Convergence of explicit iterative algorithms for solving a class of variational inequalities

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Abstract: Tian proposed a general iterative algorithm for finding a solution for variational inequalities over the set of fixed points of a nonexpansive mapping on Hilbert spaces and obtained the strong convergence theorem [M. Tian, Nonlinear Analysis, 73 (2010) 689-694]. Zhou et al. proposed a simpler explicit iterative algorithm for finding a solution for variational inequalities over the set of common fixed points of a finite family of nonexpansive mappings on Hilbert spaces and proved the strong convergence [H. Y. Zhou, P. Y. Wang, J. Optim. Theory Appl.09 November 2013]. In this paper, we firstly give a new proof of Tian's convergence theorem, which is much more simpler than Tian's original proof. Then we improve the main convergence result of Zhou et al., more precisely, using a recent new lemma, we prove the strong convergence of this algorithm under more weaker conditions (indeed, one of the original conditions is removed). Based on the two results, a more general algorithm is then proposed for solving a more general class of variational inequalities over the set of common fixed points of a finite family of nonexpansive mappings on Hilbert spaces and its strong convergence is proved. Finally, some extensions to our main results have been obtained. Our results in this paper extend and improve ones of Tian and Zhou et al.

Key–Words: Variational inequalities, Hybrid steepest-descent method, Nonexpansive mappings, Common fixed points

1 Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, let C be a nonempty closed convex subset of H, and let $F : C \to H$ be a nonlinear operator. The variational inequality problem $VI(C, F)$ can mathematically be formulated as the problem of finding a point $x^* \in C$ with the property

$$\langle Fx^*, x - x^* \rangle \geq 0, \ \forall \ x \in C. \qquad (1)$$

The variational inequality problem was initially introduced and studied by Stampacchia [1] in 1964. Afterward, some mathematicians extended the main result of [1] to the more general framework of locally convex topological vector spaces, see, for instance, [2-6]. Ever since, variational inequalities have been widely investigated by many authors because they cover a large variety of problems arising in structural analysis, economics, optimization, operations research and engineering sciences, see [7-30] and the references therein.

Recall the characteristic inequality on projection operators: given a point $z \in H$ and $u \in C$, then the inequality

$$\langle z - u, v - u \rangle \leq 0, \ \forall \ v \in C$$

holds if and only if $u = P_C z$, where P_C is the metric projection operator of H onto the closed convex set C; that is, u is the unique point in C such that

$$\|u - z\| = \inf_{v \in C} \|v - z\|.$$

It is well known that the projection P_C is nonexpansive; namely,

$$\|P_C x - P_C y\| \leq \|x - y\|, \ \forall \ x, y \in H.$$

In fact, P_C is also a firmly nonexpansive mapping, that is, the relation

$$\|P_C x - P_C y\|^2 \leq \|x - y\|^2 - \|(I - P_C)x - (I - P_C)y\|^2$$

holds for all $x, y \in H$.

Using the characteristic inequality on projection operators, it is very easy to show that the variational inequality problem is equivalent to the fixed point problem: $x^* \in C$ is a solution of variational inequality (1) if and only if $x^* \in C$ satisfies the fixed-point relation:

$$x^* = P_C(I - \lambda F)x^*,$$

where λ is an arbitrary positive constant.

Recall that an operator $F : C \to H$ is called monotone, if

$$\langle Fx - Fy, x - y\rangle \geq 0 \quad \forall x, y \in C.$$

Moreover, a monotone operator F is called strictly monotone if the equality '$=$' holds only when $x = y$ in the last relation. It is easy to see that $VI(C, F)$ (1) has at most one solution if F is strictly monotone.

Recall that $F : C \to H$ is said to be a $\kappa-$ Lipschitzian and η-strongly monotone operator, if there exist some positive constants κ and η such that

$$\|Fx - Fy\| \leq \kappa\|x - y\|,$$
$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2,$$

hold for all $x, y \in C$. Under these two conditions, it is not difficult to show that the operator $P_C(I - \lambda F) : C \to C$ is a contraction provided the constant λ is selected such that $0 < \lambda < 2\eta/\kappa^2$. By using the well-known Banach contraction mapping principle, $P_C(I - \lambda F)$ has a unique fixed point in C and thus $VI(C, F)$ (1) has a unique solution. Throughout the rest of this paper, I denotes the identity mapping on H and $F : H \to H$ is a $\kappa-$ Lipschitzian and η-strongly monotone operator.

Recall that $T : H \to H$ is said to be a nonexpansive mapping, if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in H$. Assume that $T : H \to H$ is a nonexpansive mapping with the nonempty set of fixed points, i.e.,$Fix\,(T) \triangleq \{x \in H : Tx = x\} \neq \emptyset$. It is well-known that $Fix(T)$ is a closed convex subset of H. Yamada [7] considered a particular class of variational inequalities: finding a point $x^* \in Fix(T)$ such that

$$\langle Fx^*, x - x^*\rangle \geq 0, \ \forall x \in Fix(T). \quad (2)$$

To solve (2), Yamada [7] introduced the hybrid steepest-descent method:

$$x_{n+1} = (I - \lambda_n \mu F)Tx_n, \ n \geq 0, \quad (3)$$

where the initial point x_0 is selected in H arbitrarily, μ is a constant such that $0 < \mu < 2\eta/\kappa^2$ and (λ_n) is a sequence in $(0, 1)$. Algorithm (3) is implementable since it has nothing to do with the metric projection operator, which is its evident advantage. Yamada [7] proved the following result.

Theorem 1 *([7]) Let $F : H \to H$ be a $\kappa-$ Lipschitzian and η-strongly monotone operator. Let the initial guess x_0 be taken in H arbitrarily. Assume that the sequence (λ_n) satisfies the conditions*

(i) $\lim_{n\to\infty} \lambda_n = 0$,

(ii) $\sum_{n=1}^{\infty} \lambda_n = \infty$,

(iii) either $\sum_{n=1}^{+\infty} |\lambda_{n+1} - \lambda_n| < +\infty$ *or* $\lim_{n\to\infty} \lambda_n/\lambda_{n+1} = 1$.

Then the sequence (x_n) generated by (3) converges strongly to the unique solution of the variational inequality (2).

