations of the multiple limit cycles of the system were investigated and the configurations of compound eyes were also obtained. In 2008, Li et al [3] investigated the bifurcations of multiple limit cycles for a rotor-active magnetic bearings (AMB) system with the time-varying stiffness. The governing nonlinear equation of motion was established for the rotor-AMB system with single-degree-of-freedom and parametric excitation. By using the method of multiple scales, the governing nonlinear equation of motion was first transformed to the average equation. Then, the bifurcation theory of planar dynamical system and the method of detection function were utilized to analyze the bifurcations of multiple limit cycles of the average equation. Four groups of parametric controlling conditions are given to obtain the configurations of compound eyes. Found that there exist respectively at least 17, 19, 21 and 22 limit cycles in the rotor-AMB system with the time-varying stiffness under the different controlling conditions. In 2009, Liu and Han [4] considered a four-dimensional system of ordinary differential equation depending on a small parameter. By using the Poincare mapping and the integral manifold theory, the invariant tori of the four-dimensional system is obtained. And with the results they discussed a nonlinearly coupled Van der Pol-Duffing oscillator system. In 2010, Malekzadeh [5] discussed the free vibration of the functionally graded round face beams subjected to thermal load. The first order shear deformation theory and the Hamilton principle were used in this article. And the effects of the boundary condition and geometrical parameters on the natural frequency were discussed. In 2011, Wang and Zhang [6] studied a neutral delay model, by using the abstract continuous theorem of k-set contractive operator, some new results on the existence of the positive periodic solution are obtained, and they established the global asymptotically stability of the positive periodic solution. In 2011, Zhang et al [7] investigated the existence and the uniqueness of periodic solution by means of contraction mapping principle. Then they discussed the global exponential stability of periodic solution for the system. In 2012, Guo et al [8] predicted the accurate bifurcating periodic solution for a general class of first-order nonlinear delay differential equation by constructing an approximate technique. The zeroth-order solution using just one Fourier term is applied by solving a set of nonlinear algebraic equations containing the delay term. The unbalanced residues due to Fourier truncation were considered iteratively by solving linear equations to improve the accuracy and increase the number of Fourier terms of the solutions successively. It is shown that the solutions are valid for a wide range of variation of the parameters by two examples. In 2012, Zhang [9] investigated the existence and stability of the positive equilibrium. In particular, they found that the system has Hopf bifurcation at the positive equilibrium; they analyzed the stability of the periodic solutions by reducing the original system on the center manifold. Finally, some numerical examples were given to verify their theoretical results. In 2012, Li et al [10] considered a type of coupled map lattice (CML). Due to the infinite property of the problem, they tried the periodic case, which can be dealt with on a finite set. The main approach for this study is the implicit existence theorem. The results indicate that if the parameters of the system satisfy some exact conditions, then there exists a periodic traveling wave solution in an exact neighborhood of a given one. In 2013, Li and Yin[12] shown the sufficient condition for the existence of the FGM load by using the multi-scale approach method. By using the Melinkov function, the nonsingular linear transformation and the Poincar map. In 2013, Wei[13] considered the existence and global exponential stability of periodic solutions for inertial Cohen-Grossberg-type BAM neural networks with time delays. With variable transformation the system is transformed to first order differential equations. Finally, an example is given to demonstrate the obtained results. In 2008 the complex formulas of computing the Lyapunov values were investigated for two planar polynomial systems by Li at el [14]. The relations between the two systems and their standard forms were discussed and the corresponding complex formulas of computing the Lyapunov values were obtained.In 2012, Yao and Han [15] considered the bifurcation of limit cycles of a class of polynomial differential systems; they got the upper bound and the lower bound of the limit cycles. Li and Wei[16] investigated the behavior of iced cable with two degrees of freedom.By using the Melnikov function and the Poincare mapping, they get the sufficient condition for the existence of the periodic solution about the system.The invariant tori of the system is investigated by using transformations and average equation. Wu and Ding [17] got the average equations of the aerodynamic and thermal loads by using the multi-scale method. They studied the dynamical behavior of the system in this paper. The solutions of the system were divided into six types. They discussed the existence of the various solutions. By establishing a dynamic model of the FGM subjected to aero-dynamic load, they obtained the nonlinear ordinary differential equations which with multiple degrees of freedom.

Periodic system and periodic solution exist in many natural and man-made systems.Due to the rise of the supersonic vehicle in recent years, it is a hot issue for the research on hypersonic materials. In this paper, the periodic behavior of hypersonic functionally graded plate subjected to aero-thermal load is investigated. Firstly, by using the Melnikov function, the Maple symbolic computation software and the periodic transformation, we get the sufficient condition for the existence of periodic orbit of the system. Then, three groups of parametric controlling conditions were given to obtain the phase diagrams. We get the phase diagrams when $\varepsilon \neq 0$, in this condition, the first planar system is a Hamiltonian system. When $n_{11} \neq 3n_{13}$, $\delta_2 \neq 0$, then $v = 0$ is a 1-order weak focus for the other planar system. We also get the phase diagrams when $M(h_0) \neq 0$ and $c_{0001} \neq k$, $M'(h_0) \neq 0$. Numerical simulations are performed to verify the analytical predictions. Finally, by using the average method, blow-up transformation and discussing the eigenvalues of the matrix in two different conditions, we get the stability of the periodic orbit.

2 The Existence of the Periodic orbit

A functional gradient material is a new kind of composite material. This new material usually consists of high temperature ceramic material and the composite metal material. That leads to the new material has the good characteristics such as high temperature resistance and strong toughness. Here we give the original system of hypersonic functionally graded plate subjected to aero-thermal load is:

$$\begin{aligned}&\ddot{x}_1 + m_0x_1 + m_1\dot{x}_1 + m_2x_1^2 + m_3x_2^2 + m_4x_1x_2\\&\quad + m_5x_1x_2^2 + m_6x_1^3 + m_7\dot{x}_1^2 + m_8\dot{x}_2^2\\&\quad + m_9\dot{x}_1\dot{x}_2 + m_{10}\dot{x}_1x_2^2 + m_{11}\dot{x}_1x_1^2\\&\quad + m_{12}\dot{x}_1\dot{x}_2^2 + m_{13}\dot{x}_1^3 = 0,\end{aligned}$$

$$\begin{aligned}&\ddot{x}_2 + n_0x_2 + n_1\dot{x}_2 + n_2x_2^2 + n_3x_1^2 + n_4x_1x_2\\&\quad + n_5x_2x_1^2 + n_6x_2^3 + n_7\dot{x}_2^2 + n_8\dot{x}_1^2\\&\quad + n_9\dot{x}_1\dot{x}_2 + n_{10}\dot{x}_2x_1^2 + n_{11}\dot{x}_2x_2^2\\&\quad + n_{12}\dot{x}_2\dot{x}_1^2 + n_{13}\dot{x}_2^3 = 0. \qquad (1)\end{aligned}$$

