Some New Generalized Gronwall-Bellman Type Inequalities Arising In The Theory Of Fractional Differential-integro Equations

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Abstract: In this paper, motivated by the research on qualitative properties of solutions for fractional differential-integro equations, some new Gronwall-Bellman type inequalities in two independent variables are established. Based on these inequalities, explicit bounds for unknown functions concerned are derived. As for applications, we apply the inequalities established to research boundedness, uniqueness, and continuous dependence on the initial value and parameter for the solution to a certain fractional differential-integro equation.

Key–Words: Gronwall-Bellman type inequality; Fractional differential-integro equation; Boundedness; Qualitative analysis

1 Introduction

In the research of qualitative as well as quantitative properties of solutions of differential equations, inequalities play important roles. Many authors have paid much attention to establish various inequalities including the Ostrowski type inequalities, Grüss type inequalities, opial type inequalities and so on. Among the investigation for inequalities, the Gronwall-Bellman type inequalities are of particular importance as such inequalities provide explicit bounds for solutions of certain differential equations, and also have proved to be very effective in the stability analysis of dynamic equations. In the last few decades, a lot of generalized Gronwall-Bellman type inequalities have been presented (see [1-30] for example). We notice that most of the inequalities established so far can only be used in the qualitative and quantitative analysis of solutions of differential, difference or integral equations of integer order, while few results are concerned with fractional integral and differential equations.

In [31], Ye et al. presented a new Gronwall-Bellman type inequality as follows:

$$u(t) \le a(t) + g(t) \int_0^t (t-s)^{\beta-1} u(s)ds, \ 0 \le t < T,$$

where $u(t)$ is nonnegative and locally integrable on $0 \le t < T$ (some $T \le \infty$), $\beta > 0$, $a(t)$ is a nonnegative function locally integrable on $0 \le t < T$ and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \le t < T$, $g(t) \le M$ (constant).

Based on the inequality above, the following estimates for $u(t)$ is established:

$$\begin{aligned} u(t) \quad \le \quad & a(t) + \\ & + \int_0^t \sum_{n=1}^{\infty} [\frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s)] ds, \\ & 0 \le t < T. \end{aligned}$$

The result above has proved to be useful in the research of boundedness and continuous dependence on the order α and the initial condition for solutions to certain fractional differential equations with the fractional derivative defined in the sense of Riemann-Liouville fractional derivative.

Motivated by the works in [31], in this paper, we establish some new generalized Gronwall-Bellman type inequalities in two independent variables, which are 2D extension of the results in [31], and can be used in the research of boundedness, uniqueness, and continuous dependence on the initial value and parameter for solutions to certain fractional differential-integro equations.

The next of this paper is organized as follows. In Section 2, we present the main results, in which new Gronwall-Bellman type inequalities in two independent variables are established. In Section 3, we apply the inequalities established to research qualitative as well as quantitative properties for the solution to a certain fractional differential-integro equation. In Section 4, some conclusions are presented.

2 Main results

Theorem 1 *Assume* $\alpha,\ \beta > 0$, $a(x,y)$, $u(x,y)$ *are nonnegative functions locally integrable on* $D := \{(x,y)|0 \le x < X,\ 0 \le y < Y\}$, $b(x,y)$ *is a nondecreasing nonnegative continuous function defined on* D *with* $b(x,y) \le M$, *where* M *is a positive constant. If the following inequality satisfies:*

$$\begin{aligned}&u(x,y) \le a(x,y)\\ &+b(x,y)\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}u(s,t)dsdt,\\ &\forall(x,y)\in D,\end{aligned} \tag{1}$$

then for $(x,y)\in D$,

$$\begin{aligned}u(x,y) \le a(x,y) + \sum_{n=1}^{\infty}\frac{(\Gamma(\alpha)\Gamma(\beta))^n}{\Gamma(n\alpha)\Gamma(n\beta)}\times\\ \int_0^y\int_0^x [b^n(x,y)(x-s)^{n\alpha-1}(y-t)^{n\beta-1}a(s,t)]dsdt.\end{aligned} \tag{2}$$

Proof: Define the operator A by

$$Av(x,y) = b(x,y)\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}v(s,t)dsdt$$

where v is locally integrable on D. Then we have

$$u(x,y) \le a(x,y) + Au(x,y).$$

Furthermore,

$$u(x,y) \le \sum_{k=0}^{n-1}A^k a(x,y) + A^n u(x,y).$$

Now we prove the following inequality by use of the mathematical induction method.

$$\begin{aligned}A^n a(x,y) \le \frac{(\Gamma(\alpha)\Gamma(\beta))^n}{\Gamma(n\alpha)\Gamma(n\beta)}\\ \int_0^y\int_0^x b^n(x,y)(x-s)^{n\alpha-1}(y-t)^{n\beta-1}a(s,t)dsdt.\end{aligned} \tag{3}$$

