Spectral Analysis of Transport Operator in Lebowitz-Rubinow Model

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Abstract: In this paper, we deal with the Lebowitz-Rubinow model of an age structured proliferating cell population in L^p-space $(1 \leq p < +\infty)$. It is to prove that the C_0 semigroups generated by the transport operator is compact for the boundary operator is compact, and it is to obtain that the spectrum of the transport operators is countable and consists of, at most, isolate eigenvalues with finite algebraic multiplicity with $-\infty$ as the only possible limit point, we are to obtain that the spectrum of the transport operators only consists of finitely isolate eigenvalues with finite algebraic multiplicities in the right half plane trip for the boundary operator is not compact, also we show that the asymptotic behavior of the transport equation's solution.

Key–Words: Transport operator; Cell population, Compactness, Spectrum, Asymptotic behavior.

1 Introduction

In this paper, we deals with the mathematical model of an age structured proliferating cell population introduced by Leibowitz and Rubinow [1]. The transport equation of the model is following:

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(a,l,t) &= -\frac{\partial \varphi}{\partial a}(a,l,t) - \mu(a,l)\varphi(a,l,t) \\ &= A_K\varphi = M_K\varphi + B\varphi \end{aligned} \tag{1}$$

where the variables t and a represent respectively time and age, while l is the cycle length of cells and represents the time between cell birth and cell division, here, $l \in (l_1, l_2), 0 \leq l_1 < l_2 \leq +\infty$, and $a \in [0, l]$, the constant $l_1(l_2)$ denotes the minimum cycle length (the maximum cycle length). The function $\mu = \mu(a, l)$ is the rate of cell mortality or cell loss due to causes other than division. The function $\varphi(a, l, t)$ is the density of the population with respect to age a, and cell cycle length l at time t.

Transport equation (1) is complemented by the following boundary condition

$$\varphi(0,l,t) = K\varphi(l,l,t) = \int_{l_1}^{l_2} k(l,l')\varphi(l',l',t)dl' \tag{2}$$

where the position kernel $k(l, l')$ describes the correlation between the cycle length of a mother cell l' and that of its daughter cell l satisfying the following condition of normalization

$$\int_{l_1}^{l_2} k(l,l')dl = 1 \tag{3}$$

The model (1)-(3) was introduced for the first time by Leibowitz and Rubinow in [1], which is called L–R model. It has been studied on the continuous function space by G. F. Webb in [2,3], M. Rotenberg in [4] introduced the model that each cell is characterized by its degree of maturity and its maturation velocity, the model is called Rotenberg model. Since then it has been rarely studied. Recently, the model $(1)-(3)$ has been studied by B. Lods [5, 6], M. Boulanouar [7-17], K. Latrach [18-23], A. Jeribi [24, 25], S, H. Wang [26-28], respectively. When $0 = l_1 < l_2 < +\infty$, B. Lods in [5, 6] investigated the spectrum of the transport operator and the asymptotic behavior of the corresponding C_0 semigroup, under crucial hypothesis $l_1 > 0$, M. Boulanouar proved in [9] that the existence of a strongly continuous semigroup and the solution of transport equation is well-posed, M. Boulanouar proved in [10] that the C_0 semigroup is irreducibility and that the asymptotic behavior of solution for the model was discussed. M. Boulanouar studied in [11-13] for Rotenberg's model, considered in [14] a general biological rule corresponding to a non-compact boundary condition that it is to give the asymptotic behavior of the generated semigroup in the uniform topology. In addition, M. Boulanouar mathematically analyzed an age-cycle structured population endowed with a general biological rule in [15-17]. It has been proven that the existence of a strongly continuous semigroup and the solution of transport equation is well-posed and the C_0 semigroup is irreducibility and it is to describe the asymptotic behavior of the generated semigroup in the uniform topology, when $0 = l_1 < l_2 < +\infty$. K. Latrach in [18] showed that the spectral decomposition of the solutions into an asymptotic term and a transient one which will be estimated for smooth initial data. K. Latrach studied for a boundary value problem of Rotenberg's model in [19-23]. A. Jeribi has studied in [24-27] that the spectral decomposition of the solutions into an asymptotic term and a transient one which will be estimated for Rotenberg's model, when $l_1 = 0, l_2 = +\infty$. S. H. Wang in [29, 30] proved that the spectral analysis of the transport operator A_K and investigate the asymptotic behavior of the corresponding C_0 semigroup $V_K(t)$ to the transport equation (1) in a compact boundary condition and a non-compact boundary condition corresponding. We have noted that the models studied in [33] and [34] have similar form as equation (1) but the boundary conditions used in here is more complicated than that in there.

In this paper, when $l_1 = 0, l_2 = +\infty$, we will prove that the C_0 semigroups by the transport operator generated is compact for $l > 2a$ in the boundary operator is compact and obtain that the spectrum of the transport operators is countable and consists of, at most, isolate eigenvalues with finite algebraic multiplicity with $-\infty$ as the only possible limit point. Furthermore we will prove that the spectrum of the transport operators only consists of finitely isolate eigenvalues with finite algebraic multiplicities in the right half plane trip for the boundary operator is not compact. Also we will show that the asymptotic behavior of the transport equation solution in L^p space $(1 \le p < +\infty)$.

The paper is organized as follows. In Second 2, we give some preliminary results. In Section 3, we study the compactness of generated semigroup by the transport operator. In section 4, we study the spectrum of transport operator. In Section 5, we describe the cellular profile of the model (1)-(3) by discussing the asymptotic behavior of the generated semigroup.

