Existence and Longtime Behavior of Global Solutions for a Nonlinear Damping Petrovsky Equation

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Abstract: The initial boundary value problem for a class of nonlinearly damped Petrovsky equation $u_{tt} + \Delta^2 u + a(1 + |u_t|^r)u_t = b|u|^p u$ in a bounded domain is studied. The existence of global solutions for this problem is proved by constructing a stable set in $H_0^2(\Omega)$, and obtain the energy decay result through the use of an important lemma of V.Komornik. Meanwhile, under the conditions of the positive initial energy, it is proved that the solution blows up in the finite time and the lifespan estimates of solutions are also given.

Key–Words: Damped Petrovsky equation; Stable sets; Global solutions; Energy decay estimate.

1 Introduction

In this paper we are concerned with the global solvability and decay stability of initial boundary value problem for a Petrovsky equation with nonlinear dissipative and source term

$$u_{tt} + \Delta^2 u + a(1 + |u_t|^r)u_t = b|u|^p u,\ x \in \Omega, t \geq 0, \tag{1}$$

$$u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x),\ x \in \Omega, \tag{2}$$

$$u(x, t) = \frac{\partial}{\partial n} u(x, t) = 0,\ x \in \partial\Omega,\ t \geq 0, \tag{3}$$

where $a, b, r, p > 0$ are real numbers, Ω is a bounded domain of R^N with a smooth boundary $\partial\Omega$, Δ is the Laplace operator, and $\frac{\partial u}{\partial n}|_{\partial\Omega}$ indicates derivative of u in outward normal direction of $\partial\Omega$.

A. Guesmia [1] considered the equation

$$u_{tt} + \Delta^2 u + q(x)u + g(u_t) = 0,\ \ x \in \Omega, t \geq 0, \tag{4}$$

with initial boundary value conditions (2) and (3), where g is a continuous and increasing function with $g(0) = 0$, and $q : \Omega \longrightarrow [0, +\infty)$ is a bounded function, then he proved a global existence and a regularity result of the problem (4) with (2) and (3). Under suitable growth conditions on g, he also established decay results for weak and strong solutions. Precisely, in [1], A. Guesmia showed that the solution decays exponentially if g behaves like a linear function, whereas the decay is of a polynomial order otherwise. In addition, results similar to above system, coupled with a semilinear wave equation, are also established by A. Guesmia [2]. As $q(x)u + g(u_t)$ in (4) is replaced by $\Delta^2 u_t + \Delta g(\Delta u)$, M. Aassila and A. Guesmia [3] obtained an exponential decay theorem through the use of an important lemma of V. Komornik [4]. When there is no linear dissipative term au_t in equation (1), S. A. Messaoudi [5] set up an existence result of the problem (1)-(3), and showed that the solution continues to exist globally if $r \geq p$; however, it blows up in finite time if $r < p$.

Existence and uniqueness, as well as decay calculation, of global solutions and blow up of solutions for the initial boundary value problem and Cauchy problem of the nonlinearly damped wave equation $u_{tt} - \Delta u + a|u_t|^r u_t = bu|u|^p$ have been investigated by many people through various approaches and assumptive conditions [6, 7, 8, 9, 10, 11].

In this paper, the proof of global existence for the problem (1)-(3) is based on the use of the potential well theory introduced by D. H. Sattinger [12] and L. Payne and D. H. Sattinger [9]. See also G. Todorova [10, 13], for more recent work. Meanwhile, we obtain the energy decay evaluation of global solutions by applying the lemma of V. Komornik [4]. Meanwhile, it is proved that the solution blows up in the finite time and the lifespan assessment of solutions are also given.

We adopt the usual notation and convention. Let $H^m(\Omega)$ denote the Sobolev space with the norm

$$\|u\|_{H^m(\Omega)} = (\sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(\Omega)})^{\frac{1}{2}},$$

$H_0^m(\Omega)$ denotes the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity of notation, hereafter we denote by $\|\cdot\|_p$

the Lebesgue space $L^p(\Omega)$ norm, $\|\cdot\|$ denotes $L^2(\Omega)$ norm and we write equivalent norm $\|\Delta\cdot\|$ instead of $H_0^2(\Omega)$ norm $\|\cdot\|_{H_0^2(\Omega)}$. Moreover, M denotes various positive constants depending on the known constants and it may be difference in each appearance.

This paper is organized as follows: In the next section, we study the existence of global solutions for the problem (1)-(3). Section 3 is devoted to the proof of decay result. Section 4 give the proof of global nonexistence of solutions.

We conclude the introduction by stating a local existence result, which is known as a standard one (see [5]).

Proposition 1 *Suppose that $r, p > 0$ satisfy*

$$\begin{array}{l} 0 < p < +\infty,\ N \leq 4;\ 0 < p \leq \frac{4}{N-4},\ N > 4, \\ 0 < r < +\infty,\ N \leq 4;\ 0 < r \leq \frac{8}{N-4},\ N > 4, \end{array} \tag{5}$$

if $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$, then there exists $T > 0$ such that the problem (1)-(3) has a unique local solution $u(t)$ in the class

$$\begin{array}{l} u \in C([0,T); H_0^2(\Omega)), \\ u_t \in C([0,T); L^2(\Omega)) \cap L^{r+2}(\Omega \times [0,T)). \end{array} \tag{6}$$

Proposition 2 *Under the hypotheses of Proposition 1, if*

$$\sup_{0\leq t\leq T_{\max}} (\|u_t(t)\|^2 + \|\Delta u(t)\|^2) < +\infty,$$

then $T_{\max} = +\infty$, where $[0, T_{\max}]$ is the maximum time interval on which the solution $u(x,t)$ of problem (1)-(3) exists.

In fact, in [5], when we prove the existence of local solution by using the contraction mapping principle , we can also construct the following space

$$X_T = \{u \in C([0,T]; H_0^2(\Omega)),\ u_t \in C([0,T]; L^2(\Omega))\},$$

equipped with the norm

$$\|u(t)\|_{X_T} = \sup_{0\leq t\leq T} \frac{1}{2}(\|u_t(t)\|^2 + \|\Delta u(t)\|^2).$$

Let $\varepsilon > 0$, and

$$X_{\varepsilon,T} = \{u \in X_T :\ \|u\|_{X_T} \leq \varepsilon\},$$

we define a distance $d(u,v) = \|u - v\|_{X_T}$ on $X_{\varepsilon,T}$, then $X_{\varepsilon,T}$ is a complete metric space. This show that, for small enough ε, there exists a unique fixed point on $X_{\varepsilon,T}$ and T only depends on ε. Therefore, with the standard extension method of solution, we obtain $T_{\max} = +\infty$ for

$$\sup_{0\leq t\leq T_{\max}} (\|u_t(t)\|^2 + \|\Delta u(t)\|^2) < +\infty.$$

Here we omit the detailed proof of extension.