When $n = 1$, (3) holds obviously. Now assume that (3) holds for $n = k$. Then for $n = k+1$, we have

$$\begin{aligned}&A^{k+1}a(x,y) = A(A^k a(x,y))\\
&\le \frac{(\Gamma(\alpha)\Gamma(\beta))^k}{\Gamma(k\alpha)\Gamma(k\beta)}b(x,y)\\
&\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}dsdt\\
&\int_0^t\int_0^s (b(s,t))^k(s-\tau)^{k\alpha-1}(t-\xi)^{k\beta-1}a(\tau,\xi)d\tau d\xi\\
&\le \frac{(\Gamma(\alpha)\Gamma(\beta))^k}{\Gamma(k\alpha)\Gamma(k\beta)}b^{k+1}(x,y)\\
&\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}dsdt\\
&\int_0^t\int_0^s (s-\tau)^{k\alpha-1}(t-\xi)^{k\beta-1}a(\tau,\xi)d\tau d\xi\\
&= \frac{(\Gamma(\alpha)\Gamma(\beta))^k}{\Gamma(k\alpha)\Gamma(k\beta)}b^{k+1}(x,y)\\
&\int_0^y\int_0^x\int_0^t\int_0^s (x-s)^{\alpha-1}(y-t)^{\beta-1}\\
&(s-\tau)^{k\alpha-1}(t-\xi)^{k\beta-1}a(\tau,\xi)d\tau d\xi dsdt\\
&= \frac{(\Gamma(\alpha)\Gamma(\beta))^k}{\Gamma(k\alpha)\Gamma(k\beta)}b^{k+1}(x,y)\\
&\int_0^y\int_0^x\int_\xi^y\int_\tau^x (x-s)^{\alpha-1}(y-t)^{\beta-1}(s-\tau)^{k\alpha-1}\\
&(t-\xi)^{k\beta-1}a(\tau,\xi)dsdtd\tau d\xi\\
&= \frac{(\Gamma(\alpha)\Gamma(\beta))^k}{\Gamma(k\alpha)\Gamma(k\beta)}b^{k+1}(x,y)\\
&\int_0^y\int_0^x [\int_\xi^y (y-t)^{\beta-1}(t-\xi)^{k\beta-1}dt]\\
&[\int_\tau^x (x-s)^{\alpha-1}(s-\tau)^{k\alpha-1}ds]a(\tau,\xi)d\tau d\xi.\end{aligned} \tag{4}$$

For the integral $\int_\tau^x (x-s)^{\alpha-1}(s-\tau)^{k\alpha-1}ds$, if we let $s = \tau + \rho(x-\tau)$, then we obtain that

$$\begin{aligned}&\int_\tau^x (x-s)^{\alpha-1}(s-\tau)^{k\alpha-1}ds\\
&= (x-\tau)^{(k+1)\alpha-1}\int_0^1 (1-\rho)^{\alpha-1}\rho^{k\alpha-1}d\rho\\
&= (x-\tau)^{(k+1)\alpha-1}B(k\alpha,\alpha)\\
&= (x-\tau)^{(k+1)\alpha-1}\frac{\Gamma(\alpha)\Gamma(k\alpha)}{\Gamma((k+1)\alpha)}.\end{aligned} \tag{5}$$

Similarly, for the integral $\int_\xi^y (y-t)^{\beta-1}(t-\xi)^{k\beta-1}dt$, if we let $t = \xi + \zeta(y-\xi)$, then we obtain that

$$\begin{aligned}&\int_{\xi}^{y}(y-t)^{\beta-1}(t-\xi)^{k\beta-1}dt\\&=(y-\xi)^{(k+1)\beta-1}\int_{0}^{1}(1-\zeta)^{\beta-1}\zeta^{k\beta-1}d\zeta\\&=(y-\xi)^{(k+1)\beta-1}B(k\beta,\beta)\\&=(y-\xi)^{(k+1)\beta-1}\frac{\Gamma(\beta)\Gamma(k\beta)}{\Gamma((k+1)\beta)}.\qquad(6)\end{aligned}$$

Combining (4)-(6) we deduce that (3) holds for $n = k+1$. So (3) is proved. On the other hand, as

$$A^n u(x,y)\le M^n\frac{(\Gamma(\alpha)\Gamma(\beta))^n}{\Gamma(n\alpha)\Gamma(n\beta)}\times$$

$$\int_0^y\int_0^x (x-s)^{n\alpha-1}(y-t)^{n\beta-1}u(s,t)dsdt.$$

Then one can see $\lim_{n\to\infty} A^n a(x,y)=0$. Therefore, the inequality (2) is proved.

