Oscillatory Criteria For Two Fractional Differential Equations

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Abstract: In this paper, by use of the properties of fractional derivative and certain inequality technique, some new oscillatory criteria are established for a certain functional fractional differential equation with damping. The fractional differential equation is defined in the sense of the modified Riemann-Liouville fractional derivative. Using a similar analytical method, we also research oscillation of a higher order fractional differential equation, and obtain some oscillatory criteria for it. These oscillatory criteria are new results so far in the literature. As for applications of the oscillatory criteria established, some examples are presented.

Key–Words: Oscillation; Modified Riemann-Liouville derivative; Fractional differential equation; Qualitative analysis; Damping; Interval oscillatory criteria.

1 Introduction

Fractional differential equations are generalizations of classical differential equations of integer order, and can find their applications in many fields of science and engineering. In the last few decades, research for various aspects of fractional differential equations, for example, the existence, uniqueness and stability of solutions of fractional differential equations, the numerical methods for fractional differential equations and so on has been paid much attention by many authors (for example, we refer the reader to see [1-12], and the references therein). In these investigations, we notice that very little attention is paid to oscillation of fractional differential equations. To our knowledge, recent results in this direction include the works in [13-17].

In [13,14], Chen researched oscillation of the following two fractional differential equations:

$$\begin{aligned}(a):\quad & [r(t)(D_-^{\alpha}y(t))^{\eta}]' \\ & -q(t)f(\int_t^{\infty}(v-t)^{-\alpha}y(v)dv)=0,\\ (b):\quad & D_-^{1+\alpha}y(t)-p(t)D_-^{\alpha}y(t)\\ & +q(t)f(\int_t^{\infty}(v-t)^{-\alpha}y(v)dv)=0,\end{aligned}$$

where $D_-^{\alpha}y(t)$ denotes the Liouville right-sided fractional derivative of order α of x. In [15], Zheng researched oscillation of the following nonlinear fractional differential equation with damping term, which is a generalization of (a):

$$\begin{aligned}(c):\quad & [a(t)(D_-^{\alpha}x(t))^{\gamma}]'+p(t)(D_-^{\alpha}x(t))^{\gamma}\\ & -q(t)f(\int_t^{\infty}(\xi-t)^{-\alpha}x(\xi)d\xi)=0,\ t\in[t_0,\infty).\end{aligned}$$

In [16], Grace et al. researched oscillation of the following nonlinear fractional differential equation under the definition of Riemann-Liouville differential operator:

$$\begin{aligned}(d):\quad & D_a^{q}x+f_1(t,x)=v(t)+f_1(t,x)\\ & -q(t)f(\int_t^{\infty}(\xi-t)^{-\alpha}x(\xi)d\xi)=0,\ t\in[t_0,\infty).\end{aligned}$$

In [17], Feng researched oscillation of the following nonlinear fractional differential equation in the sense of the modified Riemann-Liouville differential operator:

$$\begin{aligned}(e):\ & D_t^{\alpha}(r(t)k_1(x(t),D_t^{\alpha}x(t)))\\ & +p(t)k_2(x(t),D_t^{\alpha}x(t))+D_t^{\alpha}x(t)\\ & +q(t)f(x(t))=0,\ t\ge t_0\ge 0,\ 0<\alpha<1,\end{aligned}$$

where $D_t^{\alpha}(.)$ denotes the modified Riemann-Liouville derivative with respect to the variable t.

Motivated by the above works, in this paper, we are concerned with oscillation of the following functional fractional differential equation with damping

$$\begin{aligned} &D_t^\alpha[r(t)(D_t^\alpha x(t))^\gamma] \\ &+p(t)(D_t^\alpha x(t))^\gamma + q(t)f(x(t)) = 0, \end{aligned} \tag{1}$$

and the following higher order fractional differential equation

$$D_t^\alpha[D_t^\alpha(r(t)D_t^\alpha x(t))] + q(t)x(t) = 0, \tag{2}$$

where $t \geq t_0 > 0$, $0 < \alpha < 1$, $D_t^\alpha(.)$ denotes the modified Riemann-Liouville derivative [18] with respect to the variable t, the function $r \in C^\alpha([t_0,\infty),\mathbf{R}_+)$, $p,\ q \in C([t_0,\infty),\mathbf{R}_+)$, and C^α denotes the continuous derivative of order α, γ is the ratio of two positive integers, the function f is continuous satisfying $f(x)/x^\gamma \geq K$ for some positive constant K and $\forall x \neq 0$.

In general, a solution $x(t)$ of Eq. (1) or Eq. (2) is called oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory. Eqs. (1) and (2) are called oscillatory if all their solutions are oscillatory.

We organize the rest as follows. In Section 2, we give the definition and some important properties for the modified Riemann-Liouville derivative. Then in Section 3, we establish some new oscillatory criteria for Eq. (1) by use of the properties of fractional derivative and certain inequality technique. In Section 4, we use the analytical method in Section 3 to research oscillation of Eq. (2), and obtain some oscillatory criteria for it. In Section 5, we present some applications for the results established. Some conclusions are presented at the end of this paper.

Throughout this paper, we denote

$$\xi = \frac{t^\alpha}{\Gamma(1+\alpha)},\ \xi_i = \frac{t_i^\alpha}{\Gamma(1+\alpha)},\ i = 0,1,2,3,4,5$$

$\mathbf{R}_+ = (0,\infty)$, $r(t) = \widetilde{r}(\xi)$, $p(t) = \widetilde{p}(\xi)$, $q(t) = \widetilde{q}(\xi)$, and $A(\xi) = \exp(\int_{\xi_0}^{\xi} \frac{\widetilde{p}(\tau)}{\widetilde{r}(\tau)}d\tau)$.

2 The Jumarie's modified Riemann-Liouville derivative

The definition and some important properties for the Jumarie's modified Riemann-Liouville derivative of order α are listed as follows (see also in [19-21]):

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, \\ \qquad 0<\alpha<1, \\ (f^{(n)}(t))^{(\alpha-n)},\ n \leq \alpha < n+1,\ n \geq 1. \end{cases}$$

The derivative has the following properties:

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}, \tag{3}$$

and

$$D_t^\alpha(f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \tag{4}$$

and derivative of composed function

$$\begin{aligned} D_t^\alpha f[g(t)] &= f_g'[g(t)]D_t^\alpha g(t) \\ &= D_g^\alpha f[g(t)](g'(t))^\alpha. \end{aligned} \tag{5}$$

The modified Riemann-Liouville derivative has a lot of excellent characters in handling with various fractional calculus problems. Many authors have investigated various applications of the modified Riemann-Liouville fractional derivative (see [22-26] for example).

