

# Notes on a Particular Class of Perfect Cuboids

VALERIU DRAGAN

COMOTI National Research Institute for Gas Turbines

Computational Fluid Dynamics Department

220 D Iuliu Maniu Bd., sector 6, cod 061126, OP 76, CP174 Bucharest

ROMANIA

drvaleriu@gmail.com

*Abstract:* The current paper deals with the problem of the perfect cuboid, in particular with cuboids having only two of its edges even numbers. It is shown that in this particular class, a discussion can be made and some conclusions can be drawn to limit the existence of the perfect cuboid. In this paper Euclids formalism is used to generate Pythagorean triangles that define the faces of the cuboid. Because the pairs,  $(a; b)$ ,  $(c; d)$ ,  $(u; v)$ , used to generate the triangles were not assumed to be relative primes, the triangles are not necessarily primitives. A computerized algebraic solver is used to solve the equations which were then compared with the numerical solutions obtained independently. Further generalization is required in order to extend the conclusions of this paper to a broader class of possible cuboids.

*Key-Words:* Perfect cuboid, Euler brick, Euclid equation, Pythagorean triangles

## 1 Introduction

The problem of proving or disproving the existence of a perfect cuboid is one of the oldest open problems in mathematics. Both numerical and analytical methods have been used, during the years with good indication that such a cuboid cannot exist [1]. Reference [2] refers to semi-perfect cuboids i.e. cuboids with one of the sides or diagonals non-integers. As remarked in [2], it is known that an infinite number of semi-perfect cuboids can exist [3]. Reference [2] provides a generalization of a solution given by Bromhead [4] and disproves the existence of a perfect cuboid within the respective class.

Shapiro (2011) [5] argues that the perfect cuboid problem is linked to a class of univariate polynomials with three integers. Shapiro further argues that due to the restrictions imposed; the polynomials should be irreducible, disproving the existence of perfect cuboids. However, his study only provides numerical validation for 10000 combinations of the three integers. As indicated by [6], numerical search for perfect cuboids with all odd sides came up empty in spite of the large number of examples tested (i.e. up to 1010).

Our case refers to perfect cuboids that have their faces described by Pythagorean triangles generated through Euclids formalism. While other methods for generating such triplets exist [7]-[8], this method is simple and easier to use. Furthermore, the perfect cuboid considered in this paper must have at two even sides and the other side odd.

Further observations regarding the relations between the Pythagorean triangles formed by the sides and diagonals of the cuboid lead to properties that are proven to be inconsistent unless one of the sides (or diagonal) is equal to zero.

Since the pairs of numbers used for the reasoning are not relative primes, the proof applies to all Pythagorean triangles obtained through Euclids method.

## 2 Problem Formulation

One of the known properties of primitive perfect cuboids, should they exist, is that they must have two even edges and precisely one odd edge. It is therefore sufficient and also convenient to study the case of a primitive cuboid - since it provides us with more particular conditions to begin the argument. The next subsection offers proof of the above mentioned property of a perfect cuboid and also introduces the notations which will be used later on.

### 2.1 Preliminary discussion

If a perfect cuboid with all three sides even numbers exists, then by repeated divisions by 2, the resulting cuboids would also have to be perfect. This is because the faces diagonals would be hypotenuses of Pythagorean triangles with two even catheti and are therefore even themselves. Furthermore, the space diagonal is also the hypotenuse of a Pythagorean trian-

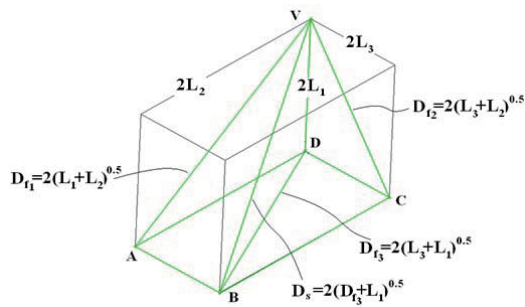


Figure 1: If all three sides ( $L_1, L_2, L_3$ ) of the cuboid are even numbers, then all diagonals ( $D_{f_1}, D_{f_2}, D_{f_3}, D_s$ ) are even as well

gle (e.g. VDB) which has both the catheti even numbers hence the space diagonal is also even.

Therefore, it is possible to reach, after enough divisions by 2, a perfect cuboid with at least one odd side. Hence, if a perfect cuboid with all even sides exists, there exists also a perfect cuboid with at least one odd side. This is because the expression under the square roots must be natural or irrational (but from the hypothesis we know that it cannot be irrational).

The problem is therefore reduced to proving that a perfect cuboid with two even sides and one odd side cannot exist.