Yamada also considered the hybrid steepest-descent cycle method for solving Lipschitzian and strongly monotone variational inequalities over the set of common fixed points of a finite family of nonexpansive mappings.

Recall that $f : H \to H$ is said to be a contraction with coefficient $\alpha \in [0, 1)$, if $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, $\forall x, y \in H$. Henceforth, $f : H \to H$ always denotes a contraction with coefficient $\alpha \in (0, 1)$.

Moudafi proposed [31] the viscosity approximation method: take an initial guess $x_0 \in C$ arbitrarily and define (x_n) recursively by

$$x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n)Tx_n, \ n \geq 0,$$

where (λ_n) is a sequence in the interval $(0, 1)$. In this case, f is a contractive self mapping on C and T is a nonexpansive self mapping on C, respectively. In the setting of Hilbert spaces, Moudafi proved that If (λ_n) satisfies the same conditions (i)-(iii) as in Theorem 1, then the sequence (x_n) generated by the viscosity approximation method converges strongly to a fixed point x^* of T, which also solves the variational inequality problem: finding an element $x^* \in Fix(T)$ such that

$$\langle f(x^*) - x^*, x - x^*\rangle \leq 0, \ \ x \in Fix(T).$$

H. K. Xu studied the viscosity approximation method in the setting of Banach spaces and obtained the strong convergence theorems [32].

Yamada's algorithm (3) has very close connection with the viscosity approximation method. In fact, algorithm (3) can be rewritten as the form:

$$x_{n+1} = \lambda_n(I - \mu F)Tx_n + (1 - \lambda_n)Tx_n.$$

Noting the fact that $(I - \mu F)T : H \to H$ is a contraction with coefficient $1 - \lambda\tau$, where $\tau = \frac{1}{2}\mu(2\eta - \mu\kappa^2)$, if μ is selected such that $0 < \mu < 2\eta/\kappa^2$ (see Lemma

4 below), it is easy to see that algorithm (3) is indeed a special case of the viscosity approximation method.

Tian [8] proposed a much more general iterative algorithm: the initial guess x_0 is selected in H arbitrarily and the x_{n+1} is defined by

$$x_{n+1} = \lambda_n \gamma f(x_n) + (I - \lambda_n \mu F)Tx_n,\ n \geq 0, \quad (4)$$

where T is a nonexpansive self-mapping of H such that $Fix(T) \neq \emptyset$, μ and γ are two constants such that $\mu \in (0, \frac{2\eta}{\kappa^2})$ and $0 < \gamma < \mu(\eta - \frac{\mu\kappa^2}{2})/\alpha \triangleq \tau/\alpha$ respectively, and (λ_n) is a sequence in $(0,1)$. Algorithm (4) is an extension not only to the viscosity approximation method but also to Yamada's hybrid steepest-descent method for solving fixed points problems of nonexpansive mappings and variational inequalities problems.

Tian [8] proved the following strong convergence result.

Theorem 2 *([8]) Let $F : H \to H$ be a $\kappa-$ Lipschitzian and η-strongly monotone operator and let $f : H \to H$ be a contraction with coefficient $\alpha \in (0,1)$. Assume that the sequence (λ_n) satisfies the same conditions (i)-(iii) as in Theorem 1. Then the sequence (x_n) generated by (4) converges strongly to the unique solution of the variational inequality problem: finding a point $\tilde{x} \in Fix(T)$ such that*

$$\langle (\mu F - \gamma f)\tilde{x}, x - \tilde{x}\rangle \geq 0,\ \forall x \in Fix(T). \quad (5)$$

Very recently, Zhou et al. [9] were concerned with the following variational inequality problem: finding a point $x^* \in \bigcap_{i=1}^N Fix(T_i)$ such that

$$\langle Fx^*, x - x^*\rangle \geq 0,\ \forall x \in \bigcap_{i=1}^N Fix(T_i), \quad (6)$$

where $N \geq 1$ is an integer, $(T_i)_{i=1}^N$ is a family of finite nonexpansive self-mappings of H with the nonempty set of common fixed points, i.e., $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$.

To solve (6), Zhou et al.[9] proposed a simpler iterative algorithm: take an initial guess $x_0 \in H$ arbitrarily and define a sequence (x_n) in the manner:

$$x_{n+1} = (I - \lambda_n \mu F)T_N^n T_{N-1}^n \cdots T_1^n x_n,\quad n \geq 0, \quad (7)$$

where $\mu \in (0, \frac{2\eta}{\kappa^2})$, $(\lambda_n) \subset (0,1)$ and $T_i^n = (1-\beta_n^i)I + \beta_n^i T_i$ such that $(\beta_n^i) \subset (0,1)$ for $i = 1, 2, \cdots, N$.

For the strong convergence of algorithm (7), Zhou et al.[9] proved the following result.

Theorem 3 *([9])Let $F : H \to H$ be a $\kappa-$ Lipschitzian and η-strongly monotone operator and let $(T_i)_{i=1}^N$ be a family of finite nonexpansive self-mappings of H with the property $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$. Assume that the sequences (λ_n) and $(\beta_n^i)\,(i = 1, 2, \cdots, N)$ satisfy the conditions:*

(i) $\lim_{n\to\infty} \lambda_n = 0$,

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$,

(iii) there exist some $\beta_*, \beta^* \in (0,1)$ such that $\beta_n^k \in [\beta_*, \beta^*]$ for $i = 1, 2, \cdots, N$ and all $n \geq 0$,

(iv) for $i = 1, 2, \cdots, N$, $\lim_{n\to\infty} |\beta_{n+1}^i - \beta_n^i| = 0$.

Then the sequence (x_n), generated by (7), converges strongly to the unique solution x^* of the variational inequality (6).