2 The Global Existence

In order to state and prove our main results, we first define the following functionals

$$I(u) = I(u(t)) = \|\Delta u(t)\|^2 - b\|u(t)\|_{p+2}^{p+2},$$

$$J(u) = J(u(t)) = \frac{1}{2}\|\Delta u(t)\|^2 - \frac{b}{p+2}\|u(t)\|_{p+2}^{p+2},$$

and according to paper [9, 10] we put

$$d = \inf\{\sup_{\lambda>0} J(\lambda u),\ u \in H_0^2(\Omega)/\{0\}\},$$

then, for the problem (1)-(3), we are able to define the stable set

$$W = \{u \in H_0^2(\Omega),\ I(u) > 0\} \cup \{0\}.$$

We denote the total energy related to (1) by

$$\begin{aligned} & E(u(t)) \\ & = \tfrac{1}{2}\|u_t(t)\|^2 + \tfrac{1}{2}\|\Delta u(t)\|^2 - \tfrac{b}{p+2}\|u(t)\|_{p+2}^{p+2} \\ & = \tfrac{1}{2}\|u_t(t)\|^2 + J(u(t)) \end{aligned}$$

for $u \in H_0^2(\Omega),\ t \geq 0$, and $E(u(0)) = \frac{1}{2}\|u_1\|^2 + J(u_0)$ is the total energy of the initial data.

Lemma 3 *Let q be a number with $q \in [2, \infty)$ as $N \leq 4$ or $q \in [2, \frac{2N}{N-4}]$ as $N > 4$. Then there is a constant B_1 depending on Ω and q such that*

$$\|u\|_q \leq B_1\|\Delta u\|,\ \forall u \in H_0^2(\Omega).$$

Lemma 4 *(Young inequality) Let k, l and ε be positive constants and $\mu,\ \nu \geq 1,\ \frac{1}{\mu} + \frac{1}{\nu} = 1$. Then one has the inequality*

$$kl \leq \frac{\varepsilon^\mu k^\mu}{\mu} + \frac{l^\nu}{\nu\varepsilon^\nu}.$$

Lemma 5 *Assume that $u \in H_0^2(\Omega)$, if (5) holds, then*

$$d = \frac{p}{2(p+2)}\frac{1}{(bC_*^{p+2})^{\frac{2}{p}}}$$

is a positive constant, where C_* is the most optimal constant in Lemma 3, namely, $C_* = \sup_{u\neq 0} \frac{\|u\|_{p+2}}{\|\Delta u\|}$.

Proof Since

$$J(\lambda u) = \frac{\lambda^2}{2}\|\Delta u\|^2 - \frac{b\lambda^{p+2}}{p+2}\|u\|_{p+2}^{p+2},$$

so, we get

$$\frac{d}{d\lambda}J(\lambda u) = \lambda\|\Delta u\|^2 - b\lambda^{p+1}\|u\|_{p+2}^{p+2}.$$

Let $\frac{d}{d\lambda}J(\lambda u) = 0$, which implies that

$$\lambda_1 = b^{-\frac{1}{p}}\Big(\frac{\|u\|_{p+2}^{p+2}}{\|\Delta u\|^2}\Big)^{-\frac{1}{p}}.$$

An elementary calculation shows that

$$\frac{d^2}{d\lambda^2}J(\lambda u)\Big|_{\lambda=\lambda_1} = \Big(\|\Delta u\|^2 - b(p+1)\|u\|_{p+2}^{p+2}\Big)\Big|_{\lambda=\lambda_1} = \|\Delta u\|^2 - (p+1)\|\Delta u\|^2 = -p\|\Delta u\|^2 < 0.$$

Hence, we have from Lemma 3 that

$$\sup_{\lambda\ge 0} J(\lambda u) = J(\lambda_1 u) = \frac{p}{2(p+2)}b^{-\frac{2}{p}}\Big(\frac{\|u\|_{p+2}}{\|\Delta u\|}\Big)^{-\frac{2(p+2)}{p}} \ge \frac{p}{2(p+2)}\frac{1}{b^{\frac{2}{p}}}C^{-\frac{2(p+2)}{p}} > 0.$$

we get from the definition of d that

$$d = \frac{p}{2(p+2)}\frac{1}{(bC_*^{p+2})^{\frac{2}{p}}} > 0.$$

Lemma 6 *Let $u(t)$ be a solution of the problem (1)-(3). Then $E(u(t))$ is a nonincreasing function for $t > 0$ and*

$$\frac{d}{dt}E(u(t)) = -a(\|u_t(t)\|_{r+2}^{r+2} + \|u_t(t)\|^2). \quad (7)$$

Proof By multiplying equation (1) by u_t, we have

$$u_{tt}u_t + \Delta^2 u u_t + a(1+|u_t|^r)u_t^2 = b|u|^p u u_t,$$

which implies that

$$\frac{1}{2}\frac{d}{dt}u_t^2 + \frac{1}{2}\frac{d}{dt}(\Delta u)^2 + a(1+|u_t|^r)u_t^2 = \frac{b}{p+2}\frac{d}{dt}(|u|^{p+2}),$$

Integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + a(\|u_t(t)\|_{r+2}^{r+2} + \|u_t(t)\|^2) = \frac{b}{p+2}\frac{d}{dt}\|u\|_{p+2}^{p+2}),$$

The above formula implies that

$$\frac{d}{dt}E(u(t)) = -a(\|u_t(t)\|_{r+2}^{r+2} + \|u_t(t)\|^2) \le 0.$$

Therefore, $E(u)$ is a nonincreasing function on t. □

Theorem 7 *Suppose that (5) holds. If $u_0 \in W, u_1 \in L^2(\Omega)$ and the initial energy satisfies $E(u(0)) < d$, then $u \in W$, for each $t \in [0, T)$.*