All faces of a perfect cuboid are defined by Pythagorean triangles. From Euclid's set of equations, each of the triangles defining the faces of the cuboid must contain at least one even side. Since two faces may have a common side which can be the even one and because the third even side is perpendicular to them and therefore cannot coincide with the other two - we conclude that: a perfect cuboid must have at least two even sides. Furthermore, if a perfect cuboid exists, one of its faces will be defined by two even sides and, as a consequence, have the respective face diagonal also an even number.

Assuming a perfect cuboid exists, having all its edges even numbers, the following will apply:

Let  $E_1 = 2^n L_1$ ,  $E_2 = 2^n L_2$  and  $E_3 = 2^n L_3$  be the edges of the perfect cuboid, where  $n, L_1, L_2, L_3$  are natural numbers. The face diagonals of the cuboid will then be of the form

$$D_f = \sqrt{(2^n L_1)^2 + (2^n L_2)^2} = 2^n \sqrt{L_1^2 + L_2^2} \quad (1)$$

and the space (volume) diagonal will be

$$\begin{aligned} D_v &= \sqrt{(2^n L_1)^2 + (2^n L_2)^2 + (2^n L_3)^2} \\ &= 2^n \sqrt{L_1^2 + L_2^2 + L_3^2}. \end{aligned} \quad (2)$$

Since all diagonals are natural numbers by defini-

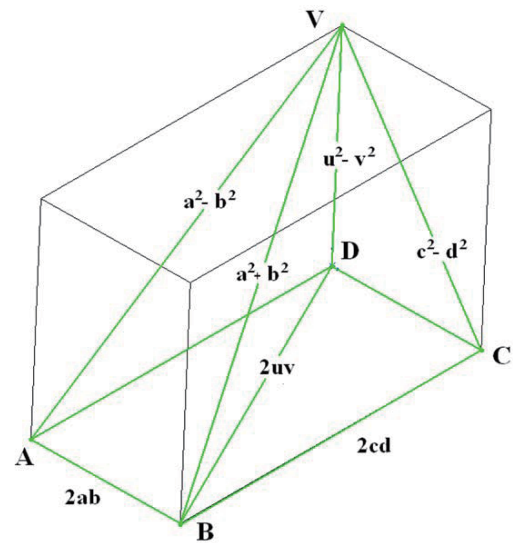


Figure 2: The assigned values of the sides and diagonals for the demonstrational

tion, it is implied that

$$\sqrt{L_1^2 + L_2^2} \in \mathbb{N} \quad (3)$$

and

$$\sqrt{L_1^2 + L_2^2 + L_3^2} \in \mathbb{N}. \quad (4)$$

This is because the square root of a natural number which is not a perfect square cannot be a rational number,

$$\sqrt{k} = w^2 \sqrt{j_1 \cdot j_2 \cdot \dots \cdot j_i} \quad (5)$$

where  $k$  is natural and is not a perfect prime,  $w$  is natural and  $j_1 \cdot \dots \cdot j_i$  are distinct prime numbers. Therefore, if the above expressions are not irrational, they must be natural numbers.

Knowing this we deduce that for any perfect cuboid which has all its edges even numbers of the form  $E=2^n L$ , there exists another perfect cuboid which has at least one of its edges odd.

### 2.2 Demonstration for a primitive cuboid

In Fig. 2, a rectangular pyramid is presented, comprising of all the sides and diagonals of a cuboid, this is done for an easier observation and has no bearing on the demonstration itself. Note that the triangles VAB, VBD and VBC are Pythagorean and have the same hypotenuse, [VB].

Let the side [VD] be odd. Using the following notations:

$$[CD] = [AB] = 2ab \quad (6)$$

$$[AV] = a^2 - b^2 \quad (7)$$

$$[VB] = a^2 + b^2 = c^2 + d^2 = u^2 + v^2 \quad (8)$$

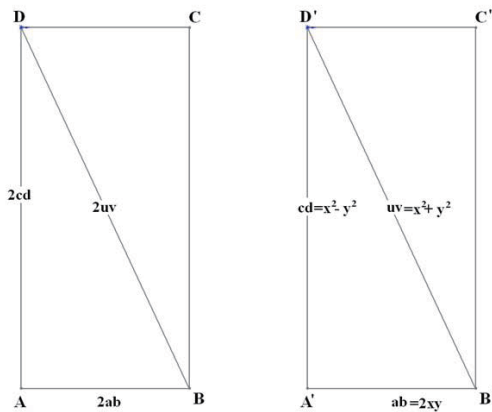


Figure 3: The ABD and A'B'D' Pythagorean triangles

$$[BD] = 2uv \tag{9}$$

$$[VD] = u^2 - v^2 \tag{10}$$

$$[BC] = [AD] = 2cd \tag{11}$$

$$[VC] = c^2 - d^2 \tag{12}$$

we convene that:  $a > b; c > d; u > v$ , in order for the sides to remain natural numbers.