3 Oscillatory criteria for Eq.(1)

In this section, we will establish some new oscillatory criteria for Eq.(1) by use of the Riccati transformation technique. First we give two lemmas for further use.

Lemma 1 *Assume $x(t)$ is an eventually positive solution of Eq.(1), and*

$$\int_{\xi_0}^{\infty} \frac{1}{[A(s)\widetilde{r}(s)]^{\frac{1}{\gamma}}}ds = \infty. \tag{6}$$

Then there exists a sufficiently large T such that $D_t^\alpha x(t) > 0$ for $t \in [T,\infty)$.

Proof. Let $x(t) = \widetilde{x}(\xi)$. Then by use of Eq.(3) we obtain $D_t^\alpha \xi(t) = 1$, and furthermore by use of the first equality in Eq. (5), we have

$$D_t^\alpha r(t) = D_t^\alpha \widetilde{r}(\xi) = \widetilde{r}'(\xi)D_t^\alpha \xi(t) = \widetilde{r}'(\xi).$$

Similarly we have $D_t^\alpha x(t) = \widetilde{x}'(\xi)$. So Eq.(1) can be transformed to the following form:

$$[\widetilde{r}(\xi)(\widetilde{x}'(\xi))^\gamma]' + \widetilde{p}(\xi)(\widetilde{x}'(\xi))^\gamma + \widetilde{q}(\xi)f(\widetilde{x}(\xi)) = 0, \tag{7}$$

where $\xi \geq \xi_0 > 0$. Since $x(t)$ is an eventually positive solution of (1), then $\widetilde{x}(\xi)$ is an eventually positive solution of Eq. (7), and there exists $\xi_1 > \xi_0$ such that $\widetilde{x}(\xi) > 0$ on $[\xi_1,\infty)$. Furthermore, for $\xi \geq \xi_1$ we have

$$\begin{aligned} &[A(\xi)\widetilde{r}(\xi)(\widetilde{x}'(\xi))^\gamma]' \\ &= A(\xi)[\widetilde{r}(\xi)(\widetilde{x}'(\xi))^\gamma]' + A(\xi)\widetilde{p}(\xi)(\widetilde{x}'(\xi))^\gamma \end{aligned}$$

$$= -\widetilde{q}(\xi)A(\xi)f(\widetilde{x}(\xi))$$
$$\leq -KA(\xi)\widetilde{q}(\xi)(\widetilde{x}(\xi))^{\gamma} < 0. \tag{8}$$

Then $A(\xi)\widetilde{r}(\xi)(\widetilde{x}'(\xi))^{\gamma}$ is strictly decreasing on $[\xi_1, \infty)$, and thus $\widetilde{x}'(\xi)$ is eventually of one sign. We claim $\widetilde{x}'(\xi) > 0$ on $[\xi_2, \infty)$, where $\xi_2 > \xi_1$ is sufficiently large. Otherwise, assume there exists a sufficiently large $\xi_3 > \xi_2$ such that $\widetilde{x}'(\xi) < 0$ on $[\xi_3, \infty)$. Then for $\xi \in [\xi_3, \infty)$ we have

$$\widetilde{x}(\xi) - \widetilde{x}(\xi_3) = \int_{\xi_3}^{\xi} \widetilde{x}'(s)ds$$
$$= \int_{\xi_3}^{\xi} \frac{[A(s)\widetilde{r}(s)]^{\frac{1}{\gamma}}\widetilde{x}'(s)}{[A(s)\widetilde{r}(s)]^{\frac{1}{\gamma}}}ds$$
$$\leq [A(\xi_3)\widetilde{r}(\xi_3)]^{\frac{1}{\gamma}}\widetilde{x}'(\xi_3)\int_{\xi_3}^{\xi} \frac{1}{[A(s)\widetilde{r}(s)]^{\frac{1}{\gamma}}}ds.$$

By (6) we deduce that $\lim_{\xi\to\infty} \widetilde{x}(\xi) = -\infty$, which contradicts to the fact that $\widetilde{x}(\xi)$ is an eventually positive solution of Eq. (7). So $\widetilde{x}'(\xi) > 0$ on $[\xi_2, \infty)$, and furthermore $D_t^{\alpha}x(t) > 0$ on $[t_2, \infty)$. The proof is complete by setting $T = t_2$. □

Lemma 2 *[27, Theorem 41]. Assume that A and B are nonnegative real numbers. Then for all $\lambda > 1$,*

$$\lambda XY^{\lambda-1} - X^{\lambda} \leq (\lambda - 1)Y^{\lambda}.$$

Now we state the main theorems and establish oscillatory criteria for Eq. (1).

Theorem 3 *Assume (6) holds, and there exist two functions $\zeta \in C^1([t_0, \infty), \mathbf{R}_+)$ and $\rho \in C^1([t_0, \infty), [0, \infty))$ such that*

$$\int_{\xi_0}^{\infty} \Big\{KA(s)\widetilde{\zeta}(s)\widetilde{q}(s) - \widetilde{\zeta}(s)\widetilde{\rho}'(s) + \frac{\widetilde{\rho}^{1+\frac{1}{\gamma}}(s)\widetilde{\zeta}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}$$
$$- \frac{\{(\gamma+1)\widetilde{\zeta}(s)\widetilde{\rho}^{\frac{1}{\gamma}}(s) + \widetilde{\zeta}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\widetilde{\zeta}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\Big\}ds$$
$$= \infty, \tag{9}$$

where $\widetilde{\zeta}(\xi) = \zeta(t)$, $\widetilde{\rho}(\xi) = \rho(t)$. Then every solution of Eq. (1) is oscillatory.