Proof Assume that there exists a number $t^* \in [0, T)$ such that $u(t) \in W$ on $[0, t^*)$ and $u(t^*) \notin W$. Then we have

$$I(u(t^*)) = 0,\ u(t^*) \ne 0. \quad (8)$$

Since $u(t) \in W$ on $[0, t^*)$, so it holds that

$$\begin{aligned} J(u(t)) &= \frac{1}{2}\|\Delta u(t)\|^2 - \frac{b}{p+2}\|u(t)\|_{p+2}^{p+2} \\ &\ge \frac{1}{2}\|\Delta u(t)\|^2 - \frac{1}{p+2}\|\Delta u(t)\|^2 \\ &= \frac{p}{2(p+2)}\|\Delta u(t)\|^2. \end{aligned} \quad (9)$$

It follows from $I(u(t^*)) = 0$ that

$$\begin{aligned} J(u(t^*)) &= \frac{1}{2}\|\Delta u(t^*)\|^2 - \frac{b}{p+2}\|u(t^*)\|_{p+2}^{p+2} \\ &= \frac{p}{2(p+2)}\|\Delta u(t^*)\|^2. \end{aligned} \quad (10)$$

Therefore, we get from (9) and (10) that

$$\begin{aligned} \|\Delta u(t)\|^2 &\le \tfrac{2(p+2)}{p}J(u(t)) \le \tfrac{2(p+2)}{p}E(u(t)) \\ &\le \frac{2(p+2)}{p}E(u(0)),\ \forall t \in [0, t^*]. \end{aligned} \quad (11)$$

We obtain from Lemma 4 and $E(u(0)) < d$ that

$$E(u(0)) < \frac{p}{2(p+2)}\frac{1}{(bC_*^{p+2})^{\frac{2}{p}}},$$

which implies that

$$bC_*^{p+2}\Big(\frac{2(p+2)}{p}E(u(0))\Big)^{\frac{p}{2}} < 1. \quad (12)$$

By exploiting Lemma 3, (11) and (12), we easily arrive at

$$\begin{aligned} b\|u\|_{p+2}^{p+2} &\le bC^{p+2}\|\Delta u\|^{p+2} = bC^{p+2}\|\Delta u\|^p\|\Delta u\|^2 \\ &\le bC_*^{p+2}\Big(\frac{2(p+2)}{p}E(u(0))\Big)^{\frac{p}{2}}\|\Delta u\|^2 < \|\Delta u\|^2, \end{aligned} \quad (13)$$

for all $t \in [0, t^*]$. Thus, we obtain

$$I(u(t^*)) = \|\Delta u(t^*)\|^2 - b\|u(t^*)\|_{p+2}^{p+2} > 0, \quad (14)$$

which contradicts (8). Hence, we conclude that $u(t) \in W$ on $[0, T)$. □

The following result is concerned with the existence of global solution which is not related to the parameters p and r. The result reads as follows:

Theorem 8 *Assume that (5) holds, $u(t)$ is a local solution of problem (1)-(3). If $u_0 \in W$, $u_1 \in L^2(\Omega)$ and $E(u(0)) < d$, then the solution $u(t)$ is a global solution of the problem (1)-(3).*

Proof We obtain from (11) that

$$\begin{aligned} d &> E(u(0)) \geq E(u(t)) = \frac{1}{2}\|u_t(t)\|^2 + J(u(t)) \\ &\geq \frac{1}{2}\|u_t(t)\|^2 + \frac{p}{2(p+2)}\|\Delta u\|^2 \\ &\geq \frac{p}{2(p+2)}(\|u_t(t)\|^2 + \|\Delta u\|^2), \end{aligned} \tag{15}$$

Therefore

$$\|u_t(t)\|^2 + \|\Delta u\|^2 \leq \frac{2(p+2)}{p} d < +\infty.$$

It follows from Proposition 2 that $u(t)$ is the global solution of problem (1)-(3). □

Moreover, we have the other global existence result which is related to parameters p and r. The following theorem shows that the solutions obtained in Proposition 1 is a global solutions if $p \leq r$.

Theorem 9 *Assume that (5) holds. If $p \leq r$, then the local solution furnished in Proposition 1 is a global solution of the problem (1)-(3) and T may be taken arbitrarily large.*

Proof Let u be a solution to the problem (1)-(3) defined on $[0, T]$ which is obtained in Proposition 1. We define

$$E_1(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{b}{p+2}\|u\|_{p+2}^{p+2}, \tag{16}$$

then

$$E_1(t) = E(t) + \frac{2b}{p+2}\|u\|_{p+2}^{p+2}. \tag{17}$$

Our aim is to prove that the following inequality holds:

$$\begin{aligned} &\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{b}{p+2}\|u\|_{p+2}^{p+2} \\ &+a\int_0^t (\|u_t(s)\|^2 + \|u_t(s)\|_{r+2}^{r+2})ds \\ &\leq C_T, \end{aligned} \tag{18}$$

for $t \in [0, T]$, where C_T depends on $\|\Delta u_0\|^2$ and $\|u_1\|^2$.

We get from Lemma 3 and (16) that

$$\begin{aligned} &\frac{b}{p+2}\|u\|_{p+2}^{p+2} \leq \frac{bB_1^{p+2}}{p+2}\|\Delta u\|^{p+2} \\ &= \tfrac{bB_1^{p+2}}{p+2}(\|\Delta u\|^2)^{\frac{p+2}{2}} \leq C_1 E_1(t)^{\frac{p+2}{2}}, \end{aligned} \tag{19}$$

where $C_1 = \frac{b(\sqrt{2}\ B_1)^{p+2}}{p+2}$.