Remark 2 *Theorem 1 is the 2D extension of the results in [31, Theorem 1].*

Corollary 3 *Under the conditions of Theorem 1, furthermore, suppose $a(x,y)$ is nondecreasing. Then we have the following estimate:*

$$u(x,y)\le a(x,y)\sum_{n=0}^{\infty}b^n(x,y)\frac{(\Gamma(\alpha)\Gamma(\beta)x^{\alpha}y^{\beta})^n}{\Gamma(n\alpha+1)\Gamma(n\beta+1)}.\qquad(7)$$

Proof. From (2) we obtain

$$\begin{aligned}&u(x,y)\le a(x,y)+\int_0^y\int_0^x[\sum_{n=1}^{\infty}b^n(x,y)\\&\frac{(\Gamma(\alpha)\Gamma(\beta))^n}{\Gamma(n\alpha)\Gamma(n\beta)}(x-s)^{n\alpha-1}(y-t)^{n\beta-1}a(s,t)]dsdt\\&\le a(x,y)\{1+\int_0^y\int_0^x[\sum_{n=1}^{\infty}b^n(x,y)\\&\frac{(\Gamma(\alpha)\Gamma(\beta))^n}{\Gamma(n\alpha)\Gamma(n\beta)}(x-s)^{n\alpha-1}(y-t)^{n\beta-1}]dsdt\}\\&=a(x,y)\sum_{n=0}^{\infty}b^n(x,y)\frac{(\Gamma(\alpha)\Gamma(\beta)x^{\alpha}y^{\beta})^n}{\Gamma(n\alpha+1)\Gamma(n\beta+1)}.\end{aligned}$$

So the proof is complete. □

Lemma 4 *[32] Assume that $a\ge 0, p\ge q\ge 0$, and $p\ne 0$, then for any $K>0$,*

$$a^{\frac{q}{p}}\le\frac{q}{p}K^{\frac{q-p}{p}}a+\frac{p-q}{p}K^{\frac{q}{p}}.$$

Theorem 5 *Under the conditions of Theorem 1, furthermore, suppose p, q are constants with $p\ge q>0$. If the following inequality satisfies:*

$$\begin{aligned}u^p(x,y)\le\ & a(x,y)+b(x,y)\textstyle\int_0^y\int_0^x(x-s)^{\alpha-1}\\&(y-t)^{\beta-1}u^q(s,t)dsdt,\qquad(8)\end{aligned}$$

then for $(x,y)\in D$, we have

$$\begin{aligned}&u(x,y)\le\{\widehat{a}(x,y)\\&+\int_0^y\int_0^x[\sum_{n=1}^{\infty}[\frac{q}{p}K^{\frac{q-p}{p}}b(x,y)]^n\frac{(\Gamma(\alpha)\Gamma(\beta))^n}{\Gamma(n\alpha)\Gamma(n\beta)}\\&(x-s)^{n\alpha-1}(y-t)^{n\beta-1}\widehat{a}(s,t)]dsdt\}^{\frac{1}{p}},\qquad(9)\end{aligned}$$

where $K>0$ is a constant, and

$$\widehat{a}(x,y)=a(x,y)+\frac{p-q}{p}K^{\frac{q}{p}}b(x,y)\frac{x^{\alpha}y^{\beta}}{\alpha\beta}$$

Proof. Denote the right-hand side of (8) by $v(x,y)$. Then we have

$$u^p(x,y)\le v(x,y),\ (x,y)\in D,\qquad(10)$$

and by use of Lemma 4 we obtain

$$\begin{aligned}&v(x,y)\le a(x,y)+b(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}\\&(y-t)^{\beta-1}v^{\frac{q}{p}}(s,t)dsdt\\&\le a(x,y)+b(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}\\&[\frac{q}{p}K^{\frac{q-p}{p}}v(s,t)+\frac{p-q}{p}K^{\frac{q}{p}}]dsdt\\&=a(x,y)+\frac{p-q}{p}K^{\frac{q}{p}}b(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}\\&(y-t)^{\beta-1}dsdt+\frac{q}{p}K^{\frac{q-p}{p}}b(x,y)\int_0^y\int_0^x\\&(x-s)^{\alpha-1}(y-t)^{\beta-1}v(s,t)dsdt\\&=a(x,y)+\frac{p-q}{p}K^{\frac{q}{p}}b(x,y)\frac{x^{\alpha}y^{\beta}}{\alpha\beta}+\frac{q}{p}K^{\frac{q-p}{p}}b(x,y)\\&\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}v(s,t)dsdt\\&=\widehat{a}(x,y)+\frac{q}{p}K^{\frac{q-p}{p}}b(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}\\&(y-t)^{\beta-1}v(s,t)dsdt,\\&(x,y)\in D.\end{aligned}\qquad(11)$$

Using Theorem 1 in (11) we get that for $(x,y)\in D$,

$$v(x,y)\le\widehat{a}(x,y)$$

$$+\int_0^y\int_0^x[\sum_{n=1}^{\infty}[\frac{q}{p}K^{\frac{q-p}{p}}b(x,y)]^n\frac{(\Gamma(\alpha)\Gamma(\beta))^n}{\Gamma(n\alpha)\Gamma(n\beta)}$$
$$(x-s)^{n\alpha-1}(y-t)^{n\beta-1}\widehat{a}(s,t)]dsdt. \quad (12)$$

Combining (10) and (12) we deduce the inequality (9).