2 Preliminary results

In this section, we introduce some notions and notations. Let

$$\Delta = \{(a, l); 0 < a < l, 0 < l < +\infty\}.$$

and the weighted space

$$X^p_\omega = L^p(\Delta, f^p_\omega),\ (\omega \ge 0\, and\, p \ge 0)$$

with the norm

$$\|\psi\|_{\omega,p} = [\int_\Delta |(\psi f_\omega)(a,l)|^p dadl]^{\frac{1}{p}}$$
$$= [\int_0^{+\infty}\int_0^l |(\psi f_\omega)(a,l)|^p dadl]^{\frac{1}{p}}.$$

where f_ω is defined as

$$f_\omega(a, l) = e^{-\omega(l-a)}.$$

Note that $L^p(\Delta) \subset X^p_0 \subset X^p_\omega(\omega \ge 0, p \ge 1)$ and if $\omega > 0$, it has the form

$$\|\psi\|_{\omega,p} \le [\int_\Delta |\psi(a,l)|^p dadl]^{\frac{1}{p}} = \|\psi\|_p.$$

We also consider the partial derivative space

$$W^p_\omega = \{\psi \in X^p_\omega :\ \frac{1}{l}\psi \in X^p_\omega;\ \frac{\partial\psi}{\partial a} \in X^p_\omega\}$$

with the norm

$$\|\psi\|_{W^p_\omega} = [\|\psi\|^p_{\omega,p} + \|\frac{1}{l}\psi\|^p_{\omega,p} + \|\frac{\partial\psi}{\partial a}\|^p_{\omega,p}]^{\frac{1}{p}}.$$

We define trace applications as follow

$$\Gamma_1 = \{(0, l), l \in (l_1, l_2)\},$$

$$\Gamma_2 = \{(l, l), l \in (l_1, l_2)\}.$$

We also consider the trace space

$$Y_p = L^p(0, \infty), (p \ge 1)$$

with its natural norm

$$\|\psi\|_{Y_p} = [\int_0^\infty |\psi(l)|^p dl]^{\frac{1}{p}}.$$

We define the following unbounded operator

$$\begin{cases} M_K\psi = -\frac{\partial\psi}{\partial a} \text{ on the domain,} \\ D(M_K) = \{\psi \in W^p_\omega, \\ \psi(0, l) = K\psi(l, l)\}. \end{cases}$$

If $K = 0$, then it is easy to see that the operator M_0 defined by

$$\begin{cases} M_0\psi = -\frac{\partial\psi}{\partial a} \text{ on the domain,} \\ D(M_0) = \{\psi \in W^p_\omega;\ \text{satisfying } \psi(0, l) = 0.\} \end{cases}$$

In this section, we recall the some results as lemmas, which will be used in next section.

Lemma 1 *The operator M_0 generates, on $X^p_\omega(\omega \geq 0,\ p \geq 1)$, a strongly continuous semigroup $(S_0(t))_{t\geq 0}$ given by*

$$S_0(t)\psi(a,l) = \chi(a,l,t)\psi(a-t,l). \tag{4}$$

where

$$\chi(a,l,t) = \begin{cases} 1 & \text{if } 0 \leq t \leq a, \\ 0 & \text{otherwise.} \end{cases}$$

Let boundary operator K satisfies
(O_1) K bounded and $\|K\|_{\ell(Y_p)} \leq 1$.
(O_2) K compact and $\|K\|_{\ell(Y_p)} > 1$.
and define

$$\omega_0 = \begin{cases} 0 & \text{if}(O_1) \text{ holds;} \\ \lambda_0 & \text{if}(O_2) \text{ holds.} \end{cases}$$

We characterize its type $w(T(t))$ and its essential type $w_{ess}(T(t))$ by

$$w(T(t)) = \lim_{t\to\infty} \frac{\ln \|T(t)\|_{ess}}{t}.$$

and

$$w_{ess}(T(t)) = \lim_{t\to\infty} \frac{\ln \|T(t)\|_{ess}}{t}.$$

Note that $\|C\|_{ess} = 0$ if and only if C is a compact operator.

Lemma 2 *Suppose that (O_1) or (O_2) holds, then, the operator M_K generates, on $X^p_\omega(\omega \geq \omega_0, p \geq 1)$, a strongly continuous C_0 semigroup $(S_K(t))_{t\geq 0}$ given by*

$$S_K(t) = \sum_{m=0}^{\infty} U_m(t), \quad (t \geq 0). \tag{5}$$

where the operator $U_m(t)$ is defined, on $L^p(\Delta)(p \geq 1)$, by

$$\begin{aligned} & U_m(t)\psi(a,l) \\ = \; & \xi(a,l,t)h(l)\int_0^{+\infty}\cdots\int_0^{+\infty}\prod_{j=1}^{m=1} h(v_j) \\ & \times \prod_{j=1}^{m} k(v_j) \times \xi(v_{m-1}, v_{m-1}, t-a-\sum_{i=1}^{m-2} v_i) \\ & \times\chi(v_m, v_m, t-a-\sum_{i=1}^{m-1} v_i) \\ & \times\psi(a+\sum_{i=1}^{m} v_i - t, v_m)dv_1\cdots dv_m. \end{aligned}$$

Proof; $\forall\varphi \geq 0$ and consequently for $\varphi \in X_p$, since

$$X_p^+ - X_p^- = X_p.$$

For $\varphi \in X^p_\omega, \lambda \in C, \psi \in D(A_K)$, we consider the resolvent equation for A_K

$$(\lambda - A_K)\psi = \varphi. \tag{6}$$

The solution of (6) is formally given by

$$\begin{aligned} \psi(a,l) &= \psi(0,l)e^{-\int_0^a(\lambda+\mu(s,l))\,ds} \\ &+ \int_0^a e^{-\int_s^a(\lambda+\mu(\tau,l))\,d\tau}.\varphi(s,l)\,ds. \end{aligned} \tag{7}$$