Note that, regardless of the side chosen to be odd, the same general arrangement is reached, only with different notations (the fact remains that either  $b$  or  $a; c$  or  $d; u$  or  $v$ ; must be odd and the other one even).

We observe that, because  $[VD]$  is odd and  $[AD]$  is even,  $[AV]$  will also have to be odd; a similar argument can be made for  $[VC]$  and  $[VB]$  which are also odd.

Figure 3 presents the DAB triangle, the one which contains the three even sides of the three above mentioned Pythagorean triangles. Triangle DAB is itself a Pythagorean triangle since all its three sides are natural numbers as well as a right angle. It is also clear that all its sides are divisible by 2, therefore another triangle,  $D'A'B'$ , exists such that

$$[A'B'] = 2xy \tag{13}$$

$$[A'D'] = x^2 - y^2 \tag{14}$$

$$[D'B'] = x^2 + y^2 \tag{15}$$

where  $x > y$ . But, since  $[AB]=2[A'B']$ , we know that

$$2xy = ab \tag{16}$$

Because  $[VA]$  is odd, we know that exactly one of  $a$  or  $b$  can be even, and if, for simplification,  $x$  and  $y$  are co-primes, we are faced with two possibilities:

$$a = 2x; \quad b = y \tag{17}$$

or

$$a = x; \quad b = 2y. \tag{17'}$$

We also know that the following equalities must apply:

$$2cd = 2(x^2 - y^2) \tag{18}$$

$$2uv = 2(x^2 + y^2) \tag{19}$$

If the first variation, Ec.17, applies, let  $p$  and  $q$  be natural numbers with the property that:

$$[VD]^2 = [AV]^2 - [AD]^2 \tag{20}$$

$$[VD]^2 = (a^2 - b^2)^2 - (2cd)^2 \tag{21}$$

$$[VD]^2 = (4x^2 - y^2)^2 - [2(x^2 - y^2)]^2 = p^2 \tag{22}$$

Rearranging,

$$12x^4 - 3y^4 = p^2. \tag{23}$$

And hence,

$$y = \sqrt[4]{\frac{12x^4 - p^2}{3}} \tag{24}$$

$$[CV]^2 = [VD]^2 + [CD]^2 = q^2 \tag{25}$$

$$p^2 + (4xy)^2 = q^2 \tag{26}$$

Substituting,

$$q^2 = 12x^4 - 3\left(\frac{12x^4 - p^2}{3}\right) + 16x^2\sqrt{\frac{12x^4 - p^2}{3}} \tag{27}$$

which reduces to

$$16x^2\sqrt{\frac{12x^4 - p^2}{3}} + p^2 - q^2 = 0. \tag{28}$$

Note that if the second variation, Ec. (17'), is applied, with the right notations, we reach an equation which has the exact shape. In this latter case, the substituted term should be  $y$  as a function of  $p$  and  $x$ , leading to (28'):

$$16y^2\sqrt{\frac{12y^4 - p^2}{3}} + p^2 - q^2 = 0, \tag{28'}$$

Since the shape of Ecs.28 and (28') are virtually identical – apart from the notations – it is enough to provide proof for one of the two variations. From this point forward, only the first variation will be used.

Using an algebraic solver [10] or a numerical solver [11], it is possible to solve Ec.28 for  $p$ . Other examples where such solvers were used in complex

mathematical and engineering problems can be found in [12-13] also, other methods exist that use simulated annealing [14] for the same purposes. The following four solutions verify Ec.28:

$$p = -\sqrt{q^2 - \frac{16}{3}x^2\sqrt{100x^4 - 3q^2} - \frac{128}{3}x^4} \quad (29)$$

$$p = \sqrt{q^2 - \frac{16}{3}x^2\sqrt{100x^4 - 3q^2} - \frac{128}{3}x^4} \quad (30)$$

$$p = -\sqrt{q^2 + \frac{16}{3}x^2\sqrt{100x^4 - 3q^2} - \frac{128}{3}x^4} \quad (31)$$

$$p = \sqrt{q^2 + \frac{16}{3}x^2\sqrt{100x^4 - 3q^2} - \frac{128}{3}x^4} \quad (32)$$

Since the expression (29) and (31) are negative, the only two remaining possibilities for  $p$  are (32) and (30). Because  $q^2$  must be a natural number, the following discussions are required in order to insure that the quantity under the square root is natural.