Proof. Assume (1) has a non-oscillatory solution x on $[t_0, \infty)$. Without loss of generality, we may assume $x(t) > 0$ on $[t_1, \infty)$, where t_1 is sufficiently large. By Lemma 1 we have $D_t^{\alpha}x(t) > 0$ on $[t_2, \infty)$ for some sufficiently large $t_2 > t_1$. Define the generalized Riccati transformation function:

$$\omega(t) = \zeta(t)\left\{\frac{A(\xi)r(t)(D_t^{\alpha}x(t))^{\gamma}}{x^{\gamma}(t)} + \rho(t)\right\}.$$

Then for $t \in [t_2, \infty)$, we have

$$D_t^{\alpha}\omega(t) = D_t^{\alpha}\zeta(t)\frac{A(\xi)r(t)(D_t^{\alpha}x(t))^{\gamma}}{x^{\gamma}(t)}$$
$$-\zeta(t)\frac{\gamma A(\xi)x^{\gamma-1}(t)r(t)(D_t^{\alpha}x(t))^{\gamma+1}}{x^{2\gamma}(t)}$$
$$+\zeta(t)\frac{D_t^{\alpha}[A(\xi)r(t)(D_t^{\alpha}x(t))^{\gamma}]}{x^{\gamma}(t)}$$
$$+D_t^{\alpha}\zeta(t)\rho(t) + \zeta(t)D_t^{\alpha}\rho(t)$$
$$= \frac{D_t^{\alpha}\zeta(t)}{\zeta(t)}\omega(t) - \gamma\frac{(\omega(t) - \zeta(t)\rho(t))^{\frac{\gamma+1}{\gamma}}}{[A(\xi)\zeta(t)r(t)]^{\frac{1}{\gamma}}}$$
$$+\zeta(t)D_t^{\alpha}\rho(t) + \zeta(t)\frac{[A(\xi)D_t^{\alpha}[r(t)(D_t^{\alpha}x(t))^{\gamma}]}{x^{\gamma}(t)}$$
$$+\zeta(t)\frac{[r(t)(D_t^{\alpha}x(t))^{\gamma}D_t^{\alpha}(A(\xi))]}{x^{\gamma}(t)}$$
$$= \frac{D_t^{\alpha}\zeta(t)}{\zeta(t)}\omega(t) - \gamma\frac{(\omega(t) - \zeta(t)\rho(t))^{\frac{\gamma+1}{\gamma}}}{[A(\xi)\zeta(t)r(t)]^{\frac{1}{\gamma}}}$$
$$+\zeta(t)D_t^{\alpha}\rho(t) + \zeta(t)\frac{[A(\xi)D_t^{\alpha}[r(t)(D_t^{\alpha}x(t))^{\gamma}]}{x^{\gamma}(t)}$$
$$+\zeta(t)\frac{[r(t)(D_t^{\alpha}x(t))^{\gamma}A(\xi)\frac{\widetilde{p}(\xi)}{\widetilde{r}(\xi)}]}{x^{\gamma}(t)}$$
$$= \frac{D_t^{\alpha}\zeta(t)}{\zeta(t)}\omega(t) - \gamma\frac{(\omega(t) - \zeta(t)\rho(t))^{\frac{\gamma+1}{\gamma}}}{[A(\xi)\zeta(t)r(t)]^{\frac{1}{\gamma}}}$$
$$+\zeta(t)D_t^{\alpha}\rho(t) + \zeta(t)\frac{[A(\xi)D_t^{\alpha}[r(t)(D_t^{\alpha}x(t))^{\gamma}]}{x^{\gamma}(t)}$$
$$+\zeta(t)\frac{p(t)(D_t^{\alpha}x(t))^{\gamma}A(\xi)]}{x^{\gamma}(t)}$$
$$= \frac{D_t^{\alpha}\zeta(t)}{\zeta(t)}\omega(t) - \gamma\frac{(\omega(t) - \zeta(t)\rho(t))^{\frac{\gamma+1}{\gamma}}}{[A(\xi)\zeta(t)r(t)]^{\frac{1}{\gamma}}}$$
$$-\zeta(t)\frac{A(\xi)q(t)f(x(t))}{x^{\gamma}(t)} + \zeta(t)D_t^{\alpha}\rho(t)$$
$$\leq \frac{D_t^{\alpha}\zeta(t)}{\zeta(t)}\omega(t) - \gamma\frac{(\omega(t) - \zeta(t)\rho(t))^{\frac{\gamma+1}{\gamma}}}{[A(\xi)\zeta(t)r(t)]^{\frac{1}{\gamma}}}$$
$$-Kq(t)\zeta(t)A(\xi) + \zeta(t)D_t^{\alpha}\rho(t). \tag{10}$$

Using the following inequality (see [28, Eq. (2.17)]):

$$(u - v)^{1+\frac{1}{\gamma}} \geq u^{1+\frac{1}{\gamma}} + \frac{1}{\gamma}v^{1+\frac{1}{\gamma}}$$
$$-(1 + \frac{1}{\gamma})v^{\frac{1}{\gamma}}u, \ \forall u, v \geq 0,$$

we obtain

$$[\omega(t) - \zeta(t)\rho(t)]^{1+\frac{1}{\gamma}}$$

$$\geq \omega^{1+\frac{1}{\gamma}}(t) + \frac{1}{\gamma}[\zeta(t)\rho(t)]^{1+\frac{1}{\gamma}} - (1+\frac{1}{\gamma})[\zeta(t)\rho(t)]^{\frac{1}{\gamma}}\omega(t). \quad (11)$$

A combination of (10) and (11) yields that

$$\begin{aligned} D_t^\alpha \omega(t) &\leq -Kq(t)\zeta(t)A(\xi) \\ &- \frac{\gamma}{[\zeta(t)r(t)A(\xi)]^{\frac{1}{\gamma}}}\Big\{\omega^{1+\frac{1}{\gamma}}(t) + \frac{1}{\gamma}[\zeta(t)\rho(t)]^{1+\frac{1}{\gamma}} \\ &- (1+\frac{1}{\gamma})[\zeta(t)\rho(t)]^{\frac{1}{\gamma}}\omega(t)\Big\} \\ &+ \frac{D^\alpha\zeta(t)}{\zeta(t)}\omega(t) + \zeta(t)D^\alpha\rho(t) \\ &= -Kq(t)\zeta(t)A(\xi) + \zeta(t)D^\alpha\rho(t) \\ &- \frac{\rho^{1+\frac{1}{\gamma}}(t)\zeta(t)}{[r(t)A(\xi)]^{\frac{1}{\gamma}}} - \frac{\gamma}{[\zeta(t)r(t)A(\xi)]^{\frac{1}{\gamma}}}\omega^{1+\frac{1}{\gamma}}(t) \\ &+ \{\frac{(\gamma+1)\rho^{\frac{1}{\gamma}}(t)}{[r(t)A(\xi)]^{\frac{1}{\gamma}}} + \frac{D^\alpha\zeta(t)}{\zeta(t)}\}\omega(t) \\ &= -Kq(t)\zeta(t)A(\xi) + \zeta(t)D^\alpha\rho(t) \\ &- \frac{\rho^{1+\frac{1}{\gamma}}(t)\zeta(t)}{[r(t)A(\xi)]^{\frac{1}{\gamma}}} - \frac{\gamma}{[\zeta(t)r(t)A(\xi)]^{\frac{1}{\gamma}}}\omega^{1+\frac{1}{\gamma}}(t) \\ &+ \{\frac{(\gamma+1)\zeta(t)\rho^{\frac{1}{\gamma}}(t) + D^\alpha\zeta(t)[r(t)A(\xi)]^{\frac{1}{\gamma}}}{\zeta(t)[r(t)A(\xi)]^{\frac{1}{\gamma}}}\}\omega(t). \end{aligned}$$