Accordingly, for $a = l$, we get

$$\begin{aligned} \psi(l,l) &= \psi(0,l)e^{-\int_0^l(\lambda+\mu(s,l))\,ds} \\ &+ \int_0^l e^{-\int_s^l(\lambda+\mu(\tau,l))\,d\tau}\varphi(s,l)\,ds. \end{aligned} \tag{8}$$

In the sequel we shall need the operators

$$\begin{cases} B_\lambda : X_p^1 \to X_p^2; \\ B_\lambda f)(l,l) = f(0,l)e^{-\int_0^l(\lambda+\mu(s,l))\,ds}. \end{cases}$$

$$\begin{cases} D_\lambda : X_p^1 \to X_\omega^p; \\ (D_\lambda f)(a,l) = f(0,l)e^{-\int_0^a(\lambda+\mu(s,l))\,ds}. \end{cases}$$

$$\begin{cases} E_\lambda : X_\omega^p \to X_p^2; \\ (E_\lambda\varphi)(l,l) = \int_0^a e^{-\int_s^l(\lambda+\mu(\tau,l))\,d\tau}\varphi(s,l)\,ds. \end{cases}$$

$$\begin{cases} F_\lambda : X_\omega^p \to X_\omega^p; \\ (F_\lambda\varphi)(a,l) = \int_0^a e^{-\int_s^a(\lambda+\mu(\tau,l))\,d\tau}\varphi(s,l)\,ds. \end{cases}$$

Let

$$\underline{\mu} = essinf\{\mu(.,.)\}.$$

Then $\forall\lambda$ with $\Re\lambda > -\underline{\mu}$, the Hölder inequality shows

$$\|B_\lambda\| \leq 1,$$

$$\|D_\lambda\| \leq \frac{1}{[p(Re\lambda + \underline{\mu})]^{\frac{1}{p}}},$$

$$\|E_\lambda\| \leq \frac{1}{(Re\lambda + \underline{\mu})^{\frac{1}{q}}},$$

$$\|F_\lambda\| \leq \frac{1}{Re\lambda + \underline{\mu}}.$$

By above the operators definite and the fact that ψ must satisfy the boundary conditions, then equations (8) and (7) become

$$\psi_{|\Gamma_2} = B_\lambda K\psi_{|\Gamma_2} + E_\lambda\varphi. \tag{9}$$

$$\psi = D_\lambda K \psi_{|\Gamma_2} + F_\lambda \varphi. \tag{10}$$

We first show that

$$(\lambda - A_K)^{-1}\varphi - (\lambda - A_0)^{-1}\varphi = \int_0^\infty e^{-\lambda t} \sum_{n\geq 0} V_n(t)\varphi dt \tag{11}$$

for $\varphi \in X_p, \varphi \geq 0$, and λ large enough. According to [27], there exists $\lambda_0 > 0$ such that

$$(\lambda - A_K)^{-1} = \sum_{n\geq 0} D_\lambda K (B_\lambda K)^n E_\lambda + (\lambda - A_0)^{-1}, \quad \forall \Re\lambda > \lambda_0.$$

Thus, it suffices to show that, for any $n \geq 0$,

$$D_\lambda K (B_\lambda K)^n E_\lambda \varphi = \int_0^\infty e^{-\lambda t} V_n(t)\varphi dt, \quad \Re\lambda > \lambda_0.$$

For $n = 0$,

$$E_\lambda\varphi(l,l) = \int_0^a e^{-\int_s^l (\mu(\tau,l)+\lambda)d\tau}\varphi(s,l)ds = \int_0^\infty e^{-\lambda t}\chi(l, l-t)\varphi(l-t,l)dt.$$

By Fubini's theorem

$$KE_\lambda\varphi(0,l) = \int_0^\infty e^{-\lambda t}dt \int_{l_1}^{l_2} k(l,l') \times\chi(l', l'-t)\varphi(l'-t,l')dl'.$$

Then

$$D_\lambda K E_\lambda \varphi(a,l) = e^{-\int_0^a \mu(s,l)ds}\int_0^\infty e^{-\lambda(t+a)}dt \times \int_{l_1}^{l_2} k(l,l')\chi(l',l'-t)\varphi(l'-t,l')dl'.$$

We make the change of variables $s = t + a$ and get

$$D_\lambda K E_\lambda\varphi(a,l) = e^{-\int_0^a \mu(\tau,l)d\tau} \times \int_0^\infty e^{-\lambda s}ds \int_{l_1}^{l_2} k(l,l') \times\chi(l', l'+a-s)\varphi(l'+a-s,l')dl'.$$

So,

$$D_\lambda K E_\lambda \varphi = \int_0^\infty e^{-\lambda s} V_0(s)\varphi ds.$$

In the same way (for the details, see [5]) one can show by induction that

$$\begin{aligned}
&(D_\lambda K)^n E_\lambda \varphi(a,l) \\
=\ & \chi(l,0)\int_0^\infty e^{-\lambda t}dt \\
&\times \int_{l_1}^{l^2} k(l,l')\chi(l',0)dl' \cdots \int_{l_1}^{l^2} dl^{(n-1)} \\
&\times k(l^{n-2}, l^{n-1}) \times \chi(l^{n-1},0)\int_{l_1}^{l^2} k(l^{n-1}, l^n). \\
&\times \chi(l^n, l + \sum_{j=1}^n l^j - t)\varphi(l + \sum_{j=1}^n l^j - t, l^n)dl^n.
\end{aligned}$$