It follows from (16) that

$$\frac{b}{p+2}\|u\|_{p+2}^{p+2} \leq E_1(t). \tag{20}$$

We obtain from (7) and (17) that

$$\begin{aligned} &E_1(t) + a\int_0^t (\|u_t(s)\|^2 + \|u_t(s)\|_{r+2}^{r+2})ds \\ &\leq E_1(0) + 2b(p+2)\int_0^t\int_\Omega |u|^{p+1}|u_t|dxds. \end{aligned} \tag{21}$$

In what following, we are going to estimate the last term in (21). Putting $Q_t = \Omega \times [0, t]$ and

$$Q_1 = \{(x, s) \in Q_t :\ |u(x, s)| \leq 1\},$$
$$Q_2 = \{(x, s) \in Q_t :\ |u(x, s)| \geq 1\}.$$

Then

$$\begin{aligned} I &= \int_0^t\int_\Omega |u|^{p+1}|u_t|dxds \\ &= \int_{Q_1} |u|^{p+1}|u_t|dxds \\ &+ \int_{Q_2} |u|^{p+1}|u_t|dxds = I_1 + I_2. \end{aligned} \tag{22}$$

Next we deal with I_1 and I_2 in (22). It is easy to see from Lemma 3 and (16) that

$$\begin{aligned} I_1 &\leq \int_{Q_1} |u_t|dxds \\ &\leq \delta|Q_t| + C_\delta\int_{Q_1}|u_t|^2dxds \\ &\leq \delta|Q_t| + C_\delta\int_0^t E_1(s)ds, \end{aligned} \tag{23}$$

for some $\delta > 0$ and in which $|Q_t|$ denotes the Lebesgue measure of Q_t.

Let $\alpha = \frac{r-p}{r+2}$, then by $p \leq r$, we have $\alpha \geq 0$ and $(p+\alpha+1)\frac{r+2}{r+1} = p+2$. Since $|u| \geq 1$ on Q_2, then we get from Lemma 4 and (20) that

$$\begin{aligned} I_2 &\leq \int_{Q_2} |u|^{p+\alpha+1}|u_t|dxds \\ &\leq \varepsilon\int_{Q_2}|u_t|^{r+2}dxds + C_\varepsilon\int_{Q_2}|u|^{p+2}dxds \\ &\leq \varepsilon\int_0^t \|u_t\|_{r+2}^{r+2}ds + \frac{(p+2)C_\varepsilon}{b}\int_0^t E_1(s)ds, \end{aligned} \tag{24}$$

for any $\varepsilon > 0$. We obtain from (22), (23) and (24) that

$$\begin{aligned} I &\leq \delta|Q_t| + \varepsilon\int_0^t \|u_t\|_{r+2}^{r+2}ds \\ &+(C_\delta + \frac{(p+2)C_\varepsilon}{b})\int_0^t E_2(s)ds. \end{aligned} \tag{25}$$

By choosing $\varepsilon > 0$ small enough such that $\varepsilon < \frac{a}{2b}$, then it follows from (21) and (25) that

$$E_1(t) + C_2 \int_0^t (\|u(s)_t\|^2 + \|u_t(s)\|_{r+2}^{r+2})ds$$
$$+ \leq E_1(0) + 2b(p+2)\delta|Q_t| + C_3 \int_0^t E_1(s)ds, \tag{26}$$

for some positive constants $C_2 = a - 2b\varepsilon > 0$ and $C_3 = 2bC_\delta + 2(p+2)C_\varepsilon > 0$.

It follows from (26) and Gronwall's inequality that

$$E_1(t) \leq (E_1(0) + 2b\delta|Q_t|)e^{C_3 t}. \tag{27}$$

Finally, we infer from (26) and (27) that

$$E_1(t) + C_2 \int_0^t (\|u(s)_t\|^2 + \|u_t(s)\|_{r+2}^{r+2})ds$$
$$\leq C_T(E_1(0) + 2b(p+2)\delta|Q_T|), \tag{28}$$

for all $0 < t \leq T$, where T is arbitrary. Thus (18) follows from (16) and (28). Therefore, the conclusion in Theorem 9 is valid according to (18) and the standard continuation argument [7, 14]. Thus, the proof of Theorem 9 is now complete. □

3 Decay Estimate

The following two lemmas play an important role in studying the decay result of global solutions for problem (1)-(3).

Lemma 10 *([4]) Let $F : R^+ \to R^+$ be a nonincreasing function and assume that there are two constants $\beta \geq 1$ and $A > 0$ such that*

$$\int_S^{+\infty} F(t)^{\frac{\beta+1}{2}} dt \leq AF(S),\ 0 \leq S < +\infty,$$

then $F(t) \leq CF(0)(1+t)^{-\frac{2}{\beta-1}}$, $\forall t \geq 0$, if $\beta > 1$, and $F(t) \leq CF(0)e^{-\omega t}$, $\forall t \geq 0$, if $\beta = 1$, where C and ω are positive constants independent of $F(0)$.

Lemma 11 *If the hypotheses in Theorem 7 hold, then*

$$b\|u(t)\|_{p+2}^{p+2} \leq (1-\theta)\|\Delta u(t)\|^2,\ \forall t \in [0, +\infty), \tag{29}$$

where

$$\theta = 1 - bC_*^{p+2}\Big(\frac{2(p+2)}{p}E(u(0))\Big)^{\frac{p}{2}} > 0.$$

Moreover, we have

$$I(u(t)) \geq \theta\|\Delta u(t)\|^2 \geq \frac{\theta}{1-\theta} b\|u(t)\|_{p+2}^{p+2},\ \forall t \in [0, +\infty).$$

Proof We get from Lemma 3 and (15)

$$\begin{aligned} b\|u\|_{p+2}^{p+2} &\leq bC^{p+2}\|\Delta u\|^{p+2} \\ &= bC^{p+2}\|\Delta u\|^p\|\Delta u\|^2 \\ &\leq bC_*^{p+2}\Big(\frac{2(p+2)}{p}E(u(0))\Big)^{\frac{p}{2}}\|\Delta u\|^2. \end{aligned} \tag{30}$$

Let

$$\theta = 1 - bC_*^{p+2}\Big(\frac{2(p+2)}{p}E(u(0))\Big)^{\frac{p}{2}}.$$

Then we have from (12) that $0 < \theta < 1$. Thus, it follows from (30) that

$$b\|u\|_{p+2}^{p+2} \leq (1-\theta)\|\Delta u\|^2. \tag{31}$$

Meanwhile, we conclude from (31) that

$$\begin{aligned} I(u) &= \|\Delta u\|^2 - b\|u\|_{p+2}^{p+2} \\ &\geq \|\Delta u\|^2 - (1-\theta)\|\Delta u\|^2 \\ &= \theta\|\Delta u\|^2 \geq \frac{\theta b}{1-\theta}\|u\|_{p+2}^{p+2}. \end{aligned}$$

This complete the proof of Lemma 11. □

Theorem 12 *If the hypotheses of Theorem 8 hold, then the global solution of problem (1)-(3) has the following energy decay evaluation*

$$E(t) \leq M(1+t)^{-\frac{2}{r}},$$

where $E(t) = E(u(t))$, $M > 0$ is a constant depending on initial energy $E(0)$.