In this paper, we shall firstly give a new proof of Theorem 2, which is much more simpler than Tian's original proof in [8]. Secondly, we improve Theorem 3. More precisely, we prove that condition (iv) is unnecessary, i.e., if condition (iv) is removed, the result of Theorem 3 still holds. Our new proof is very different from one given by Zhou et al. in [9]. In fact, we shall see that a recent new tool (see Lemma 6 below) shows very important action on our proof. Based on this result, we thirdly propose a more general algorithm than (7): the initial guess x_0 is selected in H arbitrarily and the x_{n+1} is defined by

$$x_{n+1} = \lambda_n \gamma f(x_n) + (I - \lambda_n \mu F)T_N^n T_{N-1}^n \cdots T_1^n x_n,\ n \geq 0, \quad (8)$$

where $\mu \in (0, \frac{2\eta}{\kappa^2})$, $(\lambda_n) \subset (0,1)$, γ is a constant such that $0 < \gamma < \mu(\eta - \frac{\mu\kappa^2}{2})/\alpha \triangleq \tau/\alpha$ and T_i^n, $i = 1, 2, \cdots, N; n \geq 0$ are the same as above. We shall prove that under the same conditions (i)-(iii) as in Theorem 3 (note that condition (iv) is also unnecessary), the sequence (x_n) generated by (8) converges strongly to a point in $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$, which also solves the variational inequality of finding a point $x^* \in \bigcap_{i=1}^N Fix(T_i)$ such that

$$\langle (\mu F - \gamma f)x^*, x - x^*\rangle \geq 0,\ \forall x \in \bigcap_{i=1}^N Fix(T_i). \quad (9)$$

The rest of this paper is organized as follows. Some useful lemmas are listed in the next section. In Section 3, our main results are given. In the last section, three extensions to our main results are given.

2 Preliminaries

Throughout the rest of this paper, we denote by H a real Hilbert space. We will use the notations:

- $\to$ denotes strong convergence.
- $\rightharpoonup$ denotes weak convergence.
- $\omega_w(x_n) = \{x \mid \exists \{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup x\}$ denotes the weak $\omega - limit$ set of $\{x_n\}$.

Now we collect some necessary facts and useful tools in order to establish our main convergence theorems.

Lemma 4 *([1]) Assume that $F : H \mapsto H$ is a $\kappa-$Lipschitzian and $\eta-$strongly monotone operator, $\lambda \in (0,1)$ and $\mu \in (0, \frac{2\eta}{\kappa^2})$. Then $I - \lambda\mu F$ is a contraction with coefficient $1 - \lambda\tau$, where $\tau = \frac{1}{2}\mu(2\eta - \mu\kappa^2)$.*

Lemma 5 *For all $x, y \in H$ and $\lambda \in [0,1]$, there holds the relation:*

$$\begin{aligned}\|(1-\lambda)x + \lambda y\|^2 = (1-\lambda)\|x\|^2 + \lambda\|y\|^2 \\ -\lambda(1-\lambda)\|x-y\|^2.\end{aligned}$$

This equality is trivial but in common use.

Lemma 6 *([10]) Assume (s_n) is a sequence of nonnegative real numbers such that*

$$s_{n+1} \le (1-\gamma_n)s_n + \gamma_n\delta_n,\ n \ge 0, \tag{10}$$

$$s_{n+1} \le s_n - \eta_n + \alpha_n,\ n \ge 0, \tag{11}$$

where (γ_n) is a sequence in $(0,1)$, (η_n) is a sequence of nonnegative real numbers and (δ_n) and (α_n) are two sequences in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$,

(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \delta_{n_k} \le 0$ *for any subsequence* $(n_k) \subset (n)$.

Then $\lim_{n\to\infty} s_n = 0$.

We shall see that Lemma 6 shows very important action on our proof of the main convergence result (see the proof of Theorem 11 in Section 3).

Lemma 7 *([33]) Let T be a nonexpansive mapping defined on a closed convex subset K of a Hilbert space H. Then $I-T$ is demi-closed; that is, whenever (x_n) is a sequence in K weakly convergent to some $x \in K$ and the sequence $((I-T)x_n)$ strongly converges to some $y \in H$, it follows that $(I-T)x = y$.*

Lemma 8 *([34]) Assume (a_n) is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n,\ \ n \ge 0,$$

where (γ_n) is a sequence in (0,1) and (δ_n) is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 9 *([8]) Let H be a real Hilbert space, $f : H \mapsto H$ be a contraction with coefficient $\alpha \in [0,1)$ and $F : H \mapsto H$ be a $\kappa-$Lipschitzian and $\eta-$strongly monotone operator. Then for $0 < \gamma < \mu\eta/\alpha$,*

$$\begin{aligned}&\langle(\mu F - \gamma f)x - (\mu F - \gamma f)y, x-y\rangle \\ \ge\ &(\mu\eta - \gamma\alpha)\|x-y\|^2,\ \ x, y \in H,\end{aligned}$$

that is, $\mu F - \gamma f$ is $(\mu\eta - \gamma\alpha)$-strongly monotone.

3 Iterative Algorithms

In this section, we always assume that H is a real Hilbert space, T is a nonexpansive mapping on H with $Fix(T) \ne \emptyset$, f is a contraction on H with coefficient $\alpha \in [0,1)$ and F is a $\kappa-$Lipschitzian and $\eta-$strongly monotone operator on H.

Now we give a new proof of Theorem 2, which is much more simpler than Tian's original proof in [8]. In fact, our new proof is a indirect method since it draws support from Theorem 1.

Proof: Firstly, we consider the following supplementary algorithm

$$z_{n+1} = (I - \alpha_n(\mu F - \gamma f))Tz_n,\ n \ge 0, \tag{12}$$

where the initial guess z_0 is selected in H arbitrarily. Obviously, $\mu F - \gamma f$ is a $(\mu\kappa + \gamma\alpha)$-Lipschitzian operator. By Lemma 9, we assert from the condition $0 < \gamma < \tau/\alpha(< \mu\eta/\alpha)$ that $\mu F - \gamma f$ is also $(\mu\eta - \gamma\alpha)$-strongly monotone.

Then we show that the sequence (z_n) generated by (12) converges strongly to the unique solution of variational inequality (5). To see this, we set $\hat{\kappa} = \mu\kappa + \gamma\alpha$, $\hat{\eta} = \mu\eta - \gamma\alpha$ and rewrite the algorithm (12) in the form:

$$z_{n+1} = (I - \lambda_n\hat{\mu}(\mu F - \gamma f))Tz_n,\ n \ge 0, \tag{13}$$

where $\hat{\mu}$ is a arbitrary constant such that $0 < \hat{\mu} < \frac{2\hat{\eta}}{\hat{\kappa}^2}$ and $\lambda_n = \frac{\alpha_n}{\hat{\mu}}$. Noting that (λ_n) also satisfies the conditions (i)-(iii) in Theorem 1, we have from Theorem 1 that the sequence (z_n) generated by (12) (or (13)) converges strongly to the unique solution of variational inequality (5). Thus in order to complete the proof, it suffices to show $\|x_n - z_n\| \to 0$ as $n \to \infty$.