By using the multi-scale method, we present the average equations of FGM in 1:2 internal resonances. Then applying the linear transformation

$$u_1 = \frac{\delta_1x_1 + m_1x_2}{M}, \quad u_2 = x_2,$$
$$v_1 = \frac{\delta_2x_3 + n_1x_4}{N}, \quad v_2 = x_4$$

to the average equations, we have the following forms:

$$\begin{aligned}\frac{du_1}{d\tau} &= -u_2 + M_{1a}(u_1, u_1, v_1, v_2),\\ \frac{du_2}{d\tau} &= u_1 + b_{0100}u_2 + M_{1b}(u_1, u_1, v_1, v_2),\\ \frac{dv_1}{d\tau} &= c_{0001}v_2 + M_{1c}(u_1, u_1, v_1, v_2),\\ \frac{dv_2}{d\tau} &= d_{0010}v_1 + d_{0001}v_2 + M_{1d}(u_1, u_1, v_1, v_2).\end{aligned} \qquad (2)$$

where M_{1i} are higher polynomials about $u_1, u_2, v_1, v_2,\ i = a, b, c, d$

The perturbed system of (2) can be written as

$$\begin{aligned}\frac{du}{d\tau} &= f(u) + \varepsilon P(u, v),\\ \frac{dv}{d\tau} &= g(v) + \varepsilon Q(u, v)\end{aligned} \qquad (3)$$

where

$$\begin{aligned}u &= (u_1, u_2)^T,\\ v &= (v_1, v_2)^T,\\ f(u) &= (-u_2, u_1)^T,\\ g(v) &= (c_{0001}v_2 + M_{3c}(v), d_{0010}v_1 + M_{3d}(v))^T,\\ P(u, v) &= (M_{1a}(u, v), b_{0100}u_2 + M_{1b}(u, v))^T,\\ Q(u, v) &= (M_{1c}(u, v) - M_{3c}(v), d_{0001}v_2\\ &\quad + M_{1d}(u, v) - M_{3d}(v))^T.\end{aligned}$$

$M_{3c}(v)$ and $M_{3d}(v)$ are three homogeneous polynomial about v.

The unperturbed system of the system (3) holds that the planar autonomous system

$$\frac{du}{d\tau} = f(u) \qquad (4)$$

is a Hamiltonian system and there exists an open $J \subset R$, the system(4) has a family of periodic orbits.

When $n_{11} \neq 3n_{13},\ \delta_2 \neq 0$ then $v = 0$ is a 1-order weak focus of planar autonomous system (see [12,14])

$$\frac{dv}{d\tau} = g(v). \qquad (5)$$

Lemma 1 *When $0 \ll \theta \ll 2\pi$, the system (3) can be transformed into the system of (6) with the transformation (see [12]) $u = G(\theta, h) = (\sqrt{2h}\cos\theta, \sqrt{2h}\sin\theta)^T$, $v = v$,*

$$\begin{aligned}\frac{dh}{d\theta} &= -\varepsilon\frac{f(G(\theta,h)) \wedge P(G(\theta,h), v)}{1 + \varepsilon G_h(\theta,h) \wedge P(G(\theta,h), v)},\\ \frac{dv}{d\theta} &= \frac{g(G(\theta,h), v) + \varepsilon Q(G(\theta,h), v)}{1 + \varepsilon G_h(\theta,h) \wedge P(G(\theta,h), v)},\end{aligned} \qquad (6)$$

through calculate the system (6), then we have the Melnikov function $h_1(2\pi, r)$, we denote it by $M(r)$, that is :

$$M(r) = \int_0^{2\pi} (f(G(\theta, r)) \wedge P(G(\theta, r), 0))d\theta.$$

Through using the bifurcation equations and Melnikov function, the following theorem gives the sufficient condition for the existence of the periodic orbit.

Theorem 2 *(Sufficient Condition for the Existence of Periodic Orbit)For $0 < \varepsilon << 1$*

(i) if $M(r) \neq 0$, for any $r \in J$ the system of (3) does not have any periodic orbit with period near 2π;

(ii) If there exist a $h_0 \in J$, such that $c_{0001} \neq k$, $M(h_0) = 0$ and $M'(h_0) \neq 0$, then in the neighbor of L_{h_0} , the system of (3) has a unique periodic orbit with periodic near 2π.

Proof: (i) Considering the theory of successor function, it is easy to see that,if $M(r) \neq 0$, there does not exist any periodic orbit with period near 2π.

(ii) As $M(h_0) = 0$, we know that in the neighbor of L_{h_0} the system (3) has a periodic orbit by using the theory of successor function. Hence, by Lemma 2 in [12], we have:

$$\begin{aligned}&h_1(2\pi, r) + h_2(2\pi, r)v_0 + \varepsilon h_3(2\pi, r)\\&\qquad + O(v_0, \varepsilon) = 0,\end{aligned} \qquad (7)$$

$$\begin{aligned}&(v_1(2\pi, r) - I_2)v_0 + \varepsilon v_2(2\pi, r)\\&\qquad + O(v_0, \varepsilon) = 0.\end{aligned} \qquad (8)$$

When $M(h_0) = 0$ and $M'(h_0) \neq 0$, that is $h_1(2\pi, h_0) = 0$, $h_1'(2\pi, h_0) \neq 0$, the Taylor expansion of (7) is:

$$h_1'(2\pi, h_0)(r - h_0) + \varepsilon h_3(2\pi, h_0) + O(v_0, \varepsilon) = 0. \quad (9)$$

By solving the equations of (8) and (9), we get:

$$2\pi[2b_{0100} + h_0 M_1](r - h_0) - \varepsilon h_0 + O(v_0, \varepsilon) = 0, \quad (10)$$

$$(exp(2\pi B) - I_2)v_0 + \varepsilon exp(2\pi B) + O(v_0, \varepsilon) = 0. \quad (11)$$

If $c_{0001} \neq k$ then the matrix $v_1(2\pi, h_0) - I_2$ is invertible, then through solving the system (10)-(11), we know the system of (3) has a unique periodic orbit with periodic near 2π in the neighbor of L_{h_0}.

Assuming that the periodic orbit has the form (see [12]):

$$\begin{aligned} u &= G(\theta, h_0 + \overline{h}_0(\theta, \varepsilon)), \\ v(\theta, \varepsilon) &= (v_{10}(\theta), v_{20}(\theta))^T + O(\varepsilon^2). \end{aligned}$$

3 Stability of the Periodic Orbit

We have discussed the existence of the periodic orbit in part 2. In this section, we will present the stability of the periodic solution by using the blow-up transformation, the average method and the integral manifold theory.