Remark 6 *If we apply Corollary 3 instead of Theorem 1 to (11), then we can obtain the following inequality:*

$$u(x,y)\le\{\widehat{a}(x,y)\sum_{n=0}^{\infty}[\frac{q}{p}K^{\frac{q-p}{p}}]^nb^n(x,y)$$
$$\frac{(\Gamma(\alpha)\Gamma(\beta)x^\alpha y^\beta)^n}{\Gamma(n\alpha+1)\Gamma(n\beta+1)}\}^{\frac{1}{p}}.$$

Theorem 7 *Under the conditions of Theorem 1, furthermore, suppose $p\ge 1$ is a constant, $L\in C(R_+^3,R_+)$ is nondecreasing in every variable, and satisfies $0\le L(s,t,u)-L(s,t,v)\le A(u-v)$ for $\forall u\ge v,\ s,\ t\ge 0$, where $A>0$ is a constant. If the following inequality satisfies:*

$$u^p(x,y)\le a(x,y)+b(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}$$
$$(y-t)^{\beta-1}L(s,t,u(s,t))dsdt,\ (x,y)\in D, \quad (13)$$

then for $(x,y)\in D$, we have

$$u(x,y)\le\{\widetilde{a}(x,y)+\int_0^y\int_0^x[\sum_{n=1}^{\infty}[\frac{A}{p}K^{\frac{1-p}{p}}b(x,y)]^n$$
$$\frac{(\Gamma(\alpha)\Gamma(\beta))^n}{\Gamma(n\alpha)\Gamma(n\beta)}(x-s)^{n\alpha-1}(y-t)^{n\beta-1}\widetilde{a}(s,t)]dsdt\}^{\frac{1}{p}}, \quad (14)$$

where $K>0$ is a constant, and

$$\widetilde{a}(x,y)=a(x,y)+b(x,y)L(x,y,(\frac{p-1}{p}K^{\frac{1}{p}}))\frac{x^\alpha y^\beta}{\alpha\beta}.$$

Proof. Denote the right-hand side of (13) by $v(x,y)$. Then we have

$$u^p(x,y)\le v(x,y),\ (x,y)\in D, \quad (15)$$

and by use of Lemma 4 we obtain

$$v(x,y)\le a(x,y)+b(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}$$
$$(y-t)^{\beta-1}L(s,t,v^{\frac{1}{p}}(s,t))dsdt$$
$$\le a(x,y)+b(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}$$
$$L(s,t,(\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)+\frac{p-1}{p}K^{\frac{1}{p}}))dsdt$$
$$=a(x,y)+b(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}$$
$$[L(s,t,(\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)+\frac{p-1}{p}K^{\frac{1}{p}}))$$
$$-L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})+L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})]dsdt$$
$$\le a(x,y)+b(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}$$
$$[\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)A+L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})]dsdt$$
$$=a(x,y)+b(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}$$
$$L(s,t,\frac{p-1}{p}K^{\frac{1}{p}})dsdt+\frac{A}{p}K^{\frac{1-p}{p}}b(x,y)$$
$$\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}v(s,t)dsdt.$$

Since by the assumption L is nondecreasing in every variable, then furthermore we have

$$v(x,y)\le a(x,y)+b(x,y)L(x,y,\frac{p-1}{p}K^{\frac{1}{p}})$$
$$\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}dsdt$$
$$+\frac{A}{p}K^{\frac{1-p}{p}}b(x,y)$$
$$\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}v(s,t)dsdt$$
$$=a(x,y)+b(x,y)L(x,y,\frac{p-1}{p}K^{\frac{1}{p}})\frac{x^\alpha y^\beta}{\alpha\beta}$$
$$+\frac{A}{p}K^{\frac{1-p}{p}}b(x,y)$$
$$\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}v(s,t)dsdt$$
$$=\widetilde{a}(x,y)+\frac{A}{p}K^{\frac{1-p}{p}}b(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}$$
$$(y-t)^{\beta-1}v(s,t)dsdt,\ (x,y)\in D, \quad (16)$$

where

$$\widetilde{a}(x,y)=a(x,y)+b(x,y)L(x,y,(\frac{p-1}{p}K^{\frac{1}{p}}))\frac{x^\alpha y^\beta}{\alpha\beta}$$

Using Theorem 1 in (11) we get that for $(x,y)\in D$

$$v(x,y)\le\widetilde{a}(x,y)+\int_0^y\int_0^x[\sum_{n=1}^{\infty}[\frac{A}{p}K^{\frac{1-p}{p}}b(x,y)]^n$$
$$\frac{(\Gamma(\alpha)\Gamma(\beta))^n}{\Gamma(n\alpha)\Gamma(n\beta)}(x-s)^{n\alpha-1}(y-t)^{n\beta-1}\widetilde{a}(s,t)]dsdt. \quad (17)$$

Combining (15) and (17) we deduce the desired inequality (14).