Setting $\lambda = 1 + \frac{1}{\gamma}$,

$$X^\lambda = \frac{\gamma}{[\zeta(t)r(t)A(\xi)]^{\frac{1}{\gamma}}}\omega^{1+\frac{1}{\gamma}}(t),$$

$$Y^{\lambda-1} = \gamma^{\frac{1}{\gamma+1}}\frac{(\gamma+1)\zeta(t)\rho^{\frac{1}{\gamma}}(t) + D^\alpha\zeta(t)[r(t)A(\xi)]^{\frac{1}{\gamma}}}{(\gamma+1)\zeta^{\frac{\gamma}{\gamma+1}}(t)[r(t)A(\xi)]^{\frac{1}{\gamma(\gamma+1)}}},$$

by Lemma 2 we get

$$\begin{aligned} \omega'(t) &\leq -Kq(t)\zeta(t)A(\xi) \\ &+ \zeta(t)D^\alpha\rho(t) - \frac{\rho^{1+\frac{1}{\gamma}}(t)\zeta(t)}{[r(t)A(\xi)]^{\frac{1}{\gamma}}} \\ &+ \frac{\{(\gamma+1)\zeta(t)\rho^{\frac{1}{\gamma}}(t) + D^\alpha\zeta(t)[r(t)A(\xi)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\zeta^\gamma(t)[r(t)A(\xi)]^{\frac{1}{\gamma}}}. \end{aligned} \quad (12)$$

Let $\omega(t) = \widetilde{\omega}(\xi)$. Then $D_t^\alpha w(t) = \widetilde{w}'(\xi)$, and $D_t^\alpha\zeta(t) = \widetilde{\zeta}'(\xi)$, $D_t^\alpha\rho(t) = \widetilde{\rho}'(\xi)$. So (12) can be transformed to the following form

$$\begin{aligned} \widetilde{\omega}'(\xi) &\leq -KA(\xi)\widetilde{\zeta}(\xi)\widetilde{q}(\xi) \\ &+ \widetilde{\zeta}(\xi)\widetilde{\rho}'(\xi) - \frac{\widetilde{\rho}^{1+\frac{1}{\gamma}}(\xi)\widetilde{\zeta}(\xi)}{[\widetilde{r}(\xi)A(\xi)]^{\frac{1}{\gamma}}} \\ &+ \frac{\{(\gamma+1)\widetilde{\zeta}(\xi)\widetilde{\rho}^{\frac{1}{\gamma}}(\xi) + \widetilde{\zeta}'(\xi)[\widetilde{r}(\xi)A(\xi)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\widetilde{\zeta}^\gamma(\xi)[\widetilde{r}(\xi)A(\xi)]^{\frac{1}{\gamma}}}, \\ &\xi \geq \xi_2. \end{aligned} \quad (13)$$

Substituting ξ with s in (13), an integration for (13) with respect to s from ξ_2 to ξ yields that

$$\begin{aligned} &\int_{\xi_2}^{\xi}\{KA(s)\widetilde{\zeta}(s)\widetilde{q}(s) - \widetilde{\zeta}(s)\widetilde{\rho}'(s) + \frac{\widetilde{\rho}^{1+\frac{1}{\gamma}}(s)\widetilde{\zeta}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}} \\ &- \frac{\{(\gamma+1)\widetilde{\zeta}(s)\widetilde{\rho}^{\frac{1}{\gamma}}(s) + \widetilde{\zeta}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\widetilde{\zeta}^\gamma(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds \\ &\leq \widetilde{\omega}(\xi_2) - \omega(\xi) \leq \omega(\xi_2) < \infty, \end{aligned}$$

which contradicts to (9). The proof is then complete. □

Theorem 4 *Assume (6) holds, and there exists a function $H \in C([\xi_0, \infty), \mathbf{R})$ such that $H(\xi, \xi) = 0$, for $\xi \geq \xi_0$, $H(\xi, s) > 0$, for $\xi > s \geq \xi_0$, and H has a nonpositive continuous partial derivative $H'_s(\xi, s)$. If*

$$\begin{aligned} &\limsup_{\xi\to\infty}\frac{1}{H(\xi,\xi_0)}\Big\{\int_{\xi_0}^{\xi}H(\xi,s)\{KA(s)\widetilde{\zeta}(s)\widetilde{q}(s) \\ &- \widetilde{\zeta}(s)\widetilde{\rho}'(s) + \frac{\widetilde{\rho}^{1+\frac{1}{\gamma}}(s)\widetilde{\zeta}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}} \\ &- \frac{\{(\gamma+1)\widetilde{\zeta}(s)\widetilde{\rho}^{\frac{1}{\gamma}}(s) + \widetilde{\zeta}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\widetilde{\zeta}^\gamma(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds\Big\} \\ &= \infty, \end{aligned} \quad (14)$$

where $\widetilde{\zeta}$, $\widetilde{\rho}$ are defined as in Theorem 3. Then every solution of Eq. (1) is oscillatory.