Suppose the following inequality holds: $n_{11} \neq 3n_{13}$, $\delta_2 \neq 0$. Thus, $v = 0$ is a 1-order weak focus of planar autonomous system

$$\frac{dv}{d\tau} = g(v).$$

Without loss of generality, the system of (3) can be written as:

$$\begin{aligned} \frac{du}{d\tau} &= f(u) + \varepsilon P(u, v), \\ \frac{dv_1}{d\tau} &= c_{0001}v_1 + c_{0012}v_1 v_2^2 + c_{0021}v_1^2 \\ &\quad + c_{0030}v_1^3 + c_{0003}v_2^3 + \varepsilon Q_1(u, v), \\ \frac{dv_2}{d\tau} &= d_{0010}v_1 + d_{0021}v_1^2 v_2 + d_{0003}v_2^3 \\ &\quad + d_{0012}v_1 v_2^2 + d_{0030}v_1^3 + \varepsilon Q_2(u, v). \end{aligned} \quad (12)$$

where $Q_i(u, v)$ are higher polynomials of u , v, $i = 1, 2$.

It has shown that, if there exists a $h_0 \in J$ such that $M(h_0) = 0$, $M'(h_0) \neq 0$ and $c_{0001} \neq k$, the system of (3) have a unique periodic orbit L_ε in the neighborhood of L_{h_0}.

3.1 The blow-up transformation

In order to use the average method, we should use the blow-up transformation to (6). Hence for $|\overline{u}| << 1$ and $h = h_0 + \overline{u}$, the system (6) becomes:

$$\begin{aligned} \frac{d\overline{u}}{d\theta} &= \varepsilon(a_{01}(\theta) + a_{02}(\theta)\overline{u} + a_{03}(\theta)v_1 \\ &\quad + a_{04}(\theta)v_2 + R_1(\theta, \overline{u}, v)), \\ \frac{dv_1}{d\theta} &= c_{0001}v_1 + b_{11}(\theta)\varepsilon + b_{12}(\theta)\overline{u}\varepsilon \\ &\quad + b_{14}(\theta)v_2\varepsilon + c_{0030}v_1^3 + c_{0012}v_1 v_2^2 \\ &\quad + c_{0021}v_1^2 v_2 + c_{0003}v_2^3 + b_{13}(\theta)v_1\varepsilon \\ &\quad + R_2(\theta, \overline{u}, v)), \\ \frac{dv_2}{d\theta} &= d_{0010}v_1 + b_{21}(\theta)\varepsilon + b_{22}(\theta)\varepsilon\overline{u} \\ &\quad + b_{23}(\theta)v_1\varepsilon + d_{0003}v_2^3 + d_{0021}v_1^2 v_2 \\ &\quad + d_{0012}v_1 v_2^2 + d_{0030}v_1^3 + b_{24}(\theta)v_2\varepsilon \\ &\quad + R_3(\theta, \overline{u}, v)). \end{aligned} \quad (13)$$

where

$$\begin{aligned} a_{02}(\theta) &= f_x(G(\theta, h_0))G_h' \wedge P(G(\theta, h_0), 0) \\ &\quad + f(G(\theta, h_0)) \wedge P_x'(G(\theta, h_0), 0), \\ a_{01}(\theta) &= f(G(\theta, h_0)) \wedge P(G(\theta, h_0), 0), \\ a_{03}(\theta) &= f(G(\theta, h_0)) \wedge P_{y1}'(G(\theta, h_0), 0), \\ a_{04}(\theta) &= f(G(\theta, h_0)) \wedge P_{y2}'(G(\theta, h_0), 0), \\ b_{11}(\theta) &= Q_1(G(\theta, h_0), 0), \\ b_{13}(\theta) &= Q_{1y_1}(G(\theta, h_0), 0), \\ b_{12}(\theta) &= Q_{1x}(G(\theta, h_0), 0), \\ b_{14}(\theta) &= Q_{1y_2}(G(\theta, h_0), 0), \\ b_{21}(\theta) &= Q_2(G(\theta, h_0), 0), \\ b_{22}(\theta) &= Q_{2x}(G(\theta, h_0), 0)G_h'(\theta, h_0), \\ b_{23}(\theta) &= Q_{2y_1}(G(\theta, h_0), 0), \\ b_{24}(\theta) &= Q_{2y_2}(G(\theta, h_0), 0). \end{aligned} \quad (14)$$

Let

$$\begin{aligned} \overline{u} &= \overline{h}_0 + p, \\ (v_1, v_2)^T &= \varepsilon(v_{10}, v_{20})^T + (z_1, z_2)^T. \end{aligned}$$

Then it follows that:

$$\begin{aligned} \frac{dp}{d\theta} &= \varepsilon(a_{02}(\theta)p + a_{03}(\theta)z_1 + a_{04}(\theta)z_2) \\ &\quad + \widetilde{R}_1(\theta, p, z, \varepsilon), \\ \frac{dz_1}{d\theta} &= c_{0001}z_2 + b_{12}(\theta)p\varepsilon + b_{13}(\theta)z_1\varepsilon \\ &\quad + c_{0030}(3v_{01}^2 z_1 + 3v_{01}z_1^2 + z_1^3) \\ &\quad + c_{0012}(v_{01}z_2^2 + 2v_{01}v_{02}z_2 + z_1 z_2^2 \end{aligned}$$

$$
\begin{aligned}
&+ 2v_{02}z_1z_2) + c_{0021}(v_{01}^2z_2 + z_1^2v_{02}\\
&+ 2v_{01}v_{02}z_1 + 2v_{01}z_1z_2 + z_1^2z_2)\\
&+ c_{0003}(3v_{02}^2z_2 + z_2^3 + 3v_{02}z_2^2\\
&+ b_{14}(\theta)z_2\varepsilon + \widetilde{R}_2(\theta,p,z,\varepsilon),\\
\frac{dz_2}{d\theta} &= d_{0010}z_1 + b_{22}(\theta)p\varepsilon + b_{23}(\theta)z_1\varepsilon\\
&+ d_{0003}(3v_{02}^2z_2 + 3v_{02}z_2^2 + z_2^3)\\
&+ d_{0021}(2v_{01}v_{02}z_1 + 2v_{01}z_1z_2\\
&+ v_{01}^2z_2 + z_1^2z_2 + d_{0021}(v_{01}z_2^2\\
&+ 2v_{01}v_{02}z_2 + z_1v_02^2 + z_1z_2^2\\
&+ 2v_{02}z_1z_2) + d_{0030}(3v_{01}^2z_1\\
&+ 3v_{01}z_1^2 + z_1^3) + \varepsilon b_{24}(\theta)z_2\\
&+ \widetilde{R}_3(\theta,p,z,\varepsilon). \qquad (15)
\end{aligned}
$$