In fact, (12) can also be rewritten in the another form:

$$z_{n+1} = \alpha_n \gamma f(Tz_n) + (I - \alpha_n \mu F)Tz_n, \; n \geq 0. \quad (14)$$

Combining (4) and (14), we have from Lemma 4 that

$$\begin{aligned} & \|x_{n+1} - z_{n+1}\| \\ = \; & \|\alpha_n \gamma (f(x_n) - f(Tz_n)) \\ & +((I - \alpha_n \mu F)Tx_n - (I - \alpha_n \mu F)Tz_n)\| \\ \leq \; & \alpha_n \gamma \alpha \|x_n - Tz_n\| + (1 - \tau \alpha_n)\|x_n - z_n\| \quad (15) \\ \leq \; & [1 - \alpha_n(\tau - \gamma\alpha)]\|x_n - z_n\| \\ & +\alpha_n \gamma \alpha \|z_n - Tz_n\| \\ = \; & (1 - \beta_n)\|x_n - z_n\| + \beta_n \delta_n, \end{aligned}$$

where $\beta_n = \alpha_n(\tau - \gamma\alpha)$ and $\delta_n = \frac{\gamma\alpha}{\tau - \gamma\alpha}\|z_n - Tz_n\|$. Observing that the sequence (β_n) satisfies the conditions (i)-(iii) in Theorem 1 and $\delta_n \to 0$ holds due to the fact that (z_n) converges strongly to a fixed point of T and T is continuous, it follows by using Lemma 8 that $\|x_n - z_n\| \to 0$ as $n \to \infty$. This completes the proof. □

The technique in the proof above will be used again for proving Theorem 12 below.

On the other hand, it is not difficult to see from the proof course above that Theorem 2 can be improved as follows.

Theorem 10 *Let $F : H \to H$ be a $\kappa-$ Lipschitzian and η-strongly monotone operator and let $g : H \to H$ be a $L-$ Lipschitzian mapping. Assume that μ and γ are two constants such that $\mu \in (0, \frac{2\eta}{\kappa^2})$ and $0 < \gamma < \mu(\eta - \frac{\mu\kappa^2}{2})/L \triangleq \tau/L$ respectively, and the sequence (λ_n) satisfies the same conditions (i)-(iii) as in Theorem 1. Then the sequence (x_n) generated by the scheme:*

$$x_{n+1} = \lambda_n \gamma g(x_n) + (I - \lambda_n \mu F)Tx_n, \; n \geq 0,$$

where the initial guess x_0 is selected in H arbitrarily, converges strongly to the unique solution of the variational inequality problem: finding a point $\tilde{x} \in Fix(T)$ such that

$$\langle (\mu F - \gamma g)\tilde{x}, x - \tilde{x} \rangle \geq 0, \; \forall x \in Fix(T).$$

Proof: Obviously, $\mu F - \gamma g$ is a $(\mu\kappa + \gamma L)$-Lipschitzian operator. Similar to Lemma 9, we also assert from the condition $0 < \gamma < \tau/L (< \mu\eta/L)$ that $\mu F - \gamma g$ is also $(\mu\eta - \gamma L)$-strongly monotone. By an argument very similar to the new proof of Theorem 2 above, we can accomplish the proof of this theorem.

We are now in a position to prove that if condition (iv) in Theorem 3 is removed, the result of Theorem 3 still holds, that is, an improvement to Theorem 3 is obtained as follows. Lemma 6 will play a part in the proof of our result.

Theorem 11 *Let $F : H \to H$ be a $\kappa-$ Lipschitzian and η-strongly monotone mapping. Let $(T_i)_{i=1}^N$ be a family of finite nonexpansive self-mappings of H such that $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$. Assume that the sequences $(\lambda_n) \subset (0,1)$ and $(\beta_n^i) \subset (0,1)$ $(i = 1, 2, \cdots, N)$ satisfy the conditions:*

(i) $\lim_{n\to\infty} \lambda_n = 0$,

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$,

(iii) there exist some $\beta_*, \beta^* \in (0,1)$ such that $\beta_n^k \in [\beta_*, \beta^*]$ for $i = 1, 2, \cdots, N$ and all $n \geq 0$.

Then the sequence (x_n), generated by (7), converges strongly to the unique solution x^* of the variational inequality (6).

Proof: Without loss of the generality, we prove Theorem 11 for $N = 2$ (since it is easy to see that our methods carry over the general case). We Firstly show that (x_n) is bounded. To see this, for any point $p \in \bigcap_{i=1}^N Fix(T_i)$, by (7) and Lemma 4, we have

$$\begin{aligned} & \|x_{n+1} - p\| \\ = \; & \|(I - \lambda_n \mu F)T_2^n T_1^n x_n - p\| \\ = \; & \|(I - \lambda_n \mu F)T_2^n T_1^n x_n - (I - \lambda_n \mu F)p \\ & -\lambda_n \mu F(p)\| \\ \leq \; & \|(I - \lambda_n \mu F)T_2^n T_1^n x_n - (I - \lambda_n \mu F)p\| \\ & +\lambda_n \mu \|F(p)\| \\ \leq \; & (I - \tau\lambda_n)\|T_2^n T_1^n x_n - T_2^n T_1^n p\| \\ & +\lambda_n \mu \|F(p)\| \\ \leq \; & (I - \tau\lambda_n)\|x_n - p\| + \tau\lambda_n \tfrac{\mu}{\tau}\|F(p)\| \\ \leq \; & max\{\|x_0 - p\|, \tfrac{u}{\tau}\|F(p)\|\} \triangleq M_p, \end{aligned} \quad (16)$$

for all $n \geq 0$, which shows that (x_n) is bounded, so are $(T_2^n T_1^n(x_n))$ and $(F(T_2^n T_1^n(x_n)))$.