Then, it holds that

$$\begin{aligned}
&D_\lambda K (B_\lambda K)^n E_\lambda\varphi(a,l) \\
=\ & e^{-\int_0^a \mu(s,l)ds}\int_0^\infty e^{-\lambda(t+a)}dt\int_{l_1}^{l_2} k(l,l')\chi(l',0)dl' \\
&\cdots \int_{l_1}^{l_2} k(l^{(n-1)}, l^n)\chi(l^n, 0)dl^n \\
&\cdots \int_{l_1}^{l_2} k(l^n, l^{(n+1)})\chi(l^{(n+1)}, \sum_{j=1}^{n+1} l^j - t) \\
&\times \varphi(l' + \sum_{j=2}^{n+1} l^j - t, l^{(n+1)})dl^{(n+1)}.
\end{aligned}$$

Making the change of variables $s = t + a$ yields

$$D_\lambda K (B_\lambda K)^n E\varphi = \int_0^\infty e^{-\lambda t} V_n(t)\varphi dt.$$

and this ends the proof of (11). Finally, since

$$(\lambda - A_K)^{-1}\varphi - (\lambda - A_0)^{-1}\varphi = \int_0^\infty e^{-\lambda t}(U_K(t)\varphi - U_0(t)\varphi)dt.$$

it follows that

$$U_K(t)\varphi - U_0(t)\varphi = \sum_{n=0}^\infty V_n(t)\varphi, \ (t \geq 0).$$

Therefore, equation (6) is hold and Lemma 2 is obtained. □

Lemma 3 *([10, Lemma 2.2]) Let (Ω, Σ, μ) be a positive measure space and S, T be bounded linear operators on $L_1(\Omega,)$. (a) The set of all weakly compact operators is norm-closed subset. (b) If T is weakly compact and $0 \leq S \leq T$, then S is weakly compact. (c) If S and T are weakly compact, then ST is compact.*

Lemma 4 *([10, Lemma 2.3]) Let (Ω, Σ, μ) be a positive measure space and S, T be bounded linear operators on $L_p(\Omega,)(1 < p < \infty)$ such that $0 \leq S \leq T$. If T is compact then S is also compact.*

Lemma 5 *([10, Lemma 1.1]) Let $(T(t))_{t\geq 0}$ be a positive and irreducible C_0- semigroup on the Banach lattice X satisfying the inequality $_{ess}(T(t)) <$ $_0(T(t))$. Then there exist an $\varepsilon > 0$ and an one rank projection P into X such that, for any $\eta \in (0, \varepsilon)$, there exists $M(\eta) \geq 1$ such that*

$$\| e^{-s(A)t}T(t) - P \|_{L(X)} \leq M(\eta)e^{\eta t}, \ \ t \geq 0.$$

where $s(A)$ denotes spectral bound of the generator A with semigroup $(T(t))_{t\geq 0}$.

3 Compactness of generated semigroup

In this section, we will investigate the compactness of strongly continues semigroup $S_K(t)$ and $V_K(t)$ generated by transport operator A_K. Firstly, we prove some lemmas.

Lemma 6 *Let K be the following operator*

$$K\varphi = h \int_0^{+\infty} k(l')\varphi(l')dl',$$
$$h \in C_c(0, +\infty), \ \ k \in C_c(0, +\infty).$$

Then, the operator $S_K(t)$ is weakly compact in $L^1(\Delta)$, and compact in $L^p(\Delta)(1 < p < +\infty), t > a$.

Proof: When $m = 0, t > a, S_0(t) = 0$ is compact, when $m = 1, S_1(t) = 0$, when $m \geq 2$, by (5), we have

$$\begin{aligned} U_m(t)\psi(a, l) &= \xi(a, l, t)h(l) \int_0^{+\infty} \cdots \int_0^{+\infty} \\ &\times \prod_{j=1}^{m-1} h(v_j) \times \prod_{j=1}^{m} k(v_j) \\ &\times \xi(v_{m-1}, v_{m-1}, t-a-\sum_{i=1}^{m-2} v_i) \\ &\times \chi(v_m, v_m, t - a - \sum_{i=1}^{m-1} v_i) \\ &\times \psi(a+\sum_{i=1}^{m} v_i-t, v_m)dv_1 \cdots dv_m. \end{aligned}$$

As $h \in C_c(0, +\infty)$ and $k \in C_c(0, +\infty)$, there exist $0 < l_1 < l_2 < +\infty$ such that supp $h \subset (l_1, l_2)$ and supp $k \subset (l_1, l_2)$. Next, let $t \leq (m-1)l_1$, then

$$t - a - \sum_{i=1}^{m-2} v_i \leq t - (m-2)l_1 \leq l_1$$

and

$$\xi(v_{m-1}, v_{m-1}, t - a - \sum_{i=1}^{m-2} v_i) = 0.$$

Hence, $S_m(t) = 0$ for all $m \geq \frac{t}{l_1} + 1$ and (5) becomes a finite sum, i.e.,

$$S_K(t) = S_0(t) + U_1(t) + \sum_{m=2}^{[\frac{t}{l_1}]+1} U_m(t) \tag{12}$$

Now, let

$$\begin{cases} x = a + \sum\limits_{i=1}^{m} v_i - t \\ dx = dv_{m-1} \end{cases}$$

then

$$\begin{aligned} |U_m(t)\psi(a, l)| \ \ &\leq \ \ \xi(a, l, t)h(t)\|h\|_{Y_\infty} \\ &\times \|k\|_{Y_\infty}\|h\|_{Y_q}^{m-2}\|h\|_{Y_p}^{m-2} \\ &\times \int_\Delta |k(v_m)|\psi(x, v_m)dxdv_m \end{aligned}$$

where $p^{-1} + q^{-1} = 1$ and therefore

$$\begin{aligned} &|U_m(t)\psi(a, l)| \\ \leq \ \ &\xi(a, l, t)h(l)\|h\|_{Y_\infty} \\ &\times \|k\|_{Y_\infty}\|h\|_{Y_q}^{m-2}\|h\|_{Y_p}^{m-2} \times e^{\omega l_2} \\ &\times \int_\Delta |k(v_m)|\psi(x, v_m)h_\omega(x, v_m)dxdv_m \\ = \ \ &O_m(t)T\psi(a, l) \end{aligned} \tag{13}$$

where the operator T is given by

$$\begin{aligned} T\psi(a, l) = \xi(a, l, t)h(t) \int_\Delta |k(v_m)| \\ \times \psi(x, v_m)h_\omega(x, v_m)dxdv_m \end{aligned}$$

and the constant $O_m(t)$ by

$$O_m(t) = \|h\|_{Y_\infty}\|k\|_{Y_\infty}\|h\|_{Y_q}^{m-2}\|h\|_{Y_p}^{m-2}e^{\omega l_2}.$$