From the above equations, we have

$$
\begin{aligned}
\widetilde{R}_1\theta,p,z,\varepsilon) &= R_1(\theta,\overline{h}_0+p,v_{01}+z_1,v_{02}+z_2)\\
&\quad -R_1(\theta,\overline{h}_0,v_{10},v_{20}),\\
\widetilde{R}_2\theta,p,z,\varepsilon) &= R_2(\theta,\overline{h}_0+p,v_{01}+z_1,v_{02}+z_2)\\
&\quad -R_2(\theta,\overline{h}_0,v_{10},v_{20}),\\
\widetilde{R}_3\theta,p,z,\varepsilon) &= R_3(\theta,\overline{h}_0+p,v_{01}+z_1,v_{02}+z_2)\\
&\quad -R_3(\theta,\overline{h}_0,v_{10},v_{20}).
\end{aligned}
$$

In order to use the average method, the functions should have the same scales, so we introduce the blow-up $p=\gamma\widetilde{p}, z_1=\gamma\widetilde{z}_1, z_2=\gamma\widetilde{z}_2, \gamma=\sqrt{|\varepsilon|}$. The system of (15) becomes:

$$
\begin{aligned}
\frac{d\widetilde{p}}{d\theta} &= \mu^2(a_{02}(\theta)\widetilde{p} + a_{03}\widetilde{z}_1 + a_{04}(\theta)\widetilde{z}_2)sgn(\varepsilon)\\
&+ \mu^3\widetilde{R}_{01}(\theta,\overline{u},v,\varepsilon),\\
\frac{d\widetilde{z}_1}{d\theta} &= c_{0001}\widetilde{z}_2 + sgn(\varepsilon)\mu^2(b_{12}(\theta)\widetilde{p} + b_{13}(\theta)\widetilde{z}_1\\
&+ b_{14}(\theta)\widetilde{z}_2) + c_{0030}\widetilde{z}_1^3 + c_{0012}\widetilde{z}_1\widetilde{z}_2^2\\
&+ c_{0021}\widetilde{z}_1^2\widetilde{z}_2 + c_{0003}\widetilde{z}_2^3\\
&+ \mu^3\widetilde{R}_{02}(\theta,\overline{u},v,\varepsilon),\\
\frac{d\widetilde{z}_2}{d\theta} &= d_{0010}\widetilde{z}_1 + sgn(\varepsilon)\mu^2(b_{22}(\theta)\widetilde{p} + b_{23}(\theta)\widetilde{z}_1\\
&+ b_{24}(\theta)\widetilde{z}_2) + d_{0003}\widetilde{z}_1^3 + d_{0021}\widetilde{z}_1^2\widetilde{z}_2\\
&+ d_{0012}\widetilde{z}_1\widetilde{z}_2^2 + d_{0030}\widetilde{z}_1^3\\
&+ \mu^3\widetilde{R}_{03}(\theta,\overline{u},v,\varepsilon). \qquad (16)
\end{aligned}
$$

In order to simplify the expressions, we assume

$$
\begin{aligned}
I_1(\theta,\widetilde{p},\widetilde{z}) &= sgn(\varepsilon)(a_{02}(\theta)\widetilde{p} + a_{03}(\theta)\widetilde{z}_1\\
&\quad + a_{04}(\theta)\widetilde{z}_2),\\
I_2(\theta,\widetilde{p},\widetilde{z}) &= sgn(\varepsilon)(b_{12}(\theta)\widetilde{p} + b_{13}(\theta)\widetilde{z}_1\\
&\quad + c_{0012}\widetilde{z}_1\widetilde{z}_2^2 + c_{0003}\widetilde{z}_2^3\\
&\quad + b_{14}(\theta)\widetilde{z}_2) + c_{0030}\widetilde{z}_1^3,\\
I_3(\theta,\widetilde{p},\widetilde{z}) &= sgn(\varepsilon)(b_{22}(\theta)\widetilde{p} + b_{23}(\theta)\widetilde{z}_1\\
&\quad + b_{24}(\theta)\widetilde{z}_2 + d_{0003}\widetilde{z}_2^3 + d_{0030}\widetilde{z_1^3}\\
&\quad + d_{0021}\widetilde{z}_1^2\widetilde{z}_2 + d_{0012}\widetilde{z}_1\widetilde{z}_2^2.
\end{aligned}
$$

Thus the system of (16) can be written as:

$$
\begin{aligned}
\frac{d\widetilde{p}}{d\theta} &= \mu^2I_1(\theta,\widetilde{p},\widetilde{z}) + \mu^3\widetilde{R}_{01}(\theta,\widetilde{p},\widetilde{z},\varepsilon),\\
\frac{d\widetilde{z}_1}{d\theta} &= c_{0001}\widetilde{z}_2 + \mu^2I_2(\theta,\widetilde{p},\widetilde{z}) + \mu^3\widetilde{R}_{02}(\theta,\widetilde{p},\widetilde{z},\varepsilon),\\
\frac{d\widetilde{z}_2}{d\theta} &= d_{0010}\widetilde{z}_1 + \mu^2I_3(\theta,\widetilde{p},\widetilde{z}) + \mu^3\widetilde{R}_{03}(\theta,\widetilde{p},\widetilde{z},\varepsilon).
\end{aligned} \qquad (17)
$$

Proposition 3 *Set $\widetilde{z}_1=\rho\cos\varphi, \widetilde{z}_2=-\rho\sin\varphi, \widetilde{p}=\rho\omega$. Then the system of (17) can be written as:*

$$
\begin{aligned}
\frac{d\varphi}{d\theta} &= c_{0001} - \frac{\mu^2}{\rho}(I_2\sin\varphi + I_3\cos\varphi)\\
&\quad - \frac{\mu^3}{\rho}(\widetilde{R}_{02}\sin\varphi + \widetilde{R}_{03}\cos\varphi),\\
\frac{d\rho}{d\theta} &= \mu^2(I_2\cos\varphi - I_3\sin\varphi) + \mu^3(\widetilde{R}_{02}\cos\varphi\\
&\quad - \widetilde{R}_{03}\sin\varphi),\\
\frac{d\omega}{d\theta} &= \frac{\mu^2}{\rho}[I_2 - \omega(I_1\cos\varphi - I_3\sin\varphi)]\\
&\quad + \frac{\mu^3}{\rho}(-\omega\widetilde{R}_{02}\cos\varphi + \omega\widetilde{R}_{03}\sin\psi + \widetilde{R}_{01}).
\end{aligned} \qquad (18)
$$