Proof Let $E(t) = E(u(t))$, then multiplying by $E(t)^{\frac{r}{2}}u$ both sides of the equation (1) and integrating over $\Omega \times [S, T]$, we obtain that

$$\int_S^T \int_\Omega E(t)^{\frac{r}{2}} u[u_{tt} + \Delta^2 u + a(1+|u_t|^r)u_t]dxdt$$
$$- \int_S^T \int_\Omega E(t)^{\frac{r}{2}} u[bu|u|^p]dxdt = 0 \tag{32}$$

where $0 \leq S < T < +\infty$.

Since

$$\int_S^T \int_\Omega E(t)^{\frac{r}{2}} uu_{tt}dxdt = \int_\Omega E(t)^{\frac{r}{2}} uu_t dx\Big|_S^T$$
$$- \int_S^T \int_\Omega E(t)^{\frac{r}{2}}|u_t|^2 dxdt$$
$$-\frac{r}{2}\int_S^T \int_\Omega E(t)^{\frac{r-2}{2}} E'(t)uu_t dxdt, \tag{33}$$

So, substituting the formula (33) into the left-hand side of (32), we get

$$\int_S^T\int_\Omega E(t)^{\frac{r}{2}}(|u_t|^2+|\Delta u|^2-b|u|^{p+2})dxdt$$
$$=\int_S^T\int_\Omega E(t)^{\frac{r}{2}}[2|u_t|^2-a(1+|u_t|^r)u_tu]dxdt$$
$$+\frac{r}{2}\int_S^T\int_\Omega E(t)^{\frac{r-2}{2}}E'(t)uu_tdxdt$$
$$-\int_\Omega E(t)^{\frac{r}{2}}uu_tdx\Big|_S^T. \tag{34}$$

It follows from Lemma 11, the definition of $E(t)$ and $0<\theta<1$ that

$$\int_S^T\int_\Omega E(t)^{\frac{r}{2}}(|u_t|^2+|\Delta u|^2-b|u|^{p+2})dxdt$$
$$=\int_S^T E(t)^{\frac{r}{2}}(\|u_t\|^2+I(u(t)))dt$$
$$\geq\int_S^T E(t)^{\frac{r}{2}}(\|u_t\|^2+\theta\|\Delta u\|^2)dt$$
$$\geq 2\theta\int_S^T E(t)^{\frac{r}{2}}(\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\Delta u\|^2)dt$$
$$\geq 2\theta\int_S^T E(t)^{\frac{r+2}{2}}dt, \tag{35}$$

we have from Lemma 3 and (15) that

$$\left|\frac{r}{2}\int_S^T\int_\Omega E(t)^{\frac{r-2}{2}}E'(t)uu_tdxdt\right|$$
$$\leq\frac{r}{2}\int_S^T E(t)^{\frac{r-2}{2}}|E'(t)|\Big(\frac{1}{2}\|u\|^2+\frac{1}{2}\|u_t\|^2\Big)dt$$
$$\leq-\frac{r}{2}\int_S^T E(t)^{\frac{r-2}{2}}E'(t)\Big(\frac{C^2}{2}\|\Delta u\|^2+\frac{1}{2}\|u_t\|^2\Big)dt$$
$$\leq-\frac{r}{2}\int_S^T E(t)^{\frac{r-2}{2}}E'(t)\times$$
$$\times\Big(\frac{(p+2)C^2}{p}\frac{p}{2(p+2)}\|\Delta u\|^2+\frac{1}{2}\|u_t\|^2\Big)dt$$
$$=-\frac{r}{r+2}\max\Big(\frac{(p+2)C^2}{p},1\Big)E(t)^{\frac{r+2}{2}}\Big|_S^T$$
$$\leq ME(S)^{\frac{r+2}{2}}, \tag{36}$$

Similarly, we have

$$\left|-\int_\Omega E(t)^{\frac{r}{2}}uu_tdx\Big|_S^T\right|$$
$$\leq\max\Big(\frac{(p+2)C^2}{p},1\Big)E(t)^{\frac{r+2}{2}}\Big|_S^T \tag{37}$$
$$\leq ME(S)^{\frac{r+2}{2}},$$

Substituting the form (35), (36) and (37) into (34), we conclude

$$2\theta\int_S^T E(t)^{\frac{r+2}{2}}dt\leq\int_S^T\int_\Omega E(t)^{\frac{r}{2}}[2|u_t|^2$$
$$-a(1+|u_t|^r)u_tu]dxdt+ME(S)^{\frac{r+2}{2}}. \tag{38}$$

We get from Young inequality and (7)

$$2\int_S^T\int_\Omega E(t)^{\frac{r}{2}}|u_t|^2dxdt$$
$$\leq\int_S^T\int_\Omega(\varepsilon_1E(t)^{\frac{r+2}{2}}+M(\varepsilon_1)|u_t|^{r+2})dxdt$$
$$\leq M\varepsilon_1\int_S^T E(t)^{\frac{r+2}{2}}dt$$
$$+M(\varepsilon_1)\int_S^T(\|u_t\|_{r+2}^{r+2}+\|u_t\|^2)dt$$
$$=M\varepsilon_1\int_S^T E(t)^{\frac{r+2}{2}}dt-\frac{M(\varepsilon_1)}{a}(E(T)-E(S))$$
$$\leq M\varepsilon_1\int_S^T E(t)^{\frac{r+2}{2}}dt+ME(S). \tag{39}$$