On the other hand, noting the fact that $Fix(T_i) = Fix(T_i^n)$ $(i = 1, 2; \; n \geq 0)$, we deduce from (7) and

Lemma 4 that

$$\begin{aligned} & \|x_{n+1}-x^*\|^2 \\ = & \|(I-\lambda_n\mu F)T_2^nT_1^nx_n-(I-\lambda_n\mu F)x^* \\ & -\lambda_n\mu Fx^*\|^2 \\ \leq & \|(I-\lambda_n\mu F)T_2^nT_1^nx_n-(I-\lambda_n\mu F)x^*\|^2 \\ & +2\lambda_n\mu\|Fx^*\|\cdot\|(I-\lambda_n\mu F)T_2^nT_1^nx_n \\ & -(I-\lambda_n\mu F)x^*\|+\lambda_n^2\mu^2\|Fx^*\|^2 \\ \leq & (1-\tau\lambda_n)\|T_2^nT_1^nx_n-x^*\|^2 \\ & +2\lambda_n\mu(1-\tau\lambda_n)\|Fx^*\|\cdot\|x_n-x^*\| \\ & +\lambda_n^2\mu^2\|Fx^*\|^2 \\ \leq & \|T_2^nT_1^nx_n-x^*\|^2 \\ & +\lambda_n[2\mu\|Fx^*\|\cdot\|x_n-x^*\|+\mu^2\|Fx^*\|^2] \\ \leq & \|T_2^nT_1^nx_n-x^*\|^2+\lambda_nM_1, \end{aligned} \tag{17}$$

where M_1 is a positive constant such that $M_1 \geq \sup_{n\geq 0}\{2\mu\|Fx^*\|\cdot\|x_n-x^*\|+\mu^2\|Fx^*\|^2\}$. Noting that

$$T_1^n=(1-\beta_n^1)I+\beta_n^1T_1,$$
$$T_2^n=(1-\beta_n^2)I+\beta_n^2T_2,$$

and using Lemma 5, we have

$$\begin{aligned} & \|T_2^nT_1^nx_n-x^*\|^2 \\ = & \|(1-\beta_n^2)(T_1^nx_n-x^*) \\ & +\beta_n^2(T_2T_1^nx_n-x^*)\|^2 \\ = & (1-\beta_n^2)\|T_1^nx_n-x^*\|^2 \\ & +\beta_n^2\|T_2T_1^nx_n-x^*\|^2 \\ & -\beta_n^2(1-\beta_n^2)\|T_1^nx_n-T_2T_1^nx_n\|^2 \\ \leq & \|T_1^nx_n-x^*\|^2-\beta_n^2(1-\beta_n^2)\|T_1^nx_n \\ & -T_2T_1^nx_n\|^2 \\ = & \|(1-\beta_n^1)(x_n-x^*)+\beta_n^1(T_1x_n-x^*)\|^2 \\ & -\beta_n^2(1-\beta_n^2)\|T_1^nx_n-T_2T_1^nx_n\|^2 \\ = & (1-\beta_n^1)\|x_n-x^*\|^2+\beta_n^1\|T_1x_n-x^*\|^2 \\ & -\beta_n^1(1-\beta_n^1)\|x_n-T_1x_n\|^2 \\ & -\beta_n^2(1-\beta_n^2)\|T_1^nx_n-T_2T_1^nx_n\|^2 \\ \leq & \|x_n-x^*\|^2-\beta_n^1(1-\beta_n^1)\|x_n-T_1x_n\|^2 \\ & -\beta_n^2(1-\beta_n^2)\|T_1^nx_n-T_2T_1^nx_n\|^2. \end{aligned} \tag{18}$$

Substituting (18) into (17), we get

$$\begin{aligned} & \|x_{n+1}-x^*\|^2 \\ \leq & \|x_n-x^*\|^2-\beta_n^1(1-\beta_n^1)\|x_n-T_1x_n\|^2 \\ & -\beta_n^2(1-\beta_n^2)\|T_1^nx_n-T_2T_1^nx_n\|^2 \\ & +\lambda_nM_1. \end{aligned} \tag{19}$$

Set

$$\begin{aligned} & s_n=\|x_n-x^*\|^2, \\ & \eta_n=\beta_n^1(1-\beta_n^1)\|x_n-T_1x_n\|^2 \\ & \qquad +\beta_n^2(1-\beta_n^2)\|T_1^nx_n-T_2T_1^nx_n\|^2, \\ & \alpha_n=M_1\lambda_n, \end{aligned}$$

then (19) is rewritten as follows

$$s_{n+1}\leq s_n-\eta_n+\alpha_n. \tag{20}$$

By the virtue of (7) and Lemma 4, we have

$$\begin{aligned} & \|x_{n+1}-x^*\|^2 \\ = & \|(I-\lambda_n\mu F)T_2^nT_1^nx_n-x^*\|^2 \\ = & \|(I-\lambda_n\mu F)T_2^nT_1^nx_n-(I-\lambda_n\mu F)x^* \\ & -\lambda_n\mu F(x^*)\|^2 \\ = & \|(I-\lambda_n\mu F)T_2^nT_1^nx_n-(I-\lambda_n\mu F)x^*\|^2 \\ & +\mu^2\lambda_n^2\|Fx^*\|^2 \\ & +2\lambda_n\mu\langle Fx^*,(I-\lambda_n\mu F)x^* \\ & -(I-\lambda_n\mu F)T_2^nT_1^nx_n\rangle \\ \leq & (1-\tau\lambda_n)\|x_n-x^*\|^2 \\ & +2\mu^2\lambda_n^2\|Fx^*\|\cdot\|F(T_2^nT_1^n)x_n\| \\ & +2\lambda_n\mu\langle Fx^*,x^*-x_n\rangle \\ & +2\mu\lambda_n\|Fx^*\|\cdot\|x_n-T_2^nT_1^nx_n\| \\ \leq & (1-\tau\lambda_n)\|x_n-x^*\|^2+\tau\lambda_n\delta_n, \end{aligned} \tag{21}$$