Proof: For $\widetilde{z}_1=\rho\cos\varphi, \widetilde{z}_2=-\rho\sin\varphi, \widetilde{p}=\rho\omega$, we differentiate the both sides of each formula with respect to θ, then we obtain:

$$\frac{d\widetilde{p}}{d\theta} = \frac{d\rho}{d\theta}\omega + \rho\frac{d\omega}{d\theta}, \qquad (19)$$

$$\frac{d\widetilde{z}_1}{d\theta} = \frac{d\rho}{d\theta}\cos\varphi - \rho\sin\varphi\frac{d\varphi}{d\theta}, \qquad (20)$$

$$\frac{d\widetilde{z}_2}{d\theta} = -\frac{d\rho}{d\theta}\sin\varphi - \rho\cos\varphi\frac{d\varphi}{d\theta}\ . \qquad (21)$$

Then, $(20)\times\sin\varphi + \cos\varphi\times(21)$, we get

$$\frac{d\varphi}{d\theta} = -\frac{1}{\rho}(\frac{d\widetilde{z}_1}{d\theta}\sin\varphi + \frac{d\widetilde{z}_2}{d\theta}\cos\varphi),$$

On the other hand, combing (20) with (21), we conclude that

$$
\begin{aligned}
\frac{d\varphi}{d\theta} &= c_{0001} - \frac{\mu^2}{\rho}(I_2\sin\varphi + I_3\cos\varphi)\\
&\quad -\frac{\mu^3}{\rho}(\widetilde{R}_{02}\sin\varphi + \widetilde{R}_{03}\cos\varphi).
\end{aligned}
$$

By $(20) \times \cos\varphi - \sin\varphi \times (21)$, we also have

$$\frac{d\widetilde{z}_1}{d\theta}\cos\varphi - \frac{d\widetilde{z}_2}{d\theta}\sin\varphi = \frac{d\rho}{d\theta}.$$

Combing with the equations of (20) and (21), we have

$$\begin{aligned}\frac{d\rho}{d\theta} &= \mu^2(I^2\cos\varphi - I_3\sin\varphi \\ &\quad +\mu^3(\widetilde{R}_{02}\cos\varphi - \widetilde{R}_{03}\sin\varphi).\end{aligned}$$

Combing (19) with $\frac{d\rho}{d\theta}$ gives

$$\begin{aligned}\frac{d\omega}{d\theta} &= \frac{\mu^2}{\rho}[I_1 - \omega(I_1\cos\varphi - I_3\sin\varphi)] \\ &\quad +\frac{\mu^3}{\rho}(\widetilde{R}_{01} - \omega\widetilde{R}_{02}\cos\varphi \\ &\quad +\omega\widetilde{R}_{03}\sin\varphi).\end{aligned}$$

3.2 The average method

If c_{0001} is irrational, then the method of averaging can be applied to (18) to obtain the equivalent equations

$$\begin{aligned}\frac{d\varphi}{d\theta} &= c_{0001} - \mu^2\widetilde{I}_1(\rho,\omega) + \mu^3\widetilde{R}_1(\rho,\omega,\varepsilon), \\ \frac{d\rho}{d\theta} &= \mu^2\widetilde{I}_2(\rho,\omega) + \mu^3\widetilde{R}_2(\rho,\omega,\varepsilon), \\ \frac{d\omega}{d\theta} &= \mu^2\widetilde{I}_3(\rho,\omega) + \mu^3\widetilde{R}_3(\rho,\omega,\varepsilon). \qquad (22)\end{aligned}$$

with the following forms:

$$\begin{aligned}\widetilde{I}_1(\rho,\omega) &= \frac{3\rho^2}{8}(d_{0030} - c_{0003}) + \frac{\rho^2}{8}(d_{0012} - c_{0021}) \\ &\quad +\frac{1}{4\pi}\int_0^{2\pi}(b_{23}(\theta) - b_{14}(\theta))d\theta sgn(\varepsilon), \\ \widetilde{I}_2(\rho,\omega) &= \frac{\rho^3}{8}(c_{0012} + d_{0021}) + \frac{3\rho^3}{8}(c_{0030} + d_{0003}) \\ &\quad +\frac{\rho}{4\pi}\int_0^{2\pi}(b_{13}(\theta) + b_{24}(\theta))d\theta sgn(\varepsilon), \\ \widetilde{I}_3(\rho,\omega) &= -\frac{\rho_2\omega}{8}(c_{0012} + d_{0021}) - \frac{3\rho^2\omega}{8}(c_{0030} \\ &\quad +d_{0003}) + \frac{\omega}{4\pi}\int_0^{2\pi}(2a_{02}(\theta) - b_{13}(\theta) \\ &\quad -b_{24}(\theta))d\theta sgn(\varepsilon).\end{aligned}$$

In order to use the theory of the invariant torus, we should prepare for the following works. Considering the solutions of the following equations

$$\widetilde{I}_2(\rho,\omega) = 0, \quad \widetilde{I}_3(\rho,\omega) = 0. \qquad (23)$$

It is easy to get that the equations of (23) has a zero solution $(\rho,\omega) = (0,0)$.

If $adsgn(\varepsilon) < 0$, then (23) has a nonzero solution $(\rho_0, 0)$, where

$$\begin{aligned}\rho_0 &= \sqrt{\frac{-2b}{a\pi}}, \\ a &= \int_0^{2\pi}(Q_{1_{y1}} + Q_{2_{y2}})d\theta, \\ b &= \int_0^{2\pi} a_{02}(\theta)d\theta, \\ c &= c_{0012} + d_{0021} + 3c_{0030} + 3d_{0003}.\end{aligned}$$

In order to make further reduction of the average equation, we define

$$A(\rho,\omega) = \begin{pmatrix} \frac{\partial\widetilde{I}_2(\rho,\omega)}{\partial\rho} & \frac{\partial\widetilde{I}_2(\rho,\omega)}{\partial\omega} \\ \frac{\partial\widetilde{I}_3(\rho,\omega)}{\partial\rho} & \frac{\partial\widetilde{I}_3(\rho,\omega)}{\partial\omega} \end{pmatrix}.$$

Therefore, for the zero solution $(\rho,\omega) = (0,0)$, we have

$$\begin{aligned}A(0,0) &= \begin{pmatrix} \frac{\partial\widetilde{I}_2(0,0)}{\partial\rho} & \frac{\partial\widetilde{I}_2(0,0)}{\partial\omega} \\ \frac{\partial\widetilde{I}_3(0,0)}{\partial\rho} & \frac{\partial\widetilde{I}_3(0,0)}{\partial\omega} \end{pmatrix} \\ &= \begin{pmatrix} \frac{a}{4\pi}sgn(\varepsilon) & 0 \\ 0 & \frac{2b-a}{4\pi}sgn(\varepsilon) \end{pmatrix}.\end{aligned}$$