From Young inequality, Lemma 3, (7) and (15), we receive that

$$-a\int_S^T\int_\Omega E(t)^{\frac{r}{2}}uu_t|u_t|^rdxdt$$
$$\leq a\int_S^T E(t)^{\frac{r}{2}}(\varepsilon_2\|u\|_{r+2}^{r+2}+M(\varepsilon_2)\|u_t\|_{r+2}^{r+2})dt$$
$$\leq aC^{r+2}\varepsilon_2E(0)^{\frac{r}{2}}\int_S^T\|\Delta u\|^{r+2}dt$$
$$+aM(\varepsilon_2)E(S)^{\frac{r}{2}}\int_S^T(\|u_t\|_{r+2}^{r+2}+\|u_t\|^2)dt$$
$$=aC^{r+2}\varepsilon_2E(0)^{\frac{r}{2}}\int_S^T\Big(\frac{2(p+2)}{p}E(t)\Big)^{\frac{r+2}{2}}dt$$
$$+M(\varepsilon_2)E(S)^{\frac{r}{2}}(E(S)-E(T))$$
$$\leq aC^{r+2}\varepsilon_2E(0)^{\frac{r}{2}}\Big(\frac{2(p+2)}{p}\Big)^{\frac{r+2}{2}}\times$$
$$\times\int_S^T E(t)^{\frac{r+2}{2}}dt+M(\varepsilon_2)E(S)^{\frac{r+2}{2}}, \tag{40}$$

and

$$-a\int_S^T\int_\Omega E(t)^{\frac{r}{2}}uu_tdxdt$$
$$\leq a\int_S^T E(t)^{\frac{r}{2}}\Big(\frac{1}{2}\|u\|^2+\frac{1}{2}\|u_t\|^2\Big)dt$$
$$\leq a\int_S^T E(t)^{\frac{r}{2}}\Big(\frac{C^2}{2}\|\Delta u\|^2+\frac{1}{2}\|u_t\|^2\Big)dt$$

$$
\begin{aligned}
&= a\int_S^T E(t)^{\frac{r}{2}}\left(\frac{(p+2)C^2}{p}\cdot\frac{p}{2(p+2)}\|\Delta u\|^2\right)dt\\
&+\int_S^T E(t)^{\frac{r}{2}}\left(\frac{1}{2}\|u_t\|^2\right)dt\\
&\le a\max\left(\frac{(p+2)C^2}{p},1\right)\int_S^T E(t)^{\frac{r+2}{2}}dt\\
&\le ME(S)^{\frac{r+2}{2}}
\end{aligned}
\tag{41}
$$

where $M(\varepsilon_1)$ and $M(\varepsilon_2)$ are positive constants depending on ε_1 and ε_2.

Choosing small enough ε_1 and ε_2, such that

$$
M\varepsilon_1 + aE(0)^{\frac{r}{2}}\left(\frac{2(p+2)}{p}C^2\right)^{\frac{r+2}{2}}\varepsilon_2 < 2\theta,
$$

then, substituting (39), (40) and (41) into (38), we get

$$
\begin{aligned}
\int_S^T E(t)^{\frac{r+2}{2}}dt &\le ME(S) + ME(S)^{\frac{r+2}{2}}\\
&\le M(1+E(0))^{\frac{r}{2}}E(S).
\end{aligned}
$$

Therefore, we have from Lemma 10 that

$$
E(t) \le M(E(0))(1+t)^{-\frac{r}{2}},\ t\in[0,+\infty).
$$

where $M(E(0)) > 0$ is a constant depending on $E(0)$.

The proof of Theorem 12 is thus finished. □

4 Blow-up of Solution

In this section, under suitable conditions and the positive initial energy, we shall discuss the blow-up property of the problem (1)-(3) and give the lifespan calculations of solutions.

We observe from the definition of $E(t)$ that

$$
E(t) \ge \frac{1}{2}\|\Delta u(t)\|^2 - \frac{b}{p+2}\|u(t)\|_{p+2}^{p+2}, \tag{42}
$$

for $u\in H_0^2(\Omega), t\ge 0$.

By (5) and Lemma 3, we get that

$$
\|u\|_{p+2} \le B_1\|\Delta u\|, \tag{43}
$$

where B_1 is the optimal Sobolev's constant from $H_0^2(\Omega)$ to $L^{p+2}(\Omega)$.

We have from (42) and (43) that

$$
\begin{aligned}
&E(t) \ge \frac{1}{2}\|\Delta u(t)\|^2 - \frac{bB_1^{p+2}}{p+2}(\|\Delta u(t)\|^2)^{\frac{p+2}{2}}\\
&= Q(\|\Delta u(t)\|^2),
\end{aligned}
\tag{44}
$$

where

$$
Q(\lambda) = \frac{1}{2}\lambda^2 - \frac{bB_1{}^{p+2}}{p+2}\lambda^{p+2}.
$$

Therefore, we get

$$
Q'(\lambda) = \lambda - bB_1{}^{p+2}\lambda^{p+1},
$$

$$
Q''(\lambda) = 1-(p+1)bB_1{}^{p+2}\lambda^p.
$$

Let $Q'(\lambda) = 0$, which implies that $\lambda_1 = (bB_1{}^{p+2})^{-\frac{1}{p}}$. As $\lambda=\lambda_1$, an elementary calculation shows that $Q''(\lambda) = -p < 0$. Thus, $Q(\lambda)$ has the maximum at λ_1 and the maximum value is

$$
\begin{aligned}
h = Q(\lambda_1) &= \frac{p}{2(p+2)}\left[bB_1{}^{p+2}\right]^{-\frac{2}{p}}\\
&= \frac{p}{2(p+2)}\lambda_1^2.
\end{aligned}
\tag{45}
$$

Applying the idea of E. Vitillaro [11] and S. T. Wu [15], we have the following lemma.

Lemma 13 *Assume that $u_0\in H_0^2(\Omega),\ u_1\in L^2(\Omega)$. Let u be a solution of (1)-(3) with the initial data energy satisfying $0 < E(0) < h$ and $\|\Delta u_0\| > \lambda_1$, then there exists $\lambda_2 > \lambda_1$ such that*

$$
\|\Delta u(t)\|^2 \ge \lambda_2^2, \tag{46}
$$

for $t>0$.