### 3 Problem Solution

#### 3.1 Discussion on Ec.(32)

If we consider Ec.32 as the value for  $p$ , we arrive at the conclusion that, in order for  $p$  to remain natural, the following condition must be satisfied

$$\frac{16x^4}{3} \left( \sqrt{100 - 3\frac{q^2}{x^4} - 8} \right) \in \mathbb{Z} \quad (33)$$

If, for simplification we consider that  $x$  is not divisible by 3, because  $16x^4$  is clearly a natural number, the following expression must also be an integer

$$\frac{\sqrt{100 - 3\frac{q^2}{x^4} - 8}}{3} \quad (34)$$

This leads to the condition that there exist an integer,  $k \in \mathbb{Z}$ , such that

$$100 - 3\frac{q^2}{x^4} = (8 + 3k)^2 \quad (35)$$

or, equivalently

$$\left(\frac{q}{x^2}\right)^2 = \frac{100 - (8 + 3k)^2}{3}. \quad (35')$$

Solving this equation for  $q$  yields the two solutions that verify equation (35'):

$$q = x^2\sqrt{-(k + 6)(3k - 2)} \quad (36)$$

Table 1. The results for the values of  $k$  that satisfy condition (35')

$k$	$(3k+8)^2$	$q^2/x^4 = [100 - (3k+8)^2]/3$
-1	25	25
-2	4	32
-3	1	33
-4	16	28
-5	49	17

$$q = -x^2\sqrt{-(k + 6)(3k - 2)}. \quad (36')$$

Because both  $q$  and  $x$  are natural numbers, the expression under the square root must be positive. Thus, we arrive at the conclusion that:

$$3k < 2, \quad \text{if } k > 0 \quad (37)$$

or,

$$\text{if } k < 0, \quad k > -6. \quad (37')$$

From Ec.37, it is obvious that no integer can satisfy the condition imposed. However, Ec. (37') requires the following discussion. Because of the condition imposed through Ec.35, the equivalent expression (35') must be a square rational number.

From Table 1, the only value of  $k$  that leads to a rational square number is  $k=-1$ . This is because the other possible values are either prime numbers or can only be factored in odd powers of primes.

Therefore, the following relation is deduced

$$q = 5x^2 \quad (38)$$

or, more conveniently,

$$q^2 = 25x^4 \quad (38')$$

Also,

$$\frac{16x^2}{3} \left( \sqrt{100 - 3\frac{q^2}{x^4} - 8} \right) = -16x^2 \quad (39)$$

Calculating  $p^2$  from Ec.32

$$p^2 = 25x^4 - 16x^2 \quad (40)$$

But, since by definition, from Ec.26,

$$p^2 + (4xy)^2 = q^2 \quad (41)$$

or

$$p^2 = q^2 - 16x^2y^2 \quad (42)$$

Or, further

$$p^2 = 25x^4 - 16x^2y^2 \tag{43}$$

Equalizing Ec.40 with Ec.43, and baring in mind that both  $y$  and  $x$  are positive integers, we conclude that

$$y = 1 \tag{44}$$

Equalizing Ec.23 with Ec.43, knowing that  $y=1$ , we arrive at the following

$$12x^4 - 3 = 25x^4 - 16x^2 \tag{45}$$

or

$$13x^4 - 16x^2 + 3 = 0 \tag{46}$$

which has the following solutions

$$x = \pm 1 \tag{47}$$

$$x = \pm \frac{\sqrt{39}}{13} \tag{48}$$

The only natural value of  $x$  is  $x=1$ , which implies

$$x=y=1. \tag{49}$$

Since the pair  $(x; y)$  must generate the Pythagorean triangle  $D'A'B'$ , this means that one of its sides will have to be zero (i.e. the pair  $x=y$  cannot be used to generate a proper triangle). Thus, one of the sides of the cuboid will have to be zero in order for it to exist.

### 3.2 Discussion on Ec (30)

Moving on to expression in Ec.(30), we observe the similar condition to the one in paragraph 2.1. is required in order for  $p$  to be natural, again considering that  $x$  is not divisible by 3.

$$\frac{-16x^2}{3} \left( \sqrt{100 - 3\frac{q^2}{x^4}} + 8 \right) \in \mathbb{Z} \tag{50}$$

The same reasoning as before can be made that an integer,  $m$  exists such that

$$\frac{\sqrt{100 - 3\frac{q^2}{x^4}} + 8}{3} = m \tag{51}$$

Rearranging, we obtain

$$100 - 3\frac{q^2}{x^4} - (3m - 8)^2 = 0 \tag{52}$$

The only solutions for  $q$ , that satisfy this equation are listed below

$$q = -x^2 \sqrt{-(m - 6)(3m + 2)} \tag{53}$$

Table 2. The results for the values of  $m$  that satisfy condition (55)

$m$	$(3m-8)^2$	$q^2/x^4 = [100-(3m-8)^2]/3$
1	25	25
2	4	32
3	1	33
4	16	28
5	49	17

$$q = x^2 \sqrt{-(m - 6)(3m + 2)} \tag{54}$$

Because, by definition,  $q$  and  $x$  natural numbers and  $m$  is an integer,

$$\text{if } m > 0, \text{ then } m < 6 \tag{55}$$

or,

$$\text{if } m < 0, \text{ then } |m| < 2/3, \tag{55'}$$

in order to keep the quantity under the squared root positive.