Furthermore, for the nonzero solution $(\rho_0, 0)$, we have

$$\begin{aligned}A(\rho_0,0) &= \begin{pmatrix} \frac{\partial\widetilde{I}_2(\rho_0,0)}{\partial\rho} & \frac{\partial\widetilde{I}_2(\rho_0,0)}{\partial\omega} \\ \frac{\partial\widetilde{I}_3(\rho_0,0)}{\partial\rho} & \frac{\partial\widetilde{I}_3(\rho_0,0)}{\partial\omega} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{a}{2\pi}sgn(\varepsilon) & 0 \\ 0 & \frac{b}{2\pi}sgn(\varepsilon) \end{pmatrix}.\end{aligned}$$

Hence, if $a(2b-a) \neq 0$, then system of (12) has an invariant torus S_1 (see, [11])

$$\begin{aligned}S_1 = ((\theta,\varphi,\rho,\omega) &: \rho = 0, \omega = \psi_1(\theta,\varphi,\omega), \theta \in R, \\ &\varphi \in R) \ with \ \psi_1(\theta,\varphi,0) = 0.\end{aligned}$$

When $2a\varepsilon < c\varepsilon < 0$, then S_1 is exponentially asymptotically stable. If $ab \neq 0$, then system of (12) has an invariant torus S_2:

$$\begin{aligned}S_2 = ((\theta,\varphi,\rho,\omega) &: \rho = \rho_0 + \psi_2(\theta,\varphi,\omega), \\ &\omega = \psi_3(\theta,\varphi,\omega), \theta \in R, \varphi \in R). \\ &\quad with \ \psi_i(\theta,\varphi,0) = 0, \ i = 2,3.\end{aligned}$$

The invariant torus S_2 is exponentially asymptotically stable when $b\varepsilon < 0$ and $a\varepsilon > 0$.

Summarizing the above, we are able to state our main result.

Theorem 4 *(Stability of the Periodic Orbit) If there exists a* $h_0 \in J$*, such that* $M(h_0) = 0, M(h_0)' \neq 0$ *and* $c_{0001} \neq k$*. Then in the neighbor of* L_{h_0}*, we conclude that*

(i) if $2b\varepsilon < a\varepsilon < 0$*,the periodic orbit of (3) is asymptotically stable, and the invariant torus* S_1 *exponentially asymptotically stable.*

(ii) If $b\varepsilon < 0 < a\varepsilon$*, the periodic orbit of (3), is unstable, and the invariant torus is exponentially asymptotically stable.*

(iii) If $2b\varepsilon > a\varepsilon$ *or* $0 < 2b\varepsilon < a\varepsilon$*,the periodic orbit of (3) is unstable and the invariant torus is unstable.*

4 The numerical simulations of the Periodic orbit

In this section, numerical simulations are performed to verify the analytical predictions. Three groups of parametric controlling conditions are given:

i) When $\varepsilon = 0$, the unperturbed system of (4) is a Hamiltonian system.

When $n_{11} \neq 3n_{13}$, $v = 0$ is a 1-order weak focus for the unperturbed system of (5).

The figures we refer to Fig 1.1- Fig1.6.

ii) For $0 < |\varepsilon| << 1$, if $M(r) \neq 0$, the system of (3) does not exist any periodic orbit with period near 2π in the neighborhood of L_r.

The figures we refer to Figs 2.1- Fig2.6.

iii) For $0 < |\varepsilon| << 1$, if $M(h_0) = 0$,$M'(h_0) \neq 0$ and c_{0001} is not positive integer, then in the neighbor of L_{h_0} the system (3) has a unique periodic orbit with period near 2π (see [12]).

The figures we refer to Figs 3.1-Fig3.6.

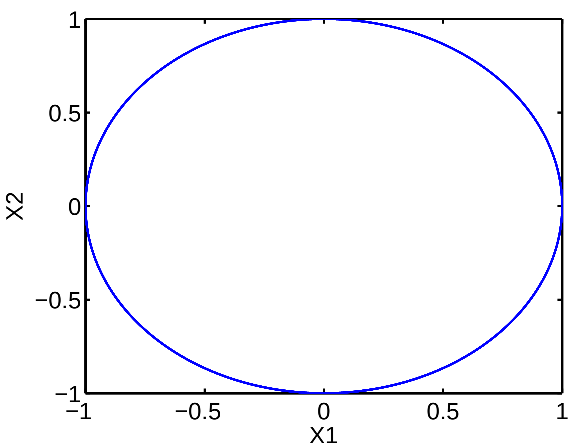

Fig1.1

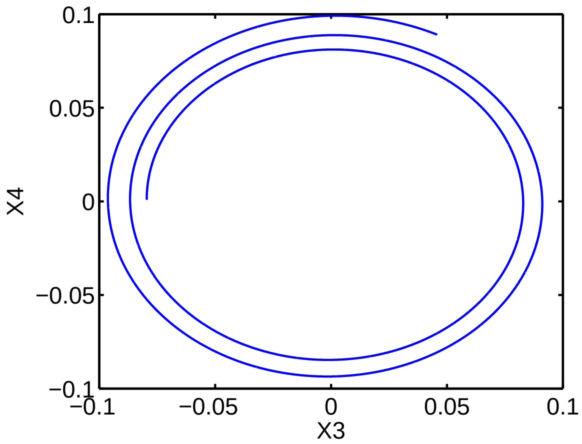

Fig1.2

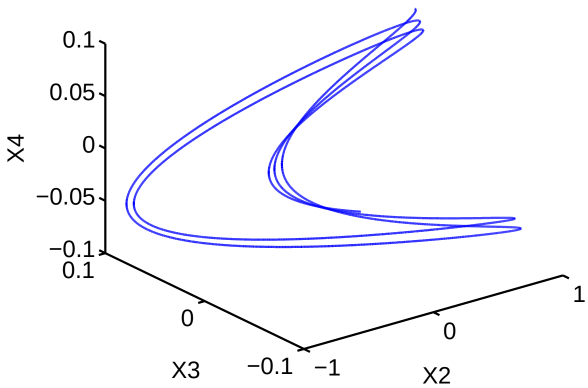

Fig1.3

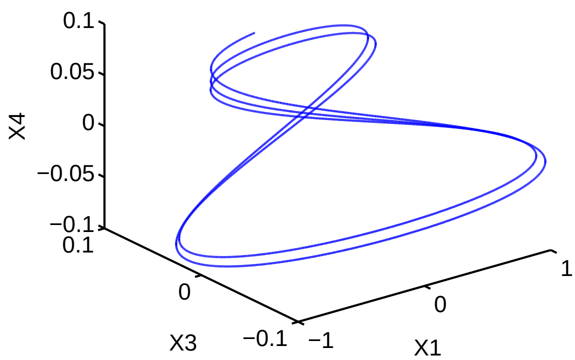

Fig1.4

5 Conclusion

The paper presents the sufficient condition for the existence of the periodic solution of FGM subjected to