Proof. Assume (1) has a non-oscillatory solution x on $[t_0, \infty)$. Without loss of generality, we may assume $x(t) > 0$ on $[t_1, \infty)$, where t_1 is sufficiently large. By Lemma 1 we have $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$ for some sufficiently large $t_2 > t_1$. Let $\omega(t)$ and $\widetilde{\omega}(\xi)$ be defined as in Theorem 3. By (11), for $\xi \geq \xi_2$ we have

$$\begin{aligned} &KA(\xi)\widetilde{\zeta}(\xi)\widetilde{q}(\xi) - \widetilde{\zeta}(\xi)\widetilde{\rho}'(\xi) + \frac{\widetilde{\rho}^{1+\frac{1}{\gamma}}(\xi)\widetilde{\zeta}(\xi)}{[\widetilde{r}(\xi)A(\xi)]^{\frac{1}{\gamma}}} \\ &- \frac{\{(\gamma+1)\widetilde{\zeta}(\xi)\widetilde{\rho}^{\frac{1}{\gamma}}(\xi) + \widetilde{\zeta}'(\xi)[\widetilde{r}(\xi)A(\xi)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\widetilde{\zeta}^\gamma(\xi)[\widetilde{r}(\xi)A(\xi)]^{\frac{1}{\gamma}}} \\ &\leq -\widetilde{\omega}(\xi). \end{aligned} \quad (15)$$

Substituting ξ with s in (16), multiplying both sides by $H(\xi,s)$ and then integrating it with respect to s from ξ_2 to ξ yields

$$\int_{\xi_2}^{\xi} H(\xi,s)\{KA(s)\widetilde{\zeta}(s)\widetilde{q}(s)-\widetilde{\zeta}(s)\widetilde{\rho}'(s) + \frac{\widetilde{\rho}^{1+\frac{1}{\gamma}}(s)\widetilde{\zeta}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}} - \frac{\{(\gamma+1)\widetilde{\zeta}(s)\widetilde{\rho}^{\frac{1}{\gamma}}(s)+\widetilde{\zeta}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\widetilde{\zeta}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds$$
$$\le -\int_{\xi_2}^{\xi} H(\xi,s)\widetilde{\omega}'(s)ds$$
$$= H(\xi,\xi_2)\omega(\xi_2) + \int_{\xi_2}^{\xi} H_s'(\xi,s)\omega(s)\Delta s$$
$$\le H(\xi,\xi_2)\omega(\xi_2) \le H(\xi,\xi_0)\omega(\xi_2).$$

Then

$$\int_{\xi_0}^{\xi} H(\xi,s)\{KA(s)\widetilde{\zeta}(s)\widetilde{q}(s)-\widetilde{\zeta}(s)\widetilde{\rho}'(s) + \frac{\widetilde{\rho}^{1+\frac{1}{\gamma}}(s)\widetilde{\zeta}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}} - \frac{\{(\gamma+1)\widetilde{\zeta}(s)\widetilde{\rho}^{\frac{1}{\gamma}}(s)+\widetilde{\zeta}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\widetilde{\zeta}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds$$
$$= \int_{\xi_0}^{\xi_2} H(\xi,s)\{KA(s)\widetilde{\zeta}(s)\widetilde{q}(s)-\widetilde{\zeta}(s)\widetilde{\rho}'(s) + \frac{\widetilde{\rho}^{1+\frac{1}{\gamma}}(s)\widetilde{\zeta}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}} - \frac{\{(\gamma+1)\widetilde{\zeta}(s)\widetilde{\rho}^{\frac{1}{\gamma}}(s)+\widetilde{\zeta}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\widetilde{\zeta}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds$$
$$+ \int_{\xi_2}^{\xi} H(\xi,s)\{KA(s)\widetilde{\zeta}(s)\widetilde{q}(s)-\widetilde{\zeta}(s)\widetilde{\rho}'(s) + \frac{\widetilde{\rho}^{1+\frac{1}{\gamma}}(s)\widetilde{\zeta}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}} - \frac{\{(\gamma+1)\widetilde{\zeta}(s)\widetilde{\rho}^{\frac{1}{\gamma}}(s)+\widetilde{\zeta}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\widetilde{\zeta}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds$$
$$\le H(\xi,\xi_0)\widetilde{\omega}(\xi_2) + H(\xi,\xi_0)\int_{\xi_0}^{\xi_2} |KA(s)\widetilde{\zeta}(s)\widetilde{q}(s)-\widetilde{\zeta}(s)\widetilde{\rho}'(s) + \frac{\widetilde{\rho}^{1+\frac{1}{\gamma}}(s)\widetilde{\zeta}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}} - \frac{\{(\gamma+1)\widetilde{\zeta}(s)\widetilde{\rho}^{\frac{1}{\gamma}}(s)+\widetilde{\zeta}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\widetilde{\zeta}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}|ds.$$

So

$$\limsup_{\xi\to\infty}\frac{1}{H(\xi,\xi_0)}\{\int_{\xi_0}^{\xi} H(\xi,s)\{KA(s)\widetilde{\zeta}(s)\widetilde{q}(s) -\widetilde{\zeta}(s)\widetilde{\rho}'(s) + \frac{\widetilde{\rho}^{1+\frac{1}{\gamma}}(s)\widetilde{\zeta}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}} - \frac{\{(\gamma+1)\widetilde{\zeta}(s)\widetilde{\rho}^{\frac{1}{\gamma}}(s)+\widetilde{\zeta}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\widetilde{\zeta}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds\}$$
$$\le \widetilde{\omega}(\xi_2) + \int_{\xi_0}^{\xi_2} |KA(s)\widetilde{\zeta}(s)\widetilde{q}(s)-\widetilde{\zeta}(s)\widetilde{\rho}'(s) + \frac{\widetilde{\rho}^{1+\frac{1}{\gamma}}(s)\widetilde{\zeta}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}} - \frac{\{(\gamma+1)\widetilde{\zeta}(s)\widetilde{\rho}^{\frac{1}{\gamma}}(s)+\widetilde{\zeta}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\widetilde{\zeta}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}|ds$$
$$< \infty,$$

which contradicts to (14). So the proof is complete. □

Based on Theorem 4, we have the following two corollaries.

Corollary 5 *Under the conditions of Theorem 4, if*

$$\limsup_{\xi\to\infty}\frac{1}{(\xi-\xi_0)^{\lambda}}\{\int_{\xi_0}^{\xi} (\xi-s)^{\lambda}\{KA(s)\widetilde{\zeta}(s)\widetilde{q}(s) -\widetilde{\zeta}(s)\widetilde{\rho}'(s) + \frac{\widetilde{\rho}^{1+\frac{1}{\gamma}}(s)\widetilde{\zeta}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}} - \frac{\{(\gamma+1)\widetilde{\zeta}(s)\widetilde{\rho}^{\frac{1}{\gamma}}(s)+\widetilde{\zeta}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\widetilde{\zeta}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds\} = \infty,$$

then every solution of Eq. (1) is oscillatory.