Since for Ec. (55') there are no integer numbers that are satisfactory, we will focus on Ec.(55). The only positive matches for  $m$  in this case are listed below:

Because

$$\left(\frac{q}{x^2}\right)^2 = \frac{100 - (3m - 8)^2}{3} \tag{56}$$

is a square rational number, the only value for  $m$  which is satisfactory is

$$m=1, \tag{57}$$

since the other values do not produce squared rational numbers – because their factors are odd powers of primes. Therefore

$$\frac{q}{x^2} = 5 \tag{58}$$

hence

$$q = 5x^2 \tag{59}$$

and also

$$\frac{-16x^2}{3} \left( \sqrt{100 - 3\frac{q^2}{x^4}} + 8 \right) = -16x^2 \frac{13}{3} \tag{60}$$

Introducing this into Ec.30, we deduce that

$$p^2 = q^2 - 16x^2 \frac{13}{3} \tag{61}$$

But, from Eq.(26) we know that

$$p^2 = q^2 - 16x^2y^2. \tag{62}$$

Because  $x, y, p$  and  $q$  are - by definition - positive, it is obvious that the only real solution for  $y$  is

$$y = \sqrt{13/3} \tag{63}$$

which cannot be a natural number. This denies the initial hypothesis that both  $x$  and  $y$  are natural numbers, therefore invalidating Ec.30 as a possible solution for  $p$ .

It can therefore be stated that neither of the four possible variations of  $p$ , i.e. Eqs.(29-32) is consistent with the hypotheses of the problem –which, in this particular case, does not have a solution.

### 4 Further work

The current section refers to a possible generalization of the previous demonstration in which we assumed that  $x$  and  $y$  are co-primes. For the generalized demonstration this assumption must be eliminated and a new parameter,  $n$ , must be introduced. There are eight possible combinations for the  $a$  and  $b$  pair, as seen below:

$$a = 2nx; \quad b = y/n \tag{64}$$

$$a = nx; \quad b = 2y/n \tag{65}$$

$$a = x/n; \quad b = 2ny \tag{66}$$

$$a = 2x/n; \quad b = ny \tag{67}$$

$$a = 2ny; \quad b = x/n \tag{68}$$

$$a = ny; \quad b = 2x/n \tag{69}$$

$$a = y/n; \quad b = 2nx \tag{70}$$

$$a = 2y/n; \quad b = nx \tag{71}$$

Knowing that, by definition  $a > b$  and  $x > y$ , we can observe that variation (69) cannot exist whereas variation (70) can only exist if  $n = 1$  and  $x < 2y$ , hence reducing the problem to the one previously studied. Moreover, the cases (63)-(66) can be paired with (67)-(70) which lead to similar equation shapes for the final equation.

Looking at the pair described by (66) we can write the following

$$[VD]^2 = x^4 \left( \frac{1}{n^4} - 4 \right) + y^4 (16n^4 - 4) = p^2 \tag{72}$$

therefore,

$$p^2 - q^2 + 16x^2 \sqrt{\frac{p^2 + x^4 \left( 4 - \frac{1}{n^4} \right)}{16n^4 - 4}} = 0 \tag{73}$$

$$y^4 = \frac{p^2 + x^4 \left( 4 - \frac{1}{n^4} \right)}{16n^4 - 4} \tag{74}$$

Since  $a > b$  let  $k > 1$  be a rational number such that  $a = kb$ . Hence the expression for  $y$  becomes

$$y^4 = \left( \frac{x}{2kn^2} \right)^4 \tag{75}$$

Merging the two above equations we write the following:

$$p^2/x^4 = \left( \frac{1}{2kn^2} \right)^4 (16n^4 - 4) + \frac{1}{n^4} - 4 \tag{76}$$

which can be re-arranged as:

$$p^2/x^4 = - \frac{(4k^4n^4 - 1)(4n^4 - 1)}{4k^4n^8} \tag{77}$$

which is a negative number.