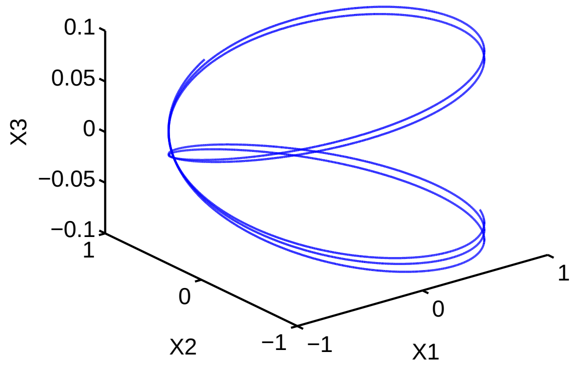

Fig1.5

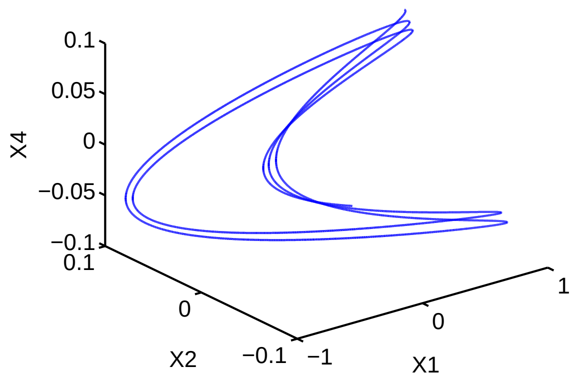

Fig1.6

Figure 1.1-1.6 Its shown that (4) is a Hamiltonian system, (5) has a 1-order weak focus when $\varepsilon = 0$

aero-thermal load with two degrees of freedom by using periodic transformation, Poincare mapping and Melnikov function. We get the results that when $M(r) \neq 0$, the system of (3) does not have any periodic solution. When $M(h_0) = 0, M'(h_0) \neq 0, c_{0001} \neq k$,the system of (3) has a unique periodic solution. Then we present three groups of figures in the next part. In Fig1.1-Fig1.6, we give the unperturbed system of (3), we can see that $\frac{du}{d\tau} = f(u) = (-u_2, u_1)^T$ is a Hamiltonian system. When $n_{11} \neq 3n_{13}$, $\delta_2 \neq 0$, then $v = 0$ is a 1-order weak focus of planar autonomous system $\frac{dv}{d\tau} = g(v)$. In the Fig2.1-Fig2.6, we get that the system of (3) does not have any periodic solution when $M(r) \neq 0$. In the Fig3.1-Fig3.6, we can have the result that if $M(h_0) = 0, M'(h_0) \neq 0, c_{0001} \neq k$,then the system of (3) has a unique periodic orbit. Then, in the