where $\delta_n=\frac{2\mu}{\tau}(\langle Fx^*,x^*-x_n\rangle+\|Fx^*\|\cdot\|x_n-T_2^nT_1^nx_n\|)+\frac{2\mu^2M_2}{\tau}\lambda_n$ and M_2 is a constant such that $M_2\geq\sup_n\|F(T_2^nT_1^nx_n)\|$. Set $\gamma_n=\tau\lambda_n$, thus (21) can also be rewritten as follows

$$s_{n+1}\leq(1-\gamma_n)s_n+\gamma_n\delta_n. \tag{22}$$

Now we use Lemma 6 to (20) and (22) to prove $s_n\to 0$ as $n\to\infty$. It is easy to see that $\gamma_n\to 0$ and $\sum_{n=0}^{\infty}\gamma_n=\infty$ hold due to conditions (i) and (ii). Therefore, in order to complete the proof, it suffices to verify that

$$\lim_{k\to\infty}\eta_{n_k}=0$$

implies

$$\limsup_{k\to\infty}\delta_{n_k}\leq 0$$

for any subsequence $(n_k)\subset(n)$. Firstly, we assert that $\eta_{n_k}\to 0$ as $k\to\infty$ implies $\|T_2^{n_k}T_1^{n_k}x_{n_k}-x_{n_k}\|\to 0$ as $k\to\infty$. Indeed, noting the following inequality

$$\begin{aligned} & \|x_{n_k}-T_2x_{n_k}\| \\ \leq & \|x_{n_k}-T_1^{n_k}x_{n_k}\| \\ & +\|T_1^{n_k}x_{n_k}-T_2T_1^{n_k}x_{n_k}\| \\ & +\|T_2T_1^{n_k}x_{n_k}-T_2x_{n_k}\| \\ \leq & 2\|x_{n_k}-T_1^{n_k}x_{n_k}\| \\ & +\|T_1^{n_k}x_{n_k}-T_2T_1^{n_k}x_{n_k}\| \\ = & 2\beta_{n_k}^1\|x_{n_k}-T_1x_{n_k}\| \\ & +\|T_1^{n_k}x_{n_k}-T_2T_1^{n_k}x_{n_k}\| \end{aligned} \tag{23}$$

and condition (iii), it is easy to see that $\eta_{n_k}\to 0$ as $k\to\infty$ implies that $\|x_{n_k}-T_1x_{n_k}\|\to 0$ and $\|x_{n_k}-T_2x_{n_k}\|\to 0$ as $k\to\infty$. Thus we obtain that

$$\begin{aligned} & \|T_2^{n_k}T_1^{n_k}x_{n_k}-x_{n_k}\| \\ \leq & \|T_2^{n_k}T_1^{n_k}x_{n_k}-T_2^{n_k}x_{n_k}\| \\ & +\|T_2^{n_k}x_{n_k}-x_{n_k}\| \\ \leq & \|T_1^{n_k}x_{n_k}-x_{n_k}\|+\|T_2^{n_k}x_{n_k}-x_{n_k}\| \\ = & \beta_{n_k}^1\|x_{n_k}-T_1x_{n_k}\|+\beta_{n_k}^2\|x_{n_k}-T_2x_{n_k}\| \\ \to & 0. \end{aligned} \tag{24}$$

Moreover, if $\eta_{n_k} \to 0$, we have by using Lemma 7 that $\omega(x_{n_k}) \subset Fix(T_1)\bigcap Fix(T_2)$ holds, noting that $\eta_{n_k} \to 0$ implies $\|x_{n_k} - T_1 x_{n_k}\| \to 0$ and $\|x_{n_k} - T_2 x_{n_k}\| \to 0$. Consequently, we assert that

$$\overline{\lim_{k\to\infty}}\langle Fx^*, x^* - x_{n_k}\rangle \le 0.$$

In fact, take a subsequence $(x_{n_{k_j}}) \subset (x_{n_k})$ such that $x_{n_{k_j}} \rightharpoonup \hat{x}\ (j \to \infty)$ and

$$\overline{\lim_{k\to\infty}}\langle Fx^*, x^* - x_{n_k}\rangle = \lim_{j\to\infty}\langle Fx^*, x^* - x_{n_{k_j}}\rangle,$$

then we have by $\omega(x_{n_k}) \subset Fix(T_1)\bigcap Fix(T_2)$ that $\hat{x} \in Fix(T_1)\bigcap Fix(T_2)$. Observing that x^* is the unique solution of (6), we have

$$\begin{array}{rl} & \overline{\lim}_{k\to\infty}\langle Fx^*, x^* - x_{n_k}\rangle \\ = & \lim_{j\to\infty}\langle Fx^*, x^* - x_{n_{k_j}}\rangle \\ = & \langle Fx^*, x^* - \hat{x}\rangle \\ \le & 0. \end{array} \qquad (25)$$

Combining (24), (25) and the condition $\lambda_n \to 0$, we have $\limsup_{k\to\infty}\delta_{n_k} \le 0$, for any subsequence $(n_k) \subset (n)$. From Lemma 6, $\lim_{n\to\infty} s_n = 0$. □

We now turn to prove the strong convergence theorem for the algorithm (8) by using the result of Theorem 11.

Theorem 12 *Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone mapping and let $f : H \to H$ be a contraction with coefficient $\alpha \in (0,1)$. Let $(T_i)_{i=1}^N$ be a family of finite nonexpansive self-mappings of H such that $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$. Assume that μ and γ are two constants such that $\mu \in (0, \frac{2\eta}{\kappa^2})$ and $0 < \gamma < \mu(\eta - \frac{\mu\kappa^2}{2})/\alpha \triangleq \tau/\alpha$ respectively, and the sequences $(\lambda_n) \subset (0,1)$ and $(\beta_n^i) \subset (0,1)\ (i = 1,2,\cdots,N)$ satisfy the conditions:*

(i) $\lim_{n\to\infty}\lambda_n = 0,$

(ii) $\sum_{n=0}^{\infty}\lambda_n = \infty,$

(iii) there exist some $\beta_*, \beta^* \in (0,1)$ such that $\beta_n^k \in [\beta_*, \beta^*]$ for $i = 1,2,\cdots,N$ and all $n \ge 0$.

Then the sequence (x_n), generated by (8), converges strongly to the unique solution x^* of the variational inequality (9).