This invalidates the pair described by Ec. (66). Consider now the pair described by Ec.(67). The following expression can be written:

$$p^2 - q^2 + 16x^2 \sqrt{\frac{p^2 + x^4 \left( 4 - \frac{1}{n^4} \right)}{n^4 - 4}} = 0 \tag{78}$$

where

$$y^4 = \frac{p^2 + x^4 \left( 4 - \frac{1}{n^4} \right)}{n^4 - 4} \tag{79}$$

Let  $k > 1$  be a rational number such that  $a = kb$ , we write

$$y^4 = \left( \frac{2x}{kn^2} \right)^4, \tag{80}$$

therefore

$$\frac{p^2}{x^4} = \left( \frac{2}{kn^2} \right)^4 (n^4 - 4) + \frac{16}{n^4} - 4 \tag{81}$$

which can be re-arranged

$$\frac{p^2}{x^4} = - \frac{4(k^4n^4 - 4)(n^4 - 4)}{k^4n^8} \tag{82}$$

Note that in order for this expression to be correct,  $n = 1$  - which reduces to the problem of  $x$  and  $y$  being co-primes. It must be said that one of the fundamental properties of primitive Pythagorean triangles is that exactly one of its cathetes is a multiple of 4 [15]. This

means that since we know that the even cathetes of triangles VAB, VBD and VBC are all contained by a single face of the cuboid, they form a Pythagorean triangle which is itself divisible by 4, or in the above formalism, the minimum value for  $n$  is  $n = 2$ .

A comment must be made, that the variations described by Ec.(68) and (69) which lead to virtual identical relations as Ec.(66) and (67) respectively. Hence the findings for Ec.(66) also apply to Ec.(68) and the findings regarding Ec. (67) apply to (69). However, not the same can be stated for Ecs (69) and (70) which although identical in form with Ecs. (63) and (64) respectively, were dismissed for different reasons.

Finally, observe the variation (64), by re-expressing the rational number  $k$  as a fraction,

$$k = g/t \tag{83}$$

thus

$$y = \frac{2n^2}{g/t} \tag{84}$$

one can write the equation giving  $q^2$

$$q^2 - p^2 - 16x^2 \left( \frac{2n^2x}{g/t} \right)^2 = 0 \tag{85}$$

which reduces to

$$q^2 - p^2 - 64x^4 \frac{n^4}{(g/t)^2} = 0 \tag{86}$$

since  $p$  is known to be an odd number-from the hypothesis, let  $p = 2P - 1$ .

$$q^2 - (2P - 1)^2 - 64x^4 \frac{n^4}{(g/t)^2} = 0 \tag{87}$$

solving this equation for  $n$  leads to the following four solutions

$$n = \frac{\left(\frac{1}{4} + \frac{i}{4}\right) \sqrt{g} \sqrt[4]{4P^2 - 4P - q^2 + 1}}{x\sqrt{t}} \tag{88}$$

$$n = -\frac{\left(\frac{1}{4} + \frac{i}{4}\right) \sqrt{g} \sqrt[4]{4P^2 - 4P - q^2 + 1}}{x\sqrt{t}} \tag{89}$$

$$n = \frac{\left(\frac{1}{4} - \frac{i}{4}\right) \sqrt{g} \sqrt[4]{4P^2 - 4P - q^2 + 1}}{x\sqrt{t}} \tag{90}$$

$$n = -\frac{\left(\frac{1}{4} - \frac{i}{4}\right) \sqrt{g} \sqrt[4]{4P^2 - 4P - q^2 + 1}}{x\sqrt{t}} \tag{91}$$

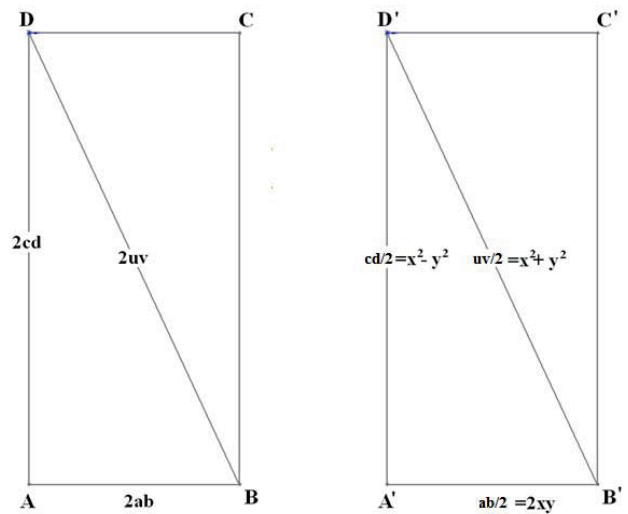


Figure 4: The ABD and A'B'D' Pythagorean triangles

Note that, also from the hypothesis,  $4P^2 - 4P - q^2 + 1$  is negative so the positive expressions above must be re-expressed as

$$n = \frac{\left(\frac{1}{4} + \frac{i}{4}\right) \sqrt{i} \sqrt{g} \sqrt[4]{q^2 - 4P^2 + 4P - 1}}{x\sqrt{t}} \tag{92}$$

and

$$n = \frac{\left(\frac{1}{4} - \frac{i}{4}\right) \sqrt{i} \sqrt{g} \sqrt[4]{q^2 - 4P^2 + 4P - 1}}{x\sqrt{t}} \tag{93}$$

respectively.