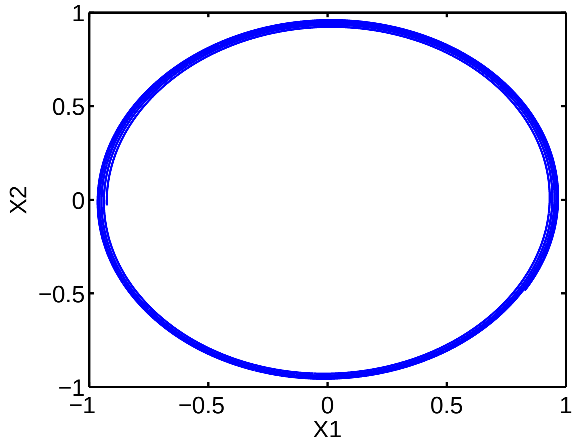

Fig2.1

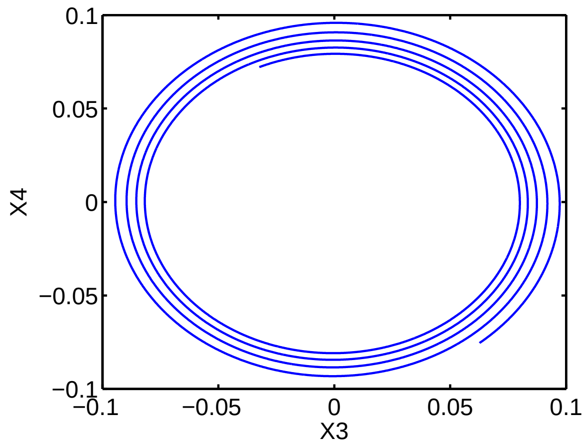

Fig2.2

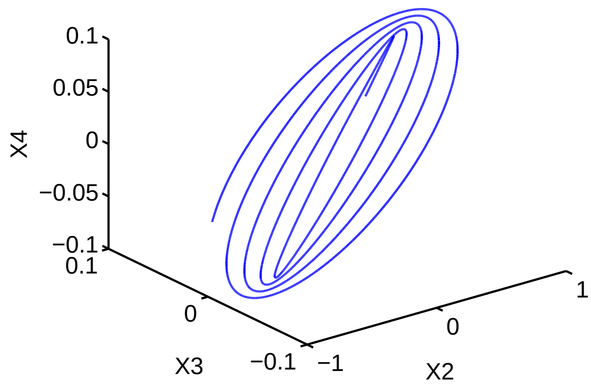

Fig2.3

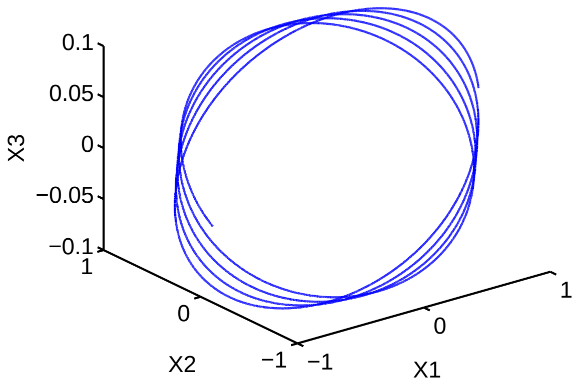

Fig2.4

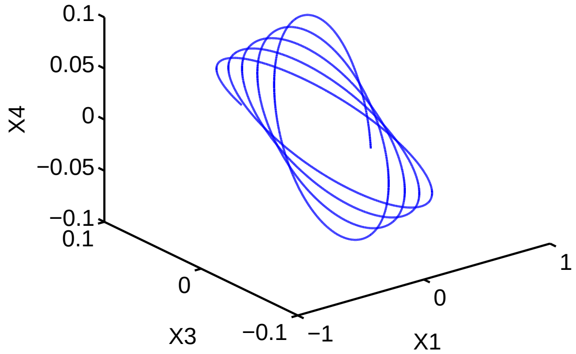

Fig2.5

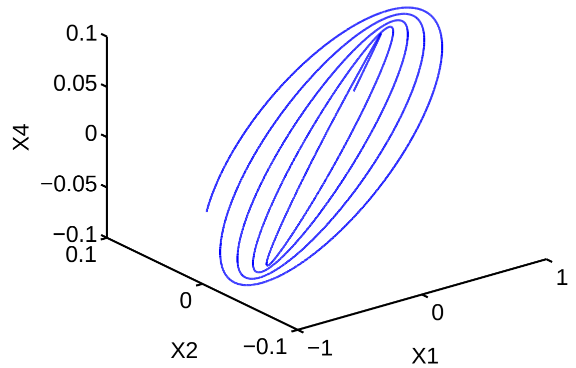

Fig2.6

Figure 2.1-2.6 In this condition there does not exist any periodic orbit in the neighborhood of L_r.

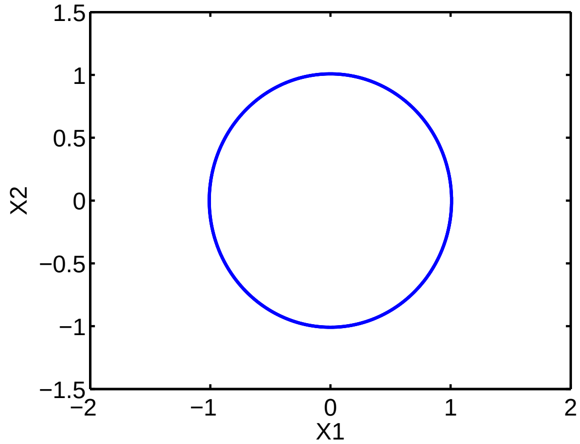

Fig3.1

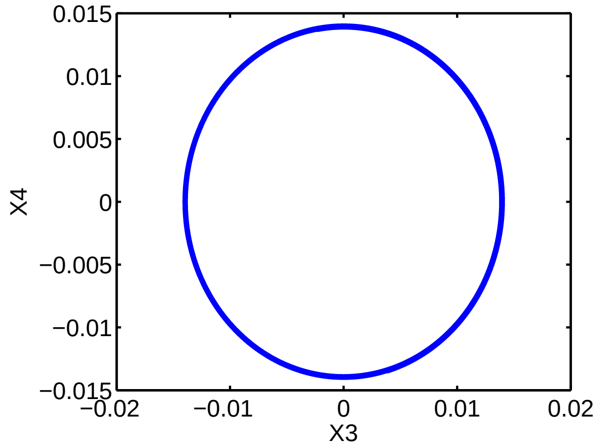

Fig3.2

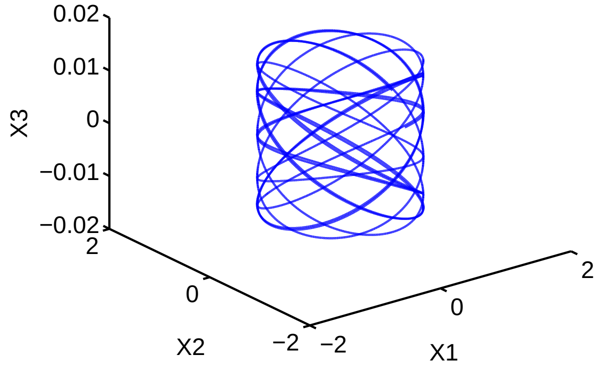

Fig3.3

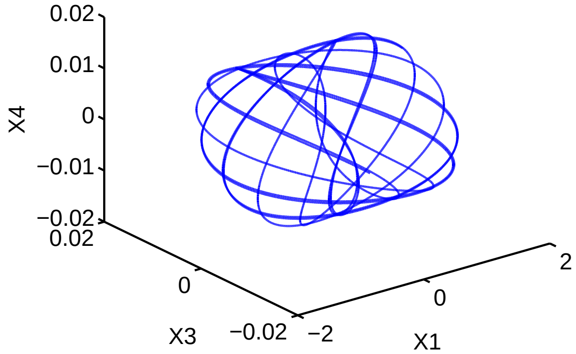

Fig3.4

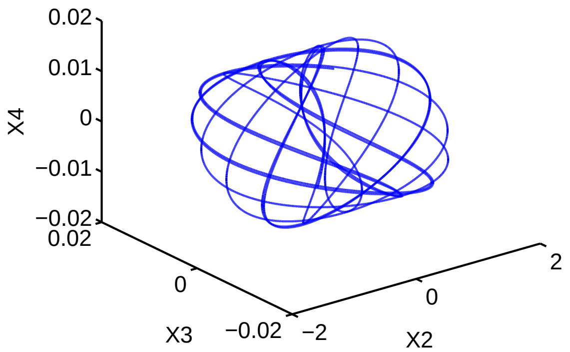

Fig3.5

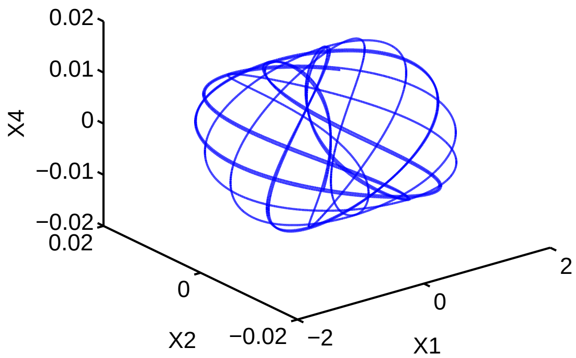

Fig3.6

Figure 3.1-3.6 Figures of system (3), in this condition there has a unique periodic orbit in the neighborhood of L_{h_0}.

next part, we analyze the stability of the periodic orbit. By using the Blow-up transformation, the average method, we get the average equations. By discussing the characteristic value of the matrices, we investigate the stability of periodic solution about the FGM subjected to aero-thermal load.

We anticipate that our proposal for the method to analyze the periodic solution for high-dimensional will contribute to improve the usage of nonlinear dynamics in the areas of new materials.

Acknowledgements: The authors gratefully acknowledge the support of the National Natural Science Foundation of China (NNSFC) through Grant Nos.11072007, 11372014, 11290152 and the Natural Science Foundation of Beijing (NSFB) through Grant No.1122001

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