The blow-up result of solution for the problem (1) reads as follows:

Theorem 14 *Assume that (5) holds, and that $u_0\in H_0^2(\Omega),\ u_1\in L^2(\Omega)$. Under the condition $p>r$, if $0<E(0)<h$ and $\|\Delta u_0\|>\lambda_1$, then the local solution of the problem (1)-(3), which is obtained in Proposition 1, blows up at a finite time. We remark that the lifespan T^* is estimated by $T* = T_{\max} = \frac{(1-\beta)G(0)^{\frac{\beta}{1-\beta}}}{\beta C_{11}}$, where $G(t)$ will be given in (63), C_{11} and β are some positive constant given in the following proof.*

Proof Let

$$
H(t) = d - E(t),\ t\ge 0, \tag{47}
$$

where $d = \frac{E(0)+h}{2}$. We see from (7) in Lemma 6 that $H'(t)\ge 0$. Thus we obtain

$$
H(t) \ge H(0) = d - E(0) > 0,\ t\ge 0. \tag{48}
$$

Let

$$
F(t) = \int_\Omega u u_t dx. \tag{49}
$$

By differentiating both sides of (49) on t, we get from the equation in (1) that

$$\begin{aligned}F'(t) &= \|u_t\|^2 - \|\Delta u\|^2 + b\|u\|_{p+2}^{p+2} \\ &-a\int_\Omega (u_t + |u_t|^r u_t)udx.\end{aligned} \tag{50}$$

We have from the definition of $E(t)$ and (50) that

$$\begin{aligned}&F'(t) \geq \frac{p+4}{2}\|u_t\|^2 + \frac{p}{2}\|\Delta u(t)\|^2 \\ &-a\int_\Omega (u_t + |u_t|^r u_t)udx \\ &+(p+2)H(t) - (p+2)d,\end{aligned} \tag{51}$$

We obtain from Lemma 13 that

$$\begin{aligned}&\frac{p}{2}\|\Delta u(t)\|^2 - (p+2)d \\ &= \frac{p(\lambda_2^2 - \lambda_1^2)}{2\lambda_2^2}\|\Delta u(t)\|^2 \\ &+\frac{p\lambda_1^2}{2}\cdot\frac{\|\Delta u(t)\|^2}{\lambda_2^2} - (p+2)d \\ &\geq \frac{p(\lambda_2^2 - \lambda_1^2)}{2\lambda_2^2}\|\Delta u(t)\|^2 + \frac{p\lambda_1^2}{2} - (p+2)d.\end{aligned} \tag{52}$$

By Lemma 13, we have that

$$\frac{\lambda_2^2 - \lambda_1^2}{\lambda_2^2} > 0, \tag{53}$$

and by (45) and (48), we see that

$$\begin{aligned}&\lambda_1^2 - (p+2)d \\ &= (p+2)(h-d) \\ &= (p+2)(d - E(0)) > 0.\end{aligned} \tag{54}$$

Combining (51)-(54), we see that

$$\begin{aligned}&F'(t) \geq \frac{p+4}{2}\|u_t\|^2 + \frac{p(\lambda_2^2 - \lambda_1^2)}{2\lambda_2^2}\|\Delta u(t)\|^2 \\ &-a\int_\Omega (u_t + |u_t|^r u_t)udx + (p+2)H(t).\end{aligned} \tag{55}$$

On the other hand, we have from Hölder inequality that

$$a\left|\int_\Omega |u_t|^r u_t u dx\right| \leq C_4\|u\|_{p+2}^{1-\frac{p+2}{r+2}}\|u\|_{p+2}^{\frac{p+2}{r+2}}\|u_t\|_{r+2}^{r+1}, \tag{56}$$

where $C_4 = a|\Omega|^{\frac{p-r}{(r+2)(p+2)}}$ in which $|\Omega|$ denotes the Lebesgue measure of Ω.

By $d = \frac{E(0)+h}{2}$, we see that $d < h$. Therefore, we get from (42), (47) and Lemma 13 that

$$\begin{aligned}&H(t) \leq d - \frac{1}{2}\|\Delta u(t)\|^2 + \frac{b}{p+2}\|u\|_{p+2}^{p+2} \\ &\leq h - \frac{1}{2}\lambda_1^2 + \frac{b}{p+2}\|u\|_{p+2}^{p+2}.\end{aligned} \tag{57}$$

By (45), we have

$$h - \frac{1}{2}\lambda_1^2 = -\frac{2}{p}h < 0, \tag{58}$$

so, we have from (48), (57) and (58) that

$$0 < H(0) \leq H(t) \leq \frac{b}{p+2}\|u\|_{p+2}^{p+2},\ t \geq 0. \tag{59}$$

We obtain from (56) and (59) that

$$\begin{aligned}&a\left|\int_\Omega |u_t|^r u_t u dx\right| \\ &\leq C_5 H(t)^{\frac{1}{p+2}-\frac{1}{r+2}}\|u\|_{p+2}^{\frac{p+2}{r+2}}\|u_t\|_{r+2}^{r+1},\end{aligned} \tag{60}$$

where $C_5 = (\frac{p+2}{b})^{\frac{r-p}{r+2}}C_4$.

We get from (47), Lemma 4, Lemma 6 and (60) that

$$\begin{aligned}&a\left|\int_\Omega |u_t|^r u_t u dx\right| \\ &\leq C_5[\varepsilon^{r+2}\|u\|_{p+2}^{p+2} + \varepsilon^{-\frac{r+2}{r+1}}H'(t)]H(t)^{-\alpha},\end{aligned} \tag{61}$$

where $\alpha = \frac{1}{r+2} - \frac{1}{p+2}$, $\varepsilon > 0$.

Let $0 < \beta < \alpha$, then we have from (48) and (61) that

$$\begin{aligned}&a\left|\int_\Omega |u_t|^r u_t u dx\right| \\ &\leq C_5[\varepsilon^{r+2}H(0)^{-\alpha}\|u\|_{p+2}^{p+2} \\ &+\varepsilon^{-\frac{r+2}{r+1}}H(0)^{\beta-\alpha}H(t)^{-\beta}H'(t)].\end{aligned} \tag{62}$$

Now, we define $G(t)$ as follows.

$$G(t) = H(t)^{1-\beta} + \rho F(t),\ t \geq 0. \tag{63}$$

where ρ is a positive constant to be determined later. By differentiating (63), then we see from (55) and (62) that

$$G'(t) = (1-\beta)H(t)^{-\beta}H'(t) + \rho F'(t)$$
$$\geq [1-\beta-\rho C_{14}\epsilon^{-\frac{m+2}{m}}H(0)^{\beta-\alpha}]H(t)^{-\beta}H'(t)$$
$$+a_1\rho\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2}[l\|\Delta u(t)\|^2 + (g\circ\Delta u)(t)]$$
$$+\rho[\frac{p+4}{2}\|u_t\|^2 + (p+2)H(t)]$$
$$\geq -C_5\rho\varepsilon^{m+2}H(0)^{-\alpha}\|u\|_{p+2}^{p+2}. \quad (64)$$