It must be said, however, that the expression

$$\left(\frac{1}{4} - \frac{i}{4}\right) \sqrt{i} = \frac{1}{2\sqrt{2}} \tag{94}$$

is still a real number.

Another approach might be dwelling on the property described by Ref.[15], the segment lengths from figure 3 must be re-expressed as seen in figure 4.

$$xy = \frac{ab}{4} \tag{95}$$

Again, by simplifying the case and imposing the condition that  $a$  and  $b$  are coprimes, we can start identifying the terms in the following variations:

$$a = 2x; b = 2y \tag{96}$$

$$ta = x; b = 4y \tag{97}$$

$$a = 4x; b = y \tag{98}$$

$$a = 4y; b = x \tag{99}$$

Note that the last two variations,

$$a = 2y; b = 2x \tag{100}$$

$$a = y; b = 4x \tag{101}$$

are impossible since  $x > y$  and  $a > b$ , by definition.

Analyzing expression(96) we write

$$[AV] = a^2 - b^2 = 4(x^2 - y^2) \tag{102}$$

however, from the geometric conditions,

$$[BC] = [AD] = 4(x^2 - y^2) \tag{103}$$

Hence, the height of the cuboid, [VD] is expressed

$$[VD] = \sqrt{[AV]^2 - [BC]^2} \tag{104}$$

we observe that the height must be zero.

Expression (97) leads to

$$[AV] = x^2 - 16y^2 \tag{105}$$

from which we deduce that

$$x > 4y \tag{106}$$

Also, we observe that the expression for [VD] can be written

$$[VD]^2 = -x^4(16 - 1) + 16y^4(16 - 1) \tag{107}$$

or

$$[VD]^2 = 15(16y^4 - x^4) \tag{108}$$

which implies that

$$x < 2y \tag{109}$$

which is a contradiction.

Therefore expression (97) must be excluded. Variation (98) may be reduced to

$$p^2 = [VD]^2 = 15(16x^4 - y^4) \tag{110}$$

and

$$[VC]^2 = p^2 + 64x^2\sqrt{16x^4 - \frac{p^2}{15}} \tag{111}$$

In the last case, corresponding to variation (99), the equations also reduce to

$$p^2 = [VD]^2 = 15(16y^4 - x^4) \tag{112}$$

and

$$[VC]^2 = p^2 + 64x^2\sqrt{\frac{p^2}{240} - \frac{x^4}{16}} \tag{113}$$

or

$$[VC]^2 = p^2 + 16x^2\sqrt{\frac{p^2}{15} - x^4} \tag{114}$$

An observation which is applicable to both (97) and (98), is that  $p$  must be divisible by 15 in order for the expression to remain integer.

More study is required for the generalized demonstration, however, this approach may prove useful due to the fact that it inserts more mathematical relations between the terms.

## 5 Conclusions

The paper provides a demonstration referring to a particular class of cuboids, proving that under certain limits no perfect cuboid can be constructed. In the initial discussion an argument is made that, if a perfect cuboid with all even edges exists, then even after successive divisions by 2 of its edges it will retain its properties (i.e. it will still remain perfect). It is further argued that, by using Euclid's model for Pythagorean triangles, a perfect cuboid cannot have less than two even sides. This leads to the particular case of a cuboid having two even edges and one odd edge as the only possibility under the Euclid formalism.

The second section establishes the only configuration for a perfect cuboid which has an odd side and two even sides. Through certain observations all the sides and diagonals of the cuboid are expressed as functions of two variables,  $x$  and  $y$ , which are natural numbers. The [VD] face diagonal is expressed in two possible non-negative solutions, Eqs.(3) and (32), that verify the conditions that arise from the setup.

Having disproved the only two possible solutions for  $q$ , expressions (32) and (30), we conclude that no rectangular parallelepiped can be constructed such that all its sides and all its diagonals are positive integers.

Since for the pairs  $(a, b)$ ,  $(c, d)$  and  $(u, v)$  there was no condition that stated they have to be relative primes, the Pythagorean triangles that could have been constructed with them are not restricted to primitive triangles. Some restrictions are however implied by the fact that the cuboid must have an odd side. This is not a serious restriction since the initial discussion establishes that if a perfect cuboid with all its sides even numbers exists, it can always be reduced to a perfect cuboid having one odd side, by successively dividing the sides by 2.