Since

$$\int_\Delta [\xi(a, l, t)|h(l)|^p]dadl = t\|h\|_{Y_p}^p < +\infty$$

for all $p \geq 1$ and

$$\begin{aligned} &\int_\Delta |k(v_m)|\psi(x, v_m)h_\omega(x, v_m)dxdv_m \\ \leq \ \ &\|k\|_\infty\|\psi\|_{\omega,1}, \ \ if \ \psi \in X_\omega^1. \end{aligned}$$

or

$$\int_{\Delta} |k(v_m)|\psi(x,v_m)h_\omega(x,v_m)dxdv_m$$
$$\leq \quad l_2^{\frac{1}{q}}\|k\|_q\|\psi\|_{\omega,p}, \quad if\ \psi \in X_\omega^p, (p>1).$$

Then, we get that the operator T is one rank in $X_\omega^p (p \geq 1)$, thus compact from (15), we have

$$0 \leq U_m(t) + O_m(t)T \leq 2O_m(t)T$$

that implies $U_m(t) + O_m(t)T$ is weakly compact in $L^1(\Delta)$ and compact in $L^p(\Delta)(p>1)$. So, the operator

$$U_m(t) = (U_m(t) + O_m(t)T) - O_m(t)T$$

is weakly compact $L^1(\Delta)$ and compact in $L^p(\Delta)(p>1)$. therefore, the operator $S_K(t)$ is weakly compact $L^1(\Delta)$ and compact in $L^p(\Delta)(p>1)$. □

Lemma 7 *Let K be the following operator*

$$K\varphi = h\int_0^{+\infty} k(l')\varphi(l')dl', \quad h \in Y_p,\ k \in Y_q.$$

where $p^{-1} + q^{-1} = 1$, then the operator $S_K(t)$ is weakly compact in $L^1(\Delta)$ and compact in $L^p(\Delta)(p>1)$ for all $t > a$.

Proof: Let $t > 0$ and $\omega \geq \omega_0$, since $h \in Y_p$ *and* $k \in Y_q$, then there exist two sequences $\{h_n\} \subset C_c(l,\infty)$ and $\{k_n\} \subset C_c(l,\infty)$. converging to h and k in Y_p and Y_q. respectively by density. Let

$$K_n\varphi = h_n\int_0^{+\infty} k_n(l')\varphi(l')dl'.$$

Then

$$|(K_n - K)\varphi| \leq |h_n - h|\int_0^{+\infty}|k_n(l')||\varphi(l')|dl'$$
$$+|h|\int_0^{+\infty}|k_n(l') - k(l')||\varphi(l')|dl'$$

and

$$\|(K_n - K)\varphi\|_{Y_p} \leq [\|h_n - h\|_{Y_p}\|k_n\|_{Y_q} + \|h\|_{Y_p}\|k_n - k\|_{Y_q}]\|\varphi\|_{Y_p}$$

Therefore

$$\lim_{n\to+\infty}\|K_n - K\|_{L(Y_p)} = 0$$

Now, Lemma 3.5 in [10] obviously implies that

$$\lim_{n\to+\infty}\|S_{K_n}(t) - S_K(t)\|_{L(L^p(\Delta))} = 0$$

and by the lemma 6 we get that $S_K(t)$ is weakly compact in $L^1(\Delta)$ and compact in $L^p(\Delta)(p>1)$. □

Lemma 8 *Let K be the following finite rank operator*

$$K\varphi = \sum_{i=1}^n h_i\int_0^{+\infty} k_i(l')\varphi(l')dl',$$
$$h_i \in Y_p,\ k_i \in Y_q, i = 1,2,\cdots,n$$

and $p^{-1} + q^{-1} = 1$, Then, the operator $S_K(t)$ is weakly compact in $L^1(\Delta)$ and compact in $L^p(\Delta)(p>1)$ for all $t > a$.

Proof: Let $t > 0$, and $\omega > \omega_0$, let

$$K_1\varphi = h\int_0^{+\infty} k(l')\varphi(l')dl'$$

where $h = \sum_{i=1}^n |h_i| \in Y_p$ and $k = \sum_{i=1}^n |k_i| \in Y_q$.

It is easy to see that K_1 is one rank operator, By Lemma 7, we know the operator $S_{K_1}(t)$ is a weakly compact in $L^1(\Delta)$ and compact in $L^p(\Delta)(p>1)$.