Lemma 8 *Suppose $u(x,y)$, $a(x,y)$, $b(x,y)$ are continuous functions with $b(x,y)\ge 0$. Then*

$$u(x,y)\le a(x,y)+\int_{y_0}^{y}\int_{x_0}^{x}b(s,t)u(s,t)dsdt$$

implies

$$u(x,y)\le a(x,y)+\int_{y_0}^{y}\int_{x_0}^{x}a(s,t)b(s,t)\exp(\int_{t}^{y}\int_{x_0}^{x}b(\tau,\xi)d\tau d\xi)dsdt$$

Furthermore, if $a(x,y)$ is nondecreasing, then we have

$$u(x,y)\le a(x,y)\exp(\int_{y_0}^{y}\int_{x_0}^{x}b(s,t)dsdt).$$

Theorem 9 *Under the conditions of Theorem 1, furthermore, suppose $a(x,y)$ is nondecreasing, $h(x,y)$ is a nonnegative function locally integrable on D, $p\ge 1$ is a constant. If for $(x,y)\in D$, the following inequality satisfies:*

$$u^p(x,y)\le a(x,y)+\int_0^y\int_0^x h(s,t)u^p(s,t)dsdt+b(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}u(s,t)dsdt, \tag{18}$$

then

$$u(x,y)\le \exp(\frac{1}{p}\int_0^y\int_0^x h(s,t)dsdt)\{H_1(x,y)+\int_0^y\int_0^x[\sum_{n=1}^{\infty}H_2^n(x,y)\frac{(\Gamma(\alpha)\Gamma(\beta))^n}{\Gamma(n\alpha)\Gamma(n\beta)}(x-s)^{n\alpha-1}(y-t)^{n\beta-1}H_1(s,t)]dsdt\}^{\frac{1}{p}},\ (x,y)\in D, \tag{19}$$

provided $H_2(x,y)\le M$, where $M>0$ is a constant, and

$$H_1(x,y)=a(x,y)+\frac{p-1}{p\alpha\beta}K^{\frac{1}{p}}x^{\alpha}y^{\beta}b(x,y)\exp(\frac{1}{p}\int_0^y\int_0^x h(s,t)dsdt), \tag{20}$$

$$H_2(x,y)=\frac{1}{p}K^{\frac{1-p}{p}}b(x,y)\exp(\frac{1}{p}\int_0^y\int_0^x h(s,t)dsdt). \tag{21}$$

Proof. Denote $a(x,y)+b(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}u(s,t)dsdt$ by $v(x,y)$. Then for $(x,y)\in D$, we have

$$u^p(x,y)\le v(x,y)+\int_0^y\int_0^x h(s,t)u^p(s,t)dsdt. \tag{22}$$

Treating $u^p(x,y)$ as one whole function, a suitable application of Lemma 8 to (22) yields:

$$u^p(x,y)\le v(x,y)+\int_0^y\int_0^x v(s,t)h(s,t)\exp(\int_t^y\int_0^x h(\tau,\xi)d\tau d\xi)dsdt,\ (x,y)\in D.$$

Since $a(x,y)$, $b(x,y)$ are both nondecreasing, then $v(x,y)$ is nondecreasing, and furthermore we obtain

$$u^p(x,y)\le v(x,y)\exp(\int_0^y\int_0^x h(s,t)dsdt). \tag{23}$$

So a combination with Lemma 4 we get that

$$\begin{aligned}v(x,y)&\le a(x,y)+b(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}[v(s,t)\exp(\int_0^t\int_0^s h(\tau,\xi)d\tau d\xi)]^{\frac{1}{p}}dsdt\\
&\le a(x,y)+b(x,y)\exp(\frac{1}{p}\int_0^y\int_0^x h(s,t)dsdt)\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}v^{\frac{1}{p}}(s,t)dsdt\\
&\le a(x,y)+b(x,y)\exp(\frac{1}{p}\int_0^y\int_0^x h(s,t)dsdt)\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}[\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)+\frac{p-1}{p}K^{\frac{1}{p}}]dsdt\\
&= a(x,y)+\frac{p-1}{p}K^{\frac{1}{p}}b(x,y)\exp(\frac{1}{p}\int_0^y\int_0^x h(s,t)dsdt)\frac{x^{\alpha}y^{\beta}}{\alpha\beta}\\
&\quad+\frac{1}{p}K^{\frac{1-p}{p}}b(x,y)\exp(\frac{1}{p}\int_0^y\int_0^x h(s,t)dsdt)\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}v(s,t)dsdt\\
&= H_1(x,y)+H_2(x,y)\int_0^y\int_0^x(x-s)^{\alpha-1}(y-t)^{\beta-1}v(s,t)dsdt,\\
&(x,y)\in D.\end{aligned} \tag{24}$$

Applying Theorem 1 to (24) we get that

$$v(x,y)\le H_1(x,y)+\int_0^y\int_0^x[\sum_{n=1}^{\infty}H_2^n(x,y)$$

$$\frac{(\Gamma(\alpha)\Gamma(\beta))^n}{\Gamma(n\alpha)\Gamma(n\beta)}(x-s)^{n\alpha-1}(y-t)^{n\beta-1}H_1(s,t)]dsdt,$$
$$(x,y) \in D. \quad (25)$$

Combining (23) and (25) we obtain the desired result.