Because one of the premises is that  $x$  and  $y$  are coprimes and that  $x$  (or,  $y$ ) is not divisible by 3, the demonstration is limited. In order for it to be complete, a generalization will need to be provided.



In the Further work section, a generalization attempt is made, consisting in introducing a term  $n$ , which is a natural number. Another property of the number  $n$  is that - in turns -  $x$  and  $y$  are divisible by it, depending on the variation. The number of possible combinations leads to eight variations, however those can be reduced to only four, as follows:

Variation 1,  $a = 2nx$  and  $b = y/n$

$$q^2 = p^2 + 16x^2 \sqrt{\frac{4x^4(4n^4 - 1) - p^2}{4 - \frac{1}{n^4}}} \quad (115)$$

Variation 2,  $a = nx$  and  $b = 2y/n$

$$q^2 = p^2 + 16x^2 \sqrt{\frac{x^4(4n^4 - 1) - p^2}{4 - \frac{16}{n^4}}} \quad (116)$$

Variation 3,  $a = x/n$  and  $b = 2ny$

$$q^2 = p^2 + 16x^2 \sqrt{\frac{x^4 \left(4 - \frac{1}{n^4}\right) + p^2}{4(4n^4 - 1)}} \quad (117)$$

Variation 4,  $a = 2x/n$  and  $b = ny$

$$q^2 = p^2 + 16x^2 \sqrt{\frac{x^4 \left(4 - \frac{16}{n^4}\right) + p^2}{n^4 - 4}} \quad (118)$$

Variation 3 is invalidated due to the fact that the expression  $p^2/x^4$  is a negative number. Also, variation 4 requires that the factor  $n = 1$ , hence falling under the discussions in the previous sections of the paper.

Although not a complete answer to the problem of the perfect cuboid, the present article provides some indication that such a mathematical construction may be impossible.

#### References:

- [1] R. K. Guy, *Is there a Perfect Cuboid? Four Squares whose Sums in Pairs are Square. Four Squares whose Differences are Square*, D18 in *Unsolved Problems in Number Theory*, 2nd ed. New York: Springer-Verlag, pp. 173-181, (1994).
- [2] W. J. A. Colman, *On certain semi-perfect cuboids*, (1986)
- [3] Leech John, *The rational cuboid revisited*, Amer. Math. Monthly 84, (1977)
- [4] T. B. Bromhead, *On square sums of squares*, Math. Gazette 44, (1960)
- [5] Sharipov Ruslan, *Perfect cuboids and irreducible polynomials*, (2011)
- [6] Eric W. Weisstein, *Perfect Cuboid*, From MathWorld—A Wolfram Web Resource. <http://mathworld.wolfram.com/PerfectCuboid.html>, Butler, pers. comm.to Eric Weissten, (2004)
- [7] Fibonacci, Leonardo Pisano, *The Book of Squares (Liber Quadratorum)*, An annotated translation into modern English by L. E. Sigler, Orlando, FL: Academic Press. ISBN 978-0-12-643130-8, (1987)
- [8] L. E. Dickson, *History of the Theory of Numbers*, Vol.II. Diophantine Analysis, Carnegie Institution of Washington, Publication No. 256, 12+803pp
- [9] J. Rukavicka, Dickson's method for generating pythagorean triples revisited, *European Journal of Pure and Applied Mathematics*, ISSN 1307-5543, Vol. 6, No. 3, (2013), pp.363-364,
- [10] \*\* <http://reference.wolfram.com/mathematica/guide/EquationSolving.html> - (retrieved Apr.2014)
- [11] \*\*Mathcad 14 User Guide- Chapter 9 Solving, pp.103-108, (2007)
- [12] Krivulin Nikolai, An algebraic approach to multidimensional minimax location problems with Chebyshev distance, *WSEAS TTransactions on Mathematics*, Issue 6, Volume 10, June (2011)
- [13] Nakabayasy Tamiyo, Nakabayasgu Keiji, Kako Fujio, Application of computer algebra approach to solve Engineering problems - A case study: Power supply stabilization loop circuit, *Applied Mathematics in Electrical and Computer Engineering*, ISBN: 978-1-61804-064-0
- [14] Hime Aguiar, Diophantine equations and fuzzy adaptive simulated annealing, *WSEAS TTransactions on Mathematics*, Volume 13, (2014)
- [15] Sierpinski, Waclaw, *Pythagorean Triangles*, Dover, ISBN 978-0-486-43278-6, (2003),