Corollary 6 *Under the conditions of Theorem 4, if*

$$\limsup_{\xi\to\infty}\frac{1}{(\ln\xi-\ln\xi_0)}\{\int_{\xi_0}^{\xi} (\ln\xi-\ln s) \{KA(s)\widetilde{\zeta}(s)\widetilde{q}(s) -\widetilde{\zeta}(s)\widetilde{\rho}'(s) + \frac{\widetilde{\rho}^{1+\frac{1}{\gamma}}(s)\widetilde{\zeta}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}} - \frac{\{(\gamma+1)\widetilde{\zeta}(s)\widetilde{\rho}^{\frac{1}{\gamma}}(s)+\widetilde{\zeta}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\widetilde{\zeta}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\}ds\} = \infty,$$

then every solution of Eq. (1) is oscillatory.

The proof of Corollaries 5-6 can be completed by choosing $H(\xi,s)=(\xi-s)^\lambda,\ \lambda>1$ or $H(\xi,s)=\ln\frac{\xi}{s}$ in Theorem 4 respectively.

Theorem 7 *Assume (6) does not hold. If (9) holds, and*

$$\int_T^\infty [\frac{1}{A(\tau)\widetilde{r}(\tau)}\int_T^\tau A(s)\widetilde{q}(s)ds]^{\frac{1}{\gamma}}d\tau=\infty. \quad (16)$$

Then every solution x of Eq.(1) is oscillatory or satisfies $\lim\limits_{t\to\infty}x(t)=0$.

Proof. Assume (1) has a non-oscillatory solution x on $[t_0,\infty)$, and let $x(t)=\widetilde{x}(\xi)$. Without loss of generality, we may assume $x(t)>0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By (8) one can see that $A(\xi)\widetilde{r}(\xi)(\widetilde{x}'(\xi))^\gamma=A(\xi)r(t)[D_t^\alpha x(t)]^\gamma$ is strictly decreasing on $[t_1,\infty)$, which implies $D_t^\alpha x(t)$ is eventually of one sign. If $D_t^\alpha x(t)>0,\ t\in[t_2,\infty)$ for some sufficiently t_2, then after carrying out the proof process similar to that of Theorem 3, we obtain that the solution x is oscillatory. Now we assume $D_t^\alpha x(t)<0,\ t\in[t_2,\infty)$ for some sufficiently t_2. Then $D_t^\alpha x(t)=D_t^\alpha\widetilde{x}(\xi)=\widetilde{x}'(\xi)D_t^\alpha\xi=\widetilde{x}'(\xi)<0,\ \xi\in[\xi_2,\infty)$. Since $\widetilde{x}(\xi)>0$, furthermore we have $\lim\limits_{\xi\to\infty}\widetilde{x}(\xi)=\beta\ge 0$. We claim $\beta=0$. Otherwise, assume $\beta>0$. Then $x(t)\ge\beta$ on $[t_2,\infty)$, and by (8) we have

$$\begin{aligned}&(A(\xi)\widetilde{r}(\xi)(\widetilde{x}'(\xi))^\gamma)'\le -KA(\xi)\widetilde{q}(\xi)(\widetilde{x}(\xi))^\gamma\\&\le -K\beta^\gamma A(\xi)\widetilde{q}(\xi),\ \xi\ge\xi_2.\end{aligned} \quad (17)$$

Substituting t with s in (17), and integrating it with respect to s from ξ_2 to ξ yields that

$$\begin{aligned}&A(\xi)\widetilde{r}(\xi)(\widetilde{x}'(\xi))^\gamma\\&\le A(\xi_2)\widetilde{r}(\xi_2)(\widetilde{x}'(\xi_2))^\gamma-K\beta^\gamma\int_{\xi_2}^\xi A(s)\widetilde{q}(s)ds\\&\le -K\beta^\gamma\int_{\xi_2}^\xi A(s)\widetilde{q}(s)ds,\end{aligned}$$

which means

$$\widetilde{x}'(\xi)<-\beta K^{\frac{1}{\gamma}}[\frac{1}{A(\xi)\widetilde{r}(\xi)}\int_{\xi_2}^\xi A(s)\widetilde{q}(s)ds]^{\frac{1}{\gamma}}. \quad (18)$$

Substituting ξ with τ in (18), and integrating it with respect to τ from ξ_2 to ξ yields that

$$\begin{aligned}&\widetilde{x}(\xi)-\widetilde{x}(\xi_2)\\&<-\beta K^{\frac{1}{\gamma}}\int_{\xi_2}^\xi[\frac{1}{A(\tau)\widetilde{r}(\tau)}\int_{\xi_2}^\tau A(s)\widetilde{q}(s)ds]^{\frac{1}{\gamma}}d\tau.\end{aligned}$$

By (16), one can see $\lim\limits_{\xi\to\infty}\widetilde{x}(\xi)=-\infty$, which is a contradiction. So the proof is complete. □

Finally, based on the theorems above, we have the following theorem.

Theorem 8 *Assume (6) does not hold. If (14) and (16) hold, then every solution x of Eq. (1) is oscillatory or satisfies $\lim\limits_{t\to\infty}x(t)=0$.*

Remark 9 *From the above process of establishing oscillatory criteria for the functional fractional differential equation (1), one can see that the most important points lie in two aspects. The first point is that Eq.(1) is transformed to a new equivalent differential equation of integer order with respect to the variable ξ by use of the properties of the modified Riemann-Liouville fractional derivative, which is Eq.(7). Since $\xi=\xi(t)$ is increasing with respect to t, then the discussion of the oscillation of the solution $x(t)$ of Eq.(1) can be transformed to the discussion of the oscillation of the solution $\widetilde{x}(\xi)$ of Eq.(7). The other point is that the construction of a suitable Riccati transformation function, which is denoted by $\omega(t)$, and plays an important role in the proof of oscillatory results. We note that following this analysis model as shown above, oscillatory criteria for other kinds of fractional differential equations in the sense of the modified Riemann-Liouville fractional derivative can also be easily obtained. So in this sense, the analytical method presented above is of general meaning, and can find its broad applications in the research of oscillation of other fractional differential equations.*

4 Oscillatory criteria for Eq. (2)

In this section, we use the analytical method summarized in Remark 1 to establish oscillatory criteria for Eq.(2). First we prove the following lemma.