On the other hand, for all $\varphi \in (Y_p)_+$, we have

$$\begin{aligned}|K\varphi| &\leq \sum_{i=1}^n |h_i|\int_0^{+\infty}|k_i(l')|\varphi(l')dl' \\ &\leq [\sum_{i=1}^n |h_i|]\int_0^{+\infty}[\sum_{i=1}^n |k_i(l')|]\varphi(l')dl' \\ &= h\int_0^{+\infty} k(l')\varphi(l')dl'\end{aligned}$$

and hence $|K\varphi| \leq K_1\varphi$. By theorem 4.3 of [10], we get that

$$|S_K(t)\varphi| \leq S_{K_1}(t)|\varphi|, \ \ t \geq 0.$$

This implies

$$0 \leq S_K(t) + S_{K_1}(t) \leq 2S_{K_1}(t)$$

and therefore, the operator $S_K(t) + S_{K_1}(t)$ is weakly compact in $L^1(\Delta)$ and compact in $L^p(\Delta)(p>1)$. By now, we get

$$S_K(t) = S_K(t) + S_{K_1}(t) - S_{K_1}(t)$$

is weakly compact in $L^1(\Delta)$ and compact in $L^p(\Delta)(p>1)$. □

Theorem 9 *Let K be a compact operator in $Y_p(p \geq 1)$. Then for all $t > 2a$, semigroup $S_K(t)$ is a compact operator in $L^p(\Delta)(p \geq 1)$.*

Proof: Let K be a compact operator in $Y_p(p \geq 1)$, so, by [11, corollary 5.3], there exists a sequence K_n of finite rank operators satisfy:

$$\lim_{n\to+\infty}\|K_n - K\|_{L(Y_1)} = 0$$

On the one hand, by Lemma 5, we know that the operator S_{K_n} is compact in $L^p(\Delta)$. and by Lemma 3.5 in [10]

$$\lim_{n\to+\infty}\|S_{K_n}(t)-S_K(t)\|_{L(L(\Delta))}=0$$

which leads to the weak compactness of the operator $S_K(t)$. therefore, for all $t>2a$, semigroup $S_K(t)$ is a compact operator in $L^p(\Delta)$. □

Let us define the following operator

$$B\varphi(a,l)=-\mu(a,l)\varphi(a,l).$$

where, we impose the following hypothesis

(O_3) $\mu\in L^\infty(\Delta)$.

So, we can know that the operator B is a bounded operator. let $A_K=M_K+B$, under the hypothesis (O_3), then it is to get that the operator A_K can generate a C_0- semigroup $V_K(t)$ (see [35]).

Theorem 10 *Let K be a compact operator in $Y_p(p\geq 1)$, B is a bounded operator, then, for all $t>2a$, semigroup $V_K(t)$ is a compact operator in $L^P(\Delta)(p\geq 1)$.*

Proof: By Theorem 9, for all $t>2a$, we have got that the semigroup $S_K(t)$ generate by M_K is compact in $L^P(\Delta)(p\geq 1)$, then for all $t>2a$, the semigroup $S_K(t)$ generate by M_K is compact in $L^P(\Delta)(p\geq 1)$. on the other hand, the operator B is bounded, we have that the semigroup $V_K(t)$ is also a compact operator in $L^P(\Delta)(p\geq 1)$, thank to the perturbation theorem (see, [35][37]).

4 Spectrum of transport operator

In this section, we are going to discuss the spectral properties of transport operator A_K, by the theorem 10 and spectral mapping principle [34, 35], we give the following theorem:

Theorem 11 *Assume that K is compact in $Y_p(p\geq 1)$, then*

(1) The spectrum $\sigma(A_K)$ of the transport operator A_K is countable and consist of, at most, isolated eigenvalues with finite algebraic multiplicity, therefore $\sigma(A_K)=\{\lambda_1,\lambda_2,\cdots\}$ with $\Re\lambda_{n+1}\leq\Re\lambda_n$ for all integer n and $\Re\lambda_n\to-\infty(n\to\infty)$, if $\sigma(A_K)$ is not finite.

(2) We denote by k_i the order of the pole λ_i of the resolvent of A_K and by P_n the associated eigenprojection, then, for any integer n

$$V_K(t) = \sum_{i=1}^{n}e^{\lambda_i t}\sum_{j=0}^{k_i-1}\frac{t^j}{j!}(A_K-\lambda_i)^jP_n +R_n(t), \qquad (t>a).$$

and, for any $\varepsilon>0$, there exist $M>0$, such that

$$\|R_n(t)\|\leq Me^{(\varepsilon+\Re\lambda_{n+1})},\quad \forall t>a.$$

Note that the boundary operator K has been used under the hypothesis: K is a compact operator, this hypothesis is fulfilled, for instance, to see [17].

An open question is : what will happen when K satisfy the non-local condition. i.e.,

$$K\varphi(l)=\alpha\varphi(l)+\beta\int_{l_1}^{l_2}k(l,l')\varphi(l')dl'$$

We can assume that

(O_4) : $K=K_1+K_2$,

with $K_i\geq 0$, $i=1,2$; K_2 is compact if $1<p<\infty$; K_2 weakly compact if $p=1$. then, we have obtained results as follow:

Theorem 12 *[30] Suppose that (O_4) holds and that there exists λ_0 such that*

$$r_\sigma(B_\lambda K_1)<1 \tag{14}$$

for all $\lambda>\lambda_0$, where $r_\sigma(T)$ denotes spectral radius of operator T. Then:

(1) The spectrum $\sigma(A_K)$ of the transport operator A_K consists of, at most, isolated eigenvalues with finite algebraic multiplicities.

(2) If $\sigma(A_K)\neq\phi$, then there exists a real leading eigenvalue $\overline{\lambda}$;

(3) If $r_\sigma(B_{\lambda_0}K_2)>1$, then $\sigma(A_K)\neq\phi$.