Proof: Without loss of the generality, we prove Theorem 12 just for the case $N = 2$. Firstly, we consider the following supplementary algorithm

$$z_{n+1} = (I - \lambda_n(\mu F - \gamma f))T_2^n T_1^n z_n,\ n \ge 0, \qquad (26)$$

where the initial guess z_0 is selected in H arbitrarily. Obviously, $\mu F - \gamma f$ is a $(\mu\kappa + \gamma\alpha)$-Lipschitzian operator. By Lemma 9, we assert from the condition $0 < \gamma < \tau/\alpha (< \mu\eta/\alpha)$ that $\mu F - \gamma f$ is also $(\mu\eta - \gamma\alpha)$-strongly monotone.

Then we show that the sequence (z_n) generated by (26) converges strongly to the unique solution of variational inequality (9). To see this, we set $\hat{\kappa} = \mu\kappa + \gamma\alpha$, $\hat{\eta} = \mu\eta - \gamma\alpha$ and rewrite the algorithm (26) in the form:

$$z_{n+1} = (I - \hat{\lambda_n}\hat{\mu}(\mu F - \gamma f))T_2^n T_1^n z_n,\ n \ge 0, \qquad (27)$$

where $\hat{\mu}$ is an arbitrary fixed constant such that $0 < \hat{\mu} < \frac{2\hat{\eta}}{\hat{\kappa}^2}$ and $\hat{\lambda_n} = \frac{\lambda_n}{\hat{\mu}}$. Noting that $(\hat{\lambda_n})$ also satisfies the conditions (i)-(ii) in Theorem 11 (there exists some integer n_0 such that $\hat{\lambda_n} \in (0,1)$ for all $n \ge 0$ due to the condition $\lambda_n \to 0$ as $n \to \infty$), we have from Theorem 11 that the sequence (z_n) generated by (26) (or (27)) converges strongly to the unique solution of variational inequality (9). Thus in order to complete the proof, it suffices to show $\|x_n - z_n\| \to 0$ as $n \to \infty$.

In fact, (26) can also be rewritten in the another form:

$$\begin{array}{rl} z_{n+1} = & \lambda_n\gamma f(T_2^n T_1^n z_n) \\ & +(I - \lambda_n\mu F)T_2^n T_1^n z_n,\ n \ge 0. \end{array} \qquad (28)$$

Combining (8) and (28), we have from Lemma 4 that

$$\begin{array}{rl} & \|x_{n+1} - z_{n+1}\| \\ = & \|\lambda_n\gamma(f(x_n) - f(T_2^n T_1^n z_n)) \\ & +((I - \lambda_n\mu F)T_2^n T_1^n x_n \\ & -(I - \lambda_n\mu F)T_2^n T_1^n z_n)\| \\ \le & \lambda_n\gamma\alpha\|x_n - T_2^n T_1^n z_n\| \\ & +(1 - \tau\lambda_n)\|x_n - z_n\| \\ \le & [1 - \lambda_n(\tau - \gamma\alpha)]\|x_n - z_n\| \\ & +\lambda_n\gamma\alpha\|z_n - T_2^n T_1^n z_n\| \\ = & (1 - \gamma_n)\|x_n - z_n\| + \gamma_n\delta_n, \end{array} \qquad (29)$$

where $\gamma_n = \lambda_n(\tau - \gamma\alpha)$ and

$$\delta_n = \frac{\gamma\alpha}{\tau - \gamma\alpha}\|z_n - T_2^n T_1^n z_n\|.$$

Obviously, $\gamma_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty}\gamma_n = \infty$. Now we verify that $\lim_{n\to\infty}\delta_n = 0$. Indeed, it follows that $\|z_n - T_1 z_n\| \to 0$ and $\|z_n - T_2 z_n\| \to 0\ (n \to \infty)$ hold due to the fact that (z_n) converges strongly to $x^* \in Fix(T_1)\bigcap Fix(T_2)$ and T_1 and T_2 are all continuous. This together with the simple relation

$$\begin{array}{rl} & \|z_n - T_2^n T_1^n z_n\| \\ \le & \|z_n - T_2^n z_n\| + \|T_2^n - T_2^n T_1^n z_n\| \\ \le & \|z_n - T_2 z_n\| + \|z_n - T_1 z_n\| \end{array}$$

leads to $\|z_n - T_2^n T_1^n z_n\| \to 0$. Consequently, $\lim_{n\to\infty} \delta_n = 0$. Thus we obtain that $\|x_n - z_n\| \to 0$ by using Lemma 8. □

Similar to Theorem 10, Theorem 12 can be easily improved as follows.

Theorem 13 *Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone mapping and let $g : H \mapsto H$ be a L- Lipschitzian mapping. Let $(T_i)_{i=1}^N$ be a family of finite nonexpansive self-mappings of H such that $\bigcap_{i=1}^N Fix(T_i) \neq \emptyset$. Assume that μ and γ are two constants such that $\mu \in (0, \frac{2\eta}{\kappa^2})$ and $0 < \gamma < \mu(\eta - \frac{\mu\kappa^2}{2})/L \triangleq \tau/L$ respectively, and the sequences $(\lambda_n) \subset (0,1)$ and $(\beta_n^i) \subset (0,1)\,(i = 1,2,\cdots,N)$ satisfy the conditions:*

(i) $\lim_{n\to\infty} \lambda_n = 0,$

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty,$

(iii) there exist some $\beta_*, \beta^* \in (0,1)$ such that $\beta_n^k \in [\beta_*, \beta^*]$ for $i = 1,2,\cdots,N$ and all $n \geq 0$.