Lemma 10 *Assume $x(t)$ is a eventually positive solution of Eq. (2), and*

$$\int_{\xi_0}^\infty\frac{1}{\widetilde{r}(s)}ds=\infty, \quad (19)$$

$$\int_{\xi_0}^\infty\frac{1}{\widetilde{r}(\zeta)}\int_\zeta^\infty\int_\tau^\infty\widetilde{q}(s)dsd\tau d\zeta=\infty. \quad (20)$$

Then there exists a sufficiently large T such that $D_t^\alpha(r(t)D_t^\alpha x(t))>0$ on $[T,\infty)$, and either $D_t^\alpha x(t)>0$ on $[T,\infty)$ or $\lim\limits_{t\to\infty}x(t)=0$.

Proof. Let $x(t) = \widetilde{x}(\xi)$, where $\xi = \frac{t^\alpha}{\Gamma(1+\alpha)}$. Then similar to the process of Lemma 1, Eq.(2) can be transformed into the following form:

$$(\widetilde{r}(\xi)\widetilde{x}'(\xi))'' + \widetilde{q}(\xi)\widetilde{x}(\xi) = 0,\ \xi \geq \xi_0 > 0. \quad (21)$$

Since $x(t)$ is a eventually positive solution of (2), then $\widetilde{x}(\xi)$ is a eventually positive solution of Eq. (21), and there exists $\xi_1 > \xi_0$ such that $\widetilde{x}(\xi) > 0$ on $[\xi_1, \infty)$. Furthermore, we have

$$(\widetilde{r}(\xi)\widetilde{x}'(\xi))'' = -\widetilde{q}(\xi)\widetilde{x}(\xi) < 0,\ \xi \geq \xi_1. \quad (22)$$

Then $(\widetilde{r}(\xi)\widetilde{x}'(\xi))'$ is strictly decreasing on $[\xi_1, \infty)$, and thus $(\widetilde{r}(\xi)\widetilde{x}'(\xi))'$ is eventually of one sign. We claim $(\widetilde{r}(\xi)\widetilde{x}'(\xi))' > 0$ on $[\xi_2, \infty)$, where $\xi_2 > \xi_1$ is sufficiently large. Otherwise, assume there exists a sufficiently large $\xi_3 > \xi_2$ such that $(\widetilde{r}(\xi)\widetilde{x}'(\xi))' < 0$ on $[\xi_3, \infty)$. Then there exists a sufficiently large ξ_4 with $\xi_4 > \xi_3$ such that $\widetilde{x}'(\xi) < 0,\ \xi \in [\xi_4, \infty)$. Furthermore,

$$\widetilde{x}(\xi) - \widetilde{x}(\xi_4) = \int_{\xi_4}^{\xi} \widetilde{x}'(s)ds = \int_{\xi_4}^{\xi} \frac{\widetilde{r}(s)\widetilde{x}'(s)}{\widetilde{r}(s)}ds$$
$$\leq \widetilde{r}(\xi_4)\widetilde{x}'(\xi_4) \int_{\xi_4}^{\xi} \frac{1}{\widetilde{r}(s)}ds.$$

By (19) we deduce that $\lim\limits_{\xi\to\infty} \widetilde{x}(\xi) = -\infty$, which contradicts to the fact that $\widetilde{x}(\xi)$ is a eventually positive solution of Eq. (21). So $(\widetilde{r}(\xi)\widetilde{x}'(\xi))' > 0$ on $[\xi_2, \infty)$, and $D_t^\alpha(r(t)D_t^\alpha x(t)) > 0$ on $[t_2, \infty)$. Thus $D_t^\alpha x(t) = \widetilde{x}'(\xi)$ is eventually of one sign. Now we assume $\widetilde{x}'(\xi) < 0,\ \xi \in [\xi_5, \infty)$ for some sufficiently large $\xi_5 > \xi_4$. Since $\widetilde{x}(\xi) > 0$, furthermore we have $\lim\limits_{\xi\to\infty} \widetilde{x}(\xi) = \beta \geq 0$. We claim $\beta = 0$. Otherwise, assume $\beta > 0$. Then $\widetilde{x}(\xi) \geq \beta$ on $[\xi_5, \infty)$, and for $\xi \in [\xi_5, \infty)$, by (21) we have

$$(\widetilde{r}(\xi)\widetilde{x}'(\xi))'' \leq -\widetilde{q}(\xi)\widetilde{x}(\xi) \leq -\widetilde{q}(\xi)\beta.$$

Substituting ξ with s in the inequality above, an integration with respect to s from ξ to ∞ yields

$$-(\widetilde{r}(\xi)\widetilde{x}'(\xi))' \leq -\lim_{\xi\to\infty}(\widetilde{r}(\xi)\widetilde{x}(\xi))' - \beta\int_{\xi}^{\infty} \widetilde{q}(s)ds$$
$$< -\beta\int_{\xi}^{\infty} \widetilde{q}(s)ds,$$

which means

$$(\widetilde{r}(\xi)\widetilde{x}'(\xi))' > \beta\int_{\xi}^{\infty} \widetilde{q}(s)ds. \quad (23)$$

Substituting ξ with τ in (23), an integration for (23) with respect to τ from ξ to ∞ yields

$$-\widetilde{r}(\xi)\widetilde{x}'(\xi)$$
$$> -\lim_{\xi\to\infty} \widetilde{r}(\xi)\widetilde{x}'(\xi) + \beta\int_{\xi}^{\infty}\int_{\tau}^{\infty} \widetilde{q}(s)dsd\tau$$
$$> \beta\int_{\xi}^{\infty}\int_{\tau}^{\infty} \widetilde{q}(s)dsd\tau,$$

that is,

$$\widetilde{x}'(\xi) < -\beta\frac{1}{\widetilde{r}(\xi)}\int_{\xi}^{\infty}\int_{\tau}^{\infty} \widetilde{q}(s)dsd\tau. \quad (24)$$

Substituting ξ with ζ in (2.7), an integration for (24) with respect to ζ from ξ_5 to ξ yields

$$\widetilde{x}(\xi) - \widetilde{x}(\xi_5) < -\beta\int_{\xi_5}^{\xi}\frac{1}{\widetilde{r}(\zeta)}\int_{\zeta}^{\infty}\int_{\tau}^{\infty} \widetilde{q}(s)dsd\tau d\zeta.$$