Remark 13 *Since K_2 is compact transition operator, then there exists a sequence of finite rank operators which converges, in the operators norm, to K_2. Hence, it suffices to establish the result for a finite rank operator, that is , $K_2=\sum_{j=1}^{n}\langle\cdot,\alpha_k\rangle\beta_k$, where $n\in N$, $\alpha_k\in X_q^2$, $\beta_k\in X_p^1$, $q=p/(p-1)$. namely, $Ku:=\langle u,\ \alpha\rangle\beta$, where $\alpha(\cdot)\in X_q^2$, $\beta(\cdot)\in X_p^1$.*

We denote by Γ_s the strip by

$$\Gamma_s=\{\lambda\in C\mid -\underline{\mu}\leq\Re\lambda\leq s(A_K)\}.$$

Theorem 14 *[30] Suppose that (O_4) holds and that there exists λ_0 such that*

$$r_\sigma(B_\lambda K_1)<1. \tag{15}$$

for all $\lambda>\lambda_0$, then the operator $(I-H_\lambda)^{-1}$ exists for all $\lambda\in\Gamma_s$ with $|Im\lambda|$ sufficiently large.

We assume that K satisfies (O_5): K is positive and some power of K is compact.

Theorem 15 *If the hypothesis (O_5) holds, then:*

(1) The spectrum $\sigma(A_K)$ of the transport operator A_K consists of, at most, isolated eigenvalues with finite algebraic multiplicities.

(2) If $\sigma(A_K) \neq \phi$, then there exists a real leading eigenvalue $\overline{\lambda}$.

(3) $\sigma(A_K) \neq \phi$ if and only if $r_\sigma(K) > 0$.

Proof: Let $N_\lambda = KB_\lambda, \lambda > -\underline{\mu}$. for $B_\lambda \leq I, \lambda > -\underline{\mu}$, then

$$N_\lambda \leq K,\ \forall \lambda > -\underline{\mu}. \quad (16)$$

Next, from (O_5) we infer that there exists an integer N such that K^N is compact, (18) implies

$$(N_\lambda)^N \leq K^N, forall \lambda > -\underline{\mu}.$$

Using Dodds-Fremlin [38], we get N_λ^N is compact for all $\lambda > -\underline{\mu}$. thus

$$(N_\lambda)^N \leq B_\lambda K^N,\ \ \forall \lambda > -\underline{\mu}.$$

Since $B_\lambda \to 0\ (\lambda \to +\infty)$, K^N is compact, we obtain

$$B_\lambda K^N \to 0\ (\lambda \to +\infty),$$

this implies that

$$(N_\lambda)^N \to 0,\ (\lambda \to +\infty).$$

Consequently, using the estimate

$$r_\sigma(N_\lambda) \leq \|(N_\lambda)^n\|^{\frac{1}{n}}\ \ , n = 1, 2, \cdots.$$

Therefore

$$r_\sigma(N_\lambda) \to 0,\ \ (\lambda \to +\infty). \quad (17)$$

(19) together with the Gohberg-Shmul'yan theorem implies that $(I - (N_\lambda)^N)$ is boundedly invertible for all $\lambda \in C$ except at a discrete set of point $S = \{\lambda_k : k = 1, 2, \cdots\}$; each λ_k is a pole of $(I - (N_\lambda)^N)^{-1}$. Using the identity

$$\begin{aligned} I - (N_\lambda)^N &= (I - N_\lambda)(I + N_\lambda + \\ &\cdots + (N_\lambda)^{N-1}),\ \lambda \in C,\ \lambda \notin S \end{aligned}$$

Hence,

$$(I - N_\lambda)^{-1} = (I + N_\lambda + \cdots + (N_\lambda)^{N-1}) \\ (I - (N_\lambda)^N)^{-1},\ \lambda \in C,\ \lambda \notin S$$

Hence, if $\lambda \in C,\ \lambda \notin S$, (9) becomes

$$\psi_{|\Gamma_2} = (I - N_\lambda)^{-1} E_\lambda \varphi.$$

Thus, (10) becomes

$$\psi = D_\lambda K (I - N_\lambda)^{-1} E_\lambda \varphi + F_\lambda \varphi$$

Accordingly, the solution of the problem is given by $\psi = (\lambda - A_K)^{-1}\varphi$. So,

$$(\lambda - A_K)^{-1} = D_\lambda K (I - N_\lambda)^{-1} E_\lambda + F_\lambda. \quad (18)$$

Thus, each $\lambda \in S$ is a pole of $(\lambda - A_K)^{-1}$ with finite rank residues, i.e., eigenvalues with finite algebraic multiplicities. therefore, $\sigma(A_K)$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities.

(2) Using (20) we get

$$\begin{aligned} &(\lambda - A_K)^{-1} \\ &= D_\lambda K (I - N_\lambda)^{-1} E_\lambda + F_\lambda \\ &= \sum_{n \geq 0} D_\lambda K (N_\lambda)^n E_\lambda + F_\lambda \end{aligned}$$

Clearly, by the positivity of the operators B_λ, E_λ, D_λ and F_λ, we deduce that $(\lambda - A_K)^{-1}$ is positive. Hence, the proof of (2) is merely a consequence of a well known result on positive resolvent operators.

(3) Let $\lambda \in \sigma(A_K) \cap R$, it follows from the spectral mapping theorem that $r_\sigma(N_\lambda)$ is a continuous strictly decreasing function. Next,

$$\lambda \in \sigma_p(A) \Leftrightarrow 1 \in \sigma_p(N_\lambda). \quad (19)$$

According (19) we get

$$\sigma(A_K) \neq \phi \Leftrightarrow \lim_{\lambda \to -\infty} r_\sigma(N_\lambda) > 1.$$

Let

$$\begin{cases} B_\lambda^* : X_p^1 \to X_p^2; \\ (B_\lambda^* f)(l, l) := f(0, l) e^{-\int_0^l (\lambda + \mu(s, l))\, ds}. \end{cases}$$

where $l \in [\varepsilon, \frac{1}{\varepsilon}] (\varepsilon > 0)$, let $N_\lambda^* = B_\lambda^* K$, then

$$B_\lambda^* \to B_\lambda,\ \varepsilon \to 0$$

$$N_\lambda^* \to N_\lambda,\ \varepsilon \to 0.$$

Hence,

$$\sigma(A_K) \neq \phi \Leftrightarrow \lim_{\lambda \to -\infty} r_\sigma(N_\lambda^\star) > 1\ (\varepsilon \to 0).$$