Then the sequence (x_n), generated by the scheme:

$$x_{n+1} = \lambda_n \gamma g(x_n) + (I - \lambda_n \mu F) T_N^n T_{N-1}^n \cdots T_1^n x_n, \; n \geq 0,$$

where the initial guess x_0 is selected in H arbitrarily, converges strongly to the unique solution of the variational inequality problem: finding a point $x^* \in \bigcap_{i=1}^N Fix(T_i)$ such that

$$\langle (\mu F - \gamma g)x^*, x - x^* \rangle \geq 0, \; \forall x \in \bigcap_{i=1}^N Fix(T_i).$$

4 Extensions

In this section, we extend above results to more broad family of strict pseudo-contractions. Recall that a mapping $S : H \to H$ is said to be a strict pseudo-contraction if there exists a constant $\gamma \in (0,1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \gamma \|(I - S)x - (I - S)y\|^2, \; \forall x, y \in H.$$

In this case, S is also called a $\gamma-$strict pseudo-contraction. In principle, every fixed point problem for strict pseudo-contractions can be transformed into a fixed point problem for nonexpansive mappings. In fact, take an arbitrary fixed constant $\theta \in [\gamma, 1)$, we can always define a nonexpansive mapping $T_\theta : H \to H$ by

$$T_\theta x = \theta x + (1-\theta) Sx, \; x \in H \tag{30}$$

such that $Fix(T_\theta) = Fix(S)$. Thus we can firstly extend Theorem 10 to $\gamma-$strict pseudo-contraction.

Theorem 14 *Let $F : H \to H$ be a $\kappa-$ Lipschitzian and η-strongly monotone operator, let $g : H \mapsto H$ be a $L-$ Lipschitzian mapping and let S be a $\gamma-$strict pseudo-contraction such that $Fix(S) \neq \emptyset$. Assume that μ and γ are two constants such that $\mu \in (0, \frac{2\eta}{\kappa^2})$ and $0 < \gamma < \mu(\eta - \frac{\mu\kappa^2}{2})/L \triangleq \tau/L$ respectively, and the sequence (λ_n) satisfies the same conditions (i)-(iii) as in Theorem 1. Then the sequence (x_n) generated by the scheme:*

$$x_{n+1} = \lambda_n \gamma g(x_n) + (I - \lambda_n \mu F) T_\theta x_n, \; n \geq 0,$$

where T_θ is given as in (30) and the initial guess x_0 is selected in H arbitrarily, converges strongly to the unique solution of the variational inequality problem: finding a point $\tilde{x} \in Fix(S)$ such that

$$\langle (\mu F - \gamma g)\tilde{x}, x - \tilde{x} \rangle \geq 0, \; x \in Fix(S).$$

Secondly, we extend Theorem 11 to the family $\{S_i\}_{i=1}^N$ of γ_i-strict pseudo-contractions. To do this, define

$$\hat{T}_i = \theta_i I + (1-\theta_i) S_i, \; i = 1,2,\cdots,N,$$

then $\{\hat{T}_i\}_{i=1}^N$ is a family of nonexpansive mappings whenever $0 < \gamma_i \leq \theta_i < 1\,(i = 1,2,\cdots,N)$. Define

$$\hat{T}_i^n = (1-\beta_n^i) I + \beta_n^i \hat{T}_i, \; i = 1,\cdots,N; \; n \geq 0; \tag{31}$$

where $(\beta_n^i) \subset (0,1)$.

Theorem 15 *Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone mapping. Let $(S_i)_{i=1}^N$ be a family of γ_i-strict pseudo-contractions such that $\bigcap_{i=1}^N Fix(S_i) \neq \emptyset$. Assume that μ is constant such that $\mu \in (0, \frac{2\eta}{\kappa^2})$ and the sequences $(\lambda_n) \subset (0,1)$ and $(\beta_n^i) \subset (0,1)\,(i = 1,2,\cdots,N)$ satisfy the conditions:*

(i) $\lim_{n\to\infty} \lambda_n = 0,$

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty,$

(iii) there exist some $\beta_*, \beta^* \in (0,1)$ such that $\beta_n^k \in [\beta_*, \beta^*]$ for $i = 1,2,\cdots,N$ and all $n \geq 0$.

Then the sequence (x_n)*, generated by the manner:*

$$x_{n+1} = (I - \lambda_n \mu F)\hat{T}_N^n \hat{T}_{N-1}^n \cdots \hat{T}_1^n x_n, \quad n \geq 0,$$

where $\hat{T}_i^n$ *is given as in (31) and the initial guess* x_0 *is selected in* H *arbitrarily, converges strongly to the unique solution of the variational inequality problem: finding a point* $x^* \in \bigcap_{i=1}^N Fix(S_i)$ *such that*

$$\langle Fx^*, x - x^* \rangle \geq 0, \ \forall x \in \bigcap_{i=1}^{N} Fix(S_i),$$

Finally, we extend Theorem 13 to the family $\{S_i\}_{i=1}^N$ of γ_i-strict pseudo-contractions.

Theorem 16 *Let* $F : H \to H$ *be a* $\kappa-$ *Lipschitzian and* η*-strongly monotone mapping and let* $g : H \mapsto H$ *be a L- Lipschitzian mapping. Let* $(S_i)_{i=1}^N$ *be a family of* γ_i-*strict pseudo-contractions such that* $\bigcap_{i=1}^N Fix(S_i) \neq \emptyset$*. Assume that* μ *and* γ *are two constants such that* $\mu \in (0, \frac{2\eta}{\kappa^2})$ *and* $0 < \gamma < \mu(\eta - \frac{\mu\kappa^2}{2})/L \triangleq \tau/L$ *respectively, and the sequences* $(\lambda_n) \subset (0,1)$ *and* $(\beta_n^i) \subset (0,1)\, (i = 1, 2, \cdots, N)$ *satisfy the conditions:*

(i) $\lim_{n\to\infty} \lambda_n = 0,$

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty,$

(iii) there exist some $\beta_*, \beta^* \in (0,1)$ *such that* $\beta_n^k \in [\beta_*, \beta^*]$ *for* $i = 1, 2, \cdots, N$ *and all* $n \geq 0$.

Then the sequence (x_n) *generated by the scheme:*

$$x_{n+1} = \lambda_n \gamma g(x_n) + (I - \lambda_n \mu F)\hat{T}_N^n \hat{T}_{N-1}^n \cdots \hat{T}_1^n x_n, \ n \geq 0,$$

where $\hat{T}_i^n$ *is given as in (31) and the initial guess* x_0 *is selected in* H *arbitrarily, converges strongly to the unique solution of the variational inequality problem of finding a point* $x^* \in \bigcap_{i=1}^N Fix(S_i)$ *such that*

$$\langle (\mu F - \gamma f)x^*, x - x^* \rangle \geq 0, \ \forall x \in \bigcap_{i=1}^{N} Fix(S_i).$$

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