Letting $k = \min\{\frac{p+2}{2},\ a_1 l\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2}\}$ and decomposing $\rho(p+2)H(t)$ in (64) by

$$\rho(p+2)H(t) = 2k\rho H(t) + \rho(p+2-2k)H(t). \quad (65)$$

Combining (47), (64) and (65), we obtain that

$$G'(t) \geq [1-\beta-\rho C_{14}\epsilon^{-\frac{r+2}{r+1}}H(0)^{\beta-\alpha}]H(t)^{-\beta}H'(t)$$
$$+\rho\Big(\frac{p+4}{2}-k\Big)\|u_t\|^2 + \rho(p+2-2k)H(t)$$
$$+\Big[\frac{2kb}{p+2} - C_{14}\varepsilon^{r+2}H(0)^{-\alpha}\Big]\rho\|u\|_{p+2}^{p+2}$$
$$+\Big[\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2} - k\Big]\rho\|\Delta u\|^2. \quad (66)$$

Choosing $\varepsilon > 0$ small enough such that $\frac{2kb}{p+2} - C_5\varepsilon^{r+2}H(0)^{-\alpha} \geq \frac{kb}{p+2}$ and $0 < \rho < \frac{1-\beta}{C_5}\varepsilon^{\frac{r+2}{r+1}}H(0)^{\alpha-\beta}$, then we have from (66) that

$$G'(t) \geq C_6\rho\Big[\|u_t\|^2 + \|\Delta u\|^2 + \|u\|_{p+2}^{p+2} + H(t) + (g\circ\Delta u)(t)\Big], \quad (67)$$

where

$$C_6 = \min\Big\{\frac{p+4}{2}-k,\ p+2-2k,\ \frac{kb}{p+2},\ \frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2}-k\Big\}.$$

Therefore, $G(t)$ is a nondecreasing function for $t \geq 0$. Letting ρ in (63) be small enough, then we get $G(0) > 0$. Consequently, we obtain that $G(t) \geq G(0) > 0$ for $t \geq 0$.

Since $0 < \beta < \alpha < 1$, it is evident that $1 < \frac{1}{1-\beta} < \frac{1}{1-\alpha}$. We deduce from (49) and (63) that

$$G(t)^{\frac{1}{1-\beta}} \leq 2^{\frac{1}{1-\beta}-1}\Big[H(t) + \Big(\rho\int_\Omega uu_t dx\Big)^{\frac{1}{1-\beta}}\Big]. \quad (68)$$

On the other hand, for $p > 0$, we have from Hölder inequality and Lemma 4 that

$$\Big(\rho\int_\Omega uu_t dx\Big)^{\frac{1}{1-\beta}} \leq C_7\|u_t\|^{\frac{1}{1-\beta}}\|u\|_{p+2}^{\frac{1}{1-\beta}} \leq C_8\Big(\|u\|_{p+2}^{\frac{\mu}{1-\beta}} + \|u_t\|^{\frac{\nu}{1-\beta}}\Big), \quad (69)$$

where $C_7 = \rho^{\frac{1}{1-\beta}}|\Omega|^{\frac{p+1}{(1-\beta)(p+2)}}$, $\frac{1}{\mu}+\frac{1}{\nu} = 1$, and C_8 is a positive constant depending on the known constants C_7, μ and ν.

Let $0 < \beta < \min\{\alpha, \frac{1}{2}-\frac{1}{p+2}\}$, $\nu = 2(1-\beta)$, then $\frac{\mu}{1-\beta} = \frac{2}{1-2\beta} < p+2$. It follows from (59) that

$$\Big(\frac{b}{(p+2)H(0)}\Big)^{\frac{1}{p+2}}\|u\|_{p+2} \geq 1. \quad (70)$$

Thus, we get from (70) that

$$\|u\|_{p+2}^{\frac{\mu}{1-\beta}} = \|u\|_{p+2}^{\frac{2}{1-2\beta}} = \|u\|_{p+2}^{\frac{2}{1-2\beta}-(p+2)}\|u\|_{p+2}^{p+2} \leq \Big(\frac{b}{(p+2)H(0)}\Big)^{1-\frac{2}{(p+2)(1-2\beta)}}\|u\|_{p+2}^{p+2}. \quad (71)$$

We obtain from (69) and (71) that

$$\Big(\rho\int_\Omega uu_t dx\Big)^{\frac{1}{1-\beta}} \leq C_{18}(\|u_t\|^2 + \|u\|_{p+2}^{p+2}), \quad (72)$$

where $C_9 = C_8\max\{1,\ (\frac{b}{(p+2)H(0)})^{1-\frac{2}{(p+2)(1-2\beta)}}\}$.

Combining (68) and (72), we find that

$$G(t)^{\frac{1}{1-\beta}} \leq C_{10}\Big[\|u_t\|^2 + \|u\|_{p+2}^{p+2} + H(t)\Big], \quad (73)$$

where $C_{10} = 2^{\frac{1}{1-\beta}-1}\max\{1,\ C_9\}$.

We obtain from (67) and (73) that

$$G'(t) \geq C_{11}G(t)^{\frac{1}{1-\beta}},\ t \geq 0, \quad (74)$$

where $C_{11} = \frac{C_6\rho}{C_{10}}$. Integrating both sides of (74) over $[0,t]$ yields that

$$G(t) \geq \Big(G(0)^{\frac{\beta}{\beta-1}} - \frac{\beta C_{11}}{1-\beta}t\Big)^{-\frac{\beta}{1-\beta}}.$$

Noting that $G(0) > 0$, then there exists $T* = T_{\max} = \frac{(1-\beta)G(0)^{\frac{\beta}{1-\beta}}}{\beta C_{11}}$ such that $G(t) \to +\infty$ as $t \to +\infty$. Namely, the solutions of the problem (1)-(3) blow up in finite time. □

5 Conclusion

In this paper, the initial boundary value problem for a class of nonlinearly damped Petrovsky equation in a bounded domain is considered. At first, the existence of global solutions which is not related to the parameters p and r is proved by constructing a stable set in $H_0^2(\Omega)$. Moreover, we have the other global existence result which is related to parameters p and r, i.e. $p \leq r$. Secondly, we obtain the energy decay estimate through the use of an important lemma of V.Komornik. At last, under the conditions of the positive initial energy, it is proved that the solution blows up in the finite time and the lifespan estimates of solutions are also given.

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