Remark 10 *If we apply Corollary 3 instead of Theorem 1 to (24), then we can obtain the following inequality:*

$$u(x,y) \le \exp(\frac{1}{p}\int_0^y \int_0^x h(s,t)dsdt)\{H_1(x,y)$$
$$\sum_{n=0}^{\infty} H_2^n(x,y)\frac{(\Gamma(\alpha)\Gamma(\beta)x^\alpha y^\beta)^n}{\Gamma(n\alpha+1)\Gamma(n\beta+1)}\}^{\frac{1}{p}},$$
$$(x,y) \in D.$$

Theorem 11 *Under the conditions of Theorem 1, furthermore, suppose $a(x,y)$ is nondecreasing, $h(x,y)$ is a nonnegative function locally integrable on D, $p \ge 1$ is a constant, $L \in C(R_+^3, R_+)$ is nondecreasing in every variable, and satisfies $0 \le L(s,t,u)-L(s,t,v) \le A(u-v)$ for $\forall u \ge v,\ s,\ t \ge 0$, where $A > 0$ is a constant. If the following inequality satisfies:*

$$u^p(x,y) \le a(x,y) + \int_0^y \int_0^x h(s,t)u^p(s,t)dsdt$$
$$+b(x,y)\int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}$$
$$L(s,t,u(s,t))dsdt,\ (x,y) \in D, \quad (26)$$

then

$$u(x,y) \le \exp(\frac{1}{p}\int_0^y \int_0^x h(s,t)dsdt)\{\widetilde{H}_1(x,y)$$
$$+\int_0^y \int_0^x [\sum_{n=1}^{\infty} \widetilde{H}_2^n(x,y)\frac{(\Gamma(\alpha)\Gamma(\beta))^n}{\Gamma(n\alpha)\Gamma(n\beta)}(x-s)^{n\alpha-1}$$
$$(y-t)^{n\beta-1}\widetilde{H}_1(s,t)]dsdt\}^{\frac{1}{p}},\ (x,y) \in D, \quad (27)$$

provided $H_2(x,y) \le M$, where $M > 0$ is a constant,

$$\widetilde{H}_1(x,y) = a(x,y) + b(x,y)\frac{x^\alpha y^\beta}{\alpha\beta}$$
$$L(x,y,\exp(\frac{1}{p}\int_0^y \int_0^x h(\tau,\xi)d\tau d\xi)\frac{p-1}{p}K^{\frac{1}{p}}), \quad (28)$$

and

$$\widetilde{H}_2(x,y) = \frac{A}{p}K^{\frac{1-p}{p}}b(x,y)\exp(\frac{1}{p}\int_0^y \int_0^x h(s,t)dsdt). \quad (29)$$

Proof. Let $v(x,y) = a(x,y) + b(x,y)\int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}L(s,t,u(s,t))dsdt$. Then

$$u^p(x,y) \le v(x,y) + \int_0^y \int_0^x h(s,t)u^p(s,t)dsdt,$$
$$(x,y) \in D. \quad (30)$$