On the other hand, for $\lambda < -\underline{\mu}$, we have

$$B_\lambda^\star \leq e^{-(\lambda + \underline{\mu}) \frac{1}{\varepsilon}}.$$

Hence

$$\begin{aligned} &r_\sigma(N_\lambda^\star) = r_\sigma(B_\lambda^\star K) \\ &\leq e^{-(\lambda + \underline{\mu}) \frac{1}{\varepsilon}} r_\sigma(K),\ \ \forall \lambda < -\underline{\mu}. \end{aligned}$$

Clearly, if $r_\sigma(K) = 0$, then $r_\sigma(N^\star_\lambda) = 0$; on the other hand, if $r_\sigma(K) > 0,\ \forall\lambda < -\underline{\mu}$, then

$$B^\star_\lambda \geq de^{-(\lambda+\underline{\mu})\varepsilon}$$

where

$$0 < d = e^{-\int_0^\varepsilon(\mu(s,l)-\underline{\mu})ds} \leq 1,$$

and $\lim\limits_{\varepsilon\to 0} d = 1$. Hence

$$N^*_\lambda \geq de^{-(\lambda+\underline{\mu})\varepsilon}K$$

and

$$r_\sigma(N^*_\lambda) \geq de^{-(\lambda+\underline{\mu})\varepsilon} r_\sigma(K).$$

If $r_\sigma(K) > 0$, then

$$\lim_{\lambda\to-\infty} r_\sigma(N^*_\lambda) = +\infty.$$

so, $\sigma(A_K) \neq \phi$ if and only if $r_\sigma(K) > 0$. on the other hand, if the upper limit $M = \frac{1}{l}\int_0^l \mu(s,l)ds$ is finite, then operator N_λ is unbounded for $\lambda < -M$. This completes the proof of the assertion (3). □

5 Asymptotic Behavior

In this section, by the mathematical description of the cellular profiles of the model, we are to give that the asymptotic behavior of the generated semigroup .

Theorem 16 *Let K be a positive and compact operator, Then,*

$$w_{ess}(V_K(t)) \leq -\underline{\mu}.$$

where μ denotes the mortality rate.

Proof: Let $t > 0$ be fixed and $\omega > \omega_0$. Using previous Lemma we get that the semigroup $(V_K(t))_{t\geq 0}$ is compact operator, then

$$w_{ess}(V_K(t)) = w_{ess}(V_0(t)) \leq -\underline{\mu}.$$

thus the theorem is obtained. □

In the case $\|K\|_{L(Y_p)} > 1$ means that the cell density is increasing during each mitotic. This case is the most observed and biologically interesting for which we give the cellular profile as follows by previous results and [10, Theorem 6.2]

Theorem 17 *If K is a positive, irreducible and compact operator with $r(K_{\overline{\mu}-\underline{\mu}}) > 1$, then there exist a one rank projection P into $X^p_w(w > w_0, p \geq 1)$ and $\varepsilon > 0$ such that for every $\eta \in (0,\varepsilon)$, there exists $M(\eta) \geq 1$ satisfying*

$$\| e^{-s(A)t}T(t) - P \|_{L(X)} \leq M(\eta)e^{\eta t},\ \ t \geq 0.$$

where the spectral bound of $s(A_K)$ of the generator A is given by

$$s(A_K) = \begin{cases} \sup\{\Re\lambda, \lambda \in \sigma(A)\}, & if \sigma(A) \neq 0, \\ -\infty, & if \sigma(A) = 0. \end{cases}$$

Proof: Since $r(K_\lambda)(K_\lambda = (\gamma_1\varepsilon_\lambda)K)$ is decreasing ([10, Lemma 5.3]), then

$$\|K\|_{L(Y_p)} \geq r(K) = r(K_0) \geq r(K_{\overline{\mu}-\underline{\mu}}) > 1.$$

where $\varepsilon_\lambda = e^{-\lambda a}$, $\overline{\mu} = sup\{\mu(a,l)\}$. which implies that the hypothesis $(O)_2$ hold. By [10, Lemma 5.3], there is a λ_0 such that $r(K_\lambda)(K_{\lambda_0} = 1$, then

$$\overline{\mu} - \underline{\mu} < \lambda_0 = w(U_K(t)).$$

Hence, the compactness of K and [10, Theorem 5.1] and Theorem 15 imply

$$w_{ess}(V_K(t)) \leq -\underline{\mu} < w(U_K(t)) - \overline{\mu} \leq w(V_K(t)).$$

Therefore, by Lemma 5, The proof is achieved. □

Remark 18 *According to the theorems 8-10, it suffices to establish the result for a finite rank operator, that ia is, $K = \sum\limits_{j=1}^{n}\langle\cdot,\alpha_k\rangle\beta_k$, where $n \in N$, $\alpha_k \in X^2_q$, $\beta_k \in X^1_p$, $q = p/(p-1)$. namely, $Ku := \langle u,\ \alpha\rangle\beta$, where $\alpha(\cdot) \in X^2_q$, $\beta(\cdot) \in X^1_p$.*

We denote by Γ_s the strip by

$$\Gamma_s = \{\lambda \in C \mid -\underline{\mu} \leq \Re\lambda \leq s(A_K)\}.$$

Remark 19 *According to theorem (8–10), if the boundary operator K is compact, then we have investigated the spectrum of transport operator A_K and the solution's asymptotic behavior of the transport equation; if the boundary operator K is not compact, then we can only discuss the spectrum of the transport operator A_K in right half-plane Γ_s, but the solution's asymptotic behavior of the transport equation is still open.*

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