Similar to the process of (22)-(23) we obtain that

$$u^p(x,y) \le v(x,y)\exp(\int_0^y \int_0^x h(s,t)dsdt). \quad (31)$$

Furthermore, by Lemma 4 we get that

$$v(x,y) \le a(x,y) + b(x,y)$$
$$\int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}$$
$$L(s,t,[v(s,t)\exp(\int_0^t \int_0^s h(\tau,\xi)d\tau d\xi)]^{\frac{1}{p}})dsdt$$
$$\le a(x,y) + b(x,y)$$
$$\int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}\times$$
$$L(s,t,\exp(\frac{1}{p}\int_0^t \int_0^s h(\tau,\xi)d\tau d\xi)$$
$$(\frac{1}{p}K^{\frac{1-p}{p}}v(s,t) + \frac{p-1}{p}K^{\frac{1}{p}}))dsdt$$
$$= a(x,y) + b(x,y)$$
$$\int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}\times$$
$$[L(s,t,\exp(\frac{1}{p}\int_0^t \int_0^s h(\tau,\xi)d\tau d\xi)$$
$$(\frac{1}{p}K^{\frac{1-p}{p}}v(s,t) + \frac{p-1}{p}K^{\frac{1}{p}}))$$
$$-L(s,t,\exp(\frac{1}{p}\int_0^t \int_0^s h(\tau,\xi)d\tau d\xi)\frac{p-1}{p}K^{\frac{1}{p}})$$
$$+L(s,t,\exp(\frac{1}{p}\int_0^t \int_0^s h(\tau,\xi)d\tau d\xi)\frac{p-1}{p}K^{\frac{1}{p}})]dsdt$$
$$\le a(x,y) + b(x,y)$$
$$\int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}$$
$$[A\exp(\frac{1}{p}\int_0^t \int_0^s h(\tau,\xi)d\tau d\xi)\frac{1}{p}K^{\frac{1-p}{p}}v(s,t)$$
$$+L(s,t,\exp(\frac{1}{p}\int_0^t \int_0^s h(\tau,\xi)d\tau d\xi)\frac{p-1}{p}K^{\frac{1}{p}})]dsdt$$
$$\le a(x,y) + b(x,y)$$
$$L(x,y,\exp(\frac{1}{p}\int_0^y \int_0^x h(\tau,\xi)d\tau d\xi)\frac{p-1}{p}K^{\frac{1}{p}})$$
$$\int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}dsdt+$$
$$\frac{A}{p}K^{\frac{1-p}{p}}b(x,y)\exp(\frac{1}{p}\int_0^y \int_0^x h(s,t)dsdt)$$
$$\int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}v(s,t)dsdt$$

$$= a(x,y) + b(x,y)\frac{x^\alpha y^\beta}{\alpha\beta} L(x,y,\exp(\frac{1}{p}\int_0^y\int_0^x h(\tau,\xi)d\tau d\xi)\frac{p-1}{p}K^{\frac{1}{p}}) + \frac{A}{p}K^{\frac{1-p}{p}} b(x,y)\exp(\frac{1}{p}\int_0^y\int_0^x h(s,t)dsdt) \int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}v(s,t)dsdt = \widetilde{H}_1(x,y) + \widetilde{H}_2(x,y)\int_0^y\int_0^x (x-s)^{\alpha-1} (y-t)^{\beta-1}v(s,t)dsdt,\ (x,y)\in D, \tag{32}$$

where $\widetilde{H}_1,\ \widetilde{H}_2$ are defined as in (28)-(29). Applying Theorem 1 to (32) we get that

$$v(x,y) \le \widetilde{H}_1(x,y) + \int_0^y\int_0^x [\sum_{n=1}^{\infty} \widetilde{H}_2^n(x,y) \frac{(\Gamma(\alpha)\Gamma(\beta))^n}{\Gamma(n\alpha)\Gamma(n\beta)}(x-s)^{n\alpha-1} (y-t)^{n\beta-1}\widetilde{H}_1(s,t)]dsdt, (x,y)\in D. \tag{33}$$

Combining (31) and (33) we obtain the desired result.

3 Applications

In this section, we apply the inequalities established above to research boundedness, uniqueness, and continuous dependence on the initial value and parameter for the solution to a fractional differential-integro equation. Let us consider the following fractional differential-integro equation:

$$D_y^\beta u(x,y) = a(x) + J_x^\alpha u(x,y), \tag{34}$$

with the initial condition

$$D_y^{\beta-1}u(x,y)|_{y=0} = K, \tag{35}$$

where $0 < \alpha,\ \beta < 1,\ u:\ D \to R$, and D is defined as in Theorem 1, $J_x^\alpha u$ denotes the Riemann-Liouville fractional partial integral with respect to the variable x defined by $J_x^\alpha u(x,y) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}u(s,y)ds,\ D_y^\beta u$ denotes the Riemann-Liouville fractional partial derivative with respect to the variable y defined by $D_y^\beta u(x,y) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dy}\int_0^y (y-t)^{-\beta}u(x,t)dt$.

Theorem 12 *For the IVP (34)-(35), we have the following estimate:*

$$u(x,y) \le H(x,y) + \int_0^y\int_0^x [\sum_{n=1}^{\infty}\frac{1}{\Gamma(n\alpha)\Gamma(n\beta)} (x-s)^{n\alpha-1}(y-t)^{n\beta-1}H(s,t)]dsdt, (x,y)\in D, \tag{36}$$

where $H(x,y) = |\frac{K}{\Gamma(\beta)}y^{\beta-1} + \frac{1}{\Gamma(\beta+1)}y^\beta a(x)|$.

Proof. The equivalent integral form of the IVP (34)-(35) can be denoted as follows:

$$u(x,y) = \frac{K}{\Gamma(\beta)}y^{\beta-1} + \frac{1}{\Gamma(\beta+1)}y^\beta a(x) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}u(s,t)dsdt.$$

So

$$|u(x,y)| \le |\frac{K}{\Gamma(\beta)}y^{\beta-1} + \frac{1}{\Gamma(\beta+1)}y^\beta a(x)| + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (x-s)^{\alpha-1} (y-t)^{\beta-1}|u(s,t)|dsdt = H(x,y) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (x-s)^{\alpha-1} (y-t)^{\beta-1}|u(s,t)|dsdt,\ (x,y)\in D. \tag{37}$$

Then a suitable application of Theorem 1 to (37) yields the desired result. □

Theorem 13 *The IVP (34)-(35) has at most one solution.*

Proof. Suppose the IVP (34)-(35) has two solutions $u_1(x,y),\ u_2(x,y)$. Then we have

$$u_1(x,y) = \frac{K}{\Gamma(\beta)}y^{\beta-1} + \frac{1}{\Gamma(\beta+1)}y^\beta a(x) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}u_1(s,t)dsdt, \tag{38}$$

$$u_2(x,y) = \frac{K}{\Gamma(\beta)}y^{\beta-1} + \frac{1}{\Gamma(\beta+1)}y^\beta a(x) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}u_2(s,t)dsdt, \tag{39}$$

Furthermore,

$$u_1(x,y) - u_2(x,y)$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} (u_1(s,t) - u_2(s,t))dsdt, \quad (40)$$

which implies

$$|u_1(x,y) - u_2(x,y)| \leq \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1} |u_1(s,t) - u_2(s,t)|dsdt, \quad (41)$$

By Theorem 1 we obtain $|u_1(x,y) - u_2(x,y)| \leq 0$. So $u_1(x,y) \equiv u_2(x,y)$, and the proof is complete. □

Now we research the continuous dependence on the initial value and parameter for the solution of the IVP (34)-(35).

Theorem 14 *Let $u(x,y)$ be the solution of the IVP (34)-(35), and $\overline{u}(x,y)$ be the solution of the following IVP:*

$$\begin{cases} D_y^\beta \overline{u}(x,y) = \overline{a}(x) + J_x^\alpha \overline{u}(x,y), \\ D_y^{\beta-1} \overline{u}(x,y)|_{y=0} = \overline{K}. \end{cases} \quad (42)$$

If $|a(x) - \overline{a}(x)| < \varepsilon$, $|K - \overline{K}| < \varepsilon$, where ε is arbitrarily small, then we have

$$|u(x,y) - \overline{u}(x,y)| \leq \varepsilon\{\frac{y^{\beta-1}}{\Gamma(\beta)} + \frac{y^\beta}{\Gamma(\beta+1)} + \sum_{n=1}^{\infty}\{\frac{1}{\Gamma(n\alpha)\Gamma(n\beta)} \frac{x^{n\alpha}}{n\alpha}[\frac{y^{(n+1)\beta}B(\beta+1, n\beta)}{\Gamma(\beta+1)} + \frac{y^{(n+1)\beta-1}B(\beta, n\beta)}{\Gamma(\beta)}]\}\}, \ (x,y) \in D, \quad (43)$$

where $B(\alpha,\beta)$ denotes the beta function with $B(\alpha,\beta) = \int_0^1 (1-u)^{\beta-1}u^{\alpha-1}du$.

Proof. Similar to Theorem 12, we can obtain the equivalent integral form of the IVP (42) as follows:

$$\overline{u}(x,y) = \frac{\overline{K}}{\Gamma(\beta)} y^{\beta-1} + \frac{1}{\Gamma(\beta+1)} y^\beta \overline{a}(x) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}\overline{u}(s,t)dsdt, \quad (44)$$

So we have

$$u(x,y) - \overline{u}(x,y) = \frac{K - \overline{K}}{\Gamma(\beta)} y^{\beta-1} + \frac{1}{\Gamma(\beta+1)} y^\beta (a(x) - \overline{a}(x)) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^y \int_0^x (x-s)^{\alpha-1} (y-t)^{\beta-1}(u(s,t) - \overline{u}(s,t))dsdt. \quad (45)$$

Furthermore,

$$|u(x,y) - \overline{u}(x,y)| \leq \frac{|K - \overline{K}|}{\Gamma(\beta)} y^{\beta-1} + \frac{1}{\Gamma(\beta+1)} y^\beta |a(x) - \overline{a}(x)| + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}|u(s,t) - \overline{u}(s,t)|dsdt,$$
$$\leq \varepsilon(\frac{y^{\beta-1}}{\Gamma(\beta)} + \frac{y^\beta}{\Gamma(\beta+1)}) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^y \int_0^x (x-s)^{\alpha-1}(y-t)^{\beta-1}|u(s,t) - \overline{u}(s,t)|dsdt. \quad (46)$$

Applying Theorem 1 to (46), after some basic computation we can deduce the desired result (43).

4 Conclusions

In this paper, some new Gronwall-Bellman type inequalities in two independent variables have been established. As for applications, we apply the results to research boundedness, uniqueness, and continuous dependence on the initial value and parameter for the solution to a certain fractional differential-integro equation. In fact, the motive to establish new Gronwall-Bellman type inequalities with such forms mostly comes from research for various fractional differential and integro equations. In particular, in order to fulfill analysis for the properties of solutions to fractional differential equations with nonlinear functions terms, it is necessary investigate how to establish new Gronwall-Bellman type inequalities, which are supposed to further research.

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