By (20), one can see $\lim_{t\to\infty} \widetilde{x}(\xi) = -\infty$, which causes a contradiction. The proof is complete. $\square$

Theorem 11 *If (19)-(20) hold, and there exist $\phi \in C^1([t_0, \infty), \mathbf{R}_+)$ and $\varphi \in C^1([t_0, \infty), [0, \infty))$ such that*

$$\int_{\xi_0}^{\infty}\{\widetilde{\phi}(s)\widetilde{q}(s) - \widetilde{\phi}(s)\widetilde{\varphi}'(s) + \frac{\widetilde{\phi}(s)\widetilde{\Delta}(s,\xi_2)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}$$
$$-\frac{[2\widetilde{\varphi}(s)\widetilde{\phi}(s)\widetilde{\Delta}(s,\xi_2) + \widetilde{r}(s)\widetilde{\phi}'(s)]^2}{4\widetilde{\phi}(s)\widetilde{\Delta}(s,\xi_2)\widetilde{r}(s)}\}ds$$
$$= \infty, \quad (25)$$

where $\widetilde{\phi}(\xi) = \phi(t)$, $\widetilde{\varphi}(\xi) = \varphi(t)$, and $\widetilde{\Delta}$ is defined as below. Then every solution of Eq. (2) is oscillatory or satisfies $\lim\limits_{t\to\infty} x(t) = 0$.

Proof. Assume (2) has a nonoscillatory solution x on $[t_0, \infty)$. Without loss of generality, we may assume $x(t) > 0$ on $[t_1, \infty)$, where t_1 is sufficiently large. By Lemma 10 we have $D_t^\alpha(r(t)D_t^\alpha x(t)) > 0,\ t \in [t_2, \infty)$, where $t_2 > t_1$ is sufficiently large, and either $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$ or $\lim\limits_{t\to\infty} x(t) = 0$. Now we assume $D_t^\alpha x(t) > 0$ on $[t_2, \infty)$. Define the generalized Riccati function:

$$\omega(t) = \phi(t)\{\frac{D_t^\alpha(r(t)D_t^\alpha x(t))}{x(t)} + \varphi(t)\}. \quad (26)$$

Then for $t \in [t_2, \infty)$, we have

$$D_t^\alpha\omega(t) = D_t^\alpha\phi(t)\frac{D_t^\alpha(r(t)D_t^\alpha x(t))}{x(t)}$$

$$+\phi(t)D_t^\alpha\{\frac{D_t^\alpha(r(t)D_t^\alpha x(t))}{x(t)}\}$$
$$+D_t^\alpha\phi(t)\varphi(t)+\phi(t)D_t^\alpha\varphi(t)$$
$$=-\phi(t)q(t)-\frac{\phi(t)D_t^\alpha x(t)D_t^\alpha(r(t)D_t^\alpha x(t))}{x^2(t)}$$
$$+\frac{D_t^\alpha\phi(t)}{\phi(t)}\omega(t)+\phi(t)D_t^\alpha\varphi(t). \quad (27)$$

On the other hands, By (22), we obtain that $(\widetilde{r}(\xi)\widetilde{x}'(\xi))'$ is strictly decreasing on $[\xi_2,\infty)$. So

$$\widetilde{r}(\xi)\widetilde{x}'(\xi)\geq\widetilde{r}(\xi)\widetilde{x}'(\xi)-\widetilde{r}(\xi_2)\widetilde{x}'(\xi_2)$$
$$=\int_{\xi_2}^{\xi}(\widetilde{r}(s)\widetilde{x}'(s))'ds\geq(\widetilde{r}(\xi)\widetilde{x}'(\xi))'(\xi-\xi_2),$$

that is

$$D_t^\alpha x(t)$$
$$\geq\frac{D_t^\alpha(r(t)D_t^\alpha x(t))}{r(t)}(\frac{t^\alpha}{\Gamma(1+\alpha)}-\frac{t_2^\alpha}{\Gamma(1+\alpha)}).$$

Denote $\Delta(t,t_2)=\frac{t^\alpha}{\Gamma(1+\alpha)}-\frac{t_2^\alpha}{\Gamma(1+\alpha)}$. Then furthermore we have

$$D_t^\alpha\omega(t)\leq-\phi(t)q(t)$$
$$-\frac{\phi(t)\Delta(t,t_2)}{r(t)}[\frac{\omega(t)}{\phi(t)}-\varphi(t)]^2$$
$$+\frac{D_t^\alpha\phi(t)}{\phi(t)}\omega(t)+\phi(t)D_t^\alpha\varphi(t)$$
$$=-\phi(t)q(t)+\phi(t)D_t^\alpha\varphi(t)$$
$$-\frac{\phi(t)\Delta(t,t_2)\varphi^2(t)}{r(t)}-\frac{\Delta(t,t_2)}{r(t)\phi(t)}\omega^2(t)$$
$$+\frac{2\varphi(t)\phi(t)\Delta(t,t_2)+r(t)D_t^\alpha\phi(t)}{r(t)\phi(t)}\omega(t)$$
$$\leq-\phi(t)q(t)+\phi(t)D_t^\alpha\varphi(t)$$
$$-\frac{\phi(t)\Delta(t,t_2)\varphi^2(t)}{r(t)}$$
$$+\frac{[2\varphi(t)\phi(t)\Delta(t,t_2)+r(t)D_t^\alpha\phi(t)]^2}{4\phi(t)\Delta(t,t_2)r(t)},$$
$$t\geq t_2. \quad (28)$$

Let $\widetilde{\Delta}(\xi,\xi_2)=\Delta(t,t_2)$, $\widetilde{\omega}(\xi)=\omega(t)$, $\widetilde{\phi}(\xi)=\phi(t)$, $\widetilde{\varphi}(\xi)=\varphi(t)$. Then (28) can be transformed to the following form:

$$\widetilde{\omega}'(\xi)\leq-\widetilde{\phi}(\xi)\widetilde{q}(\xi)+\widetilde{\phi}(\xi)\widetilde{\varphi}'(\xi)$$
$$-\frac{\widetilde{\phi}(\xi)\widetilde{\Delta}(\xi,\xi_2)\widetilde{\varphi}^2(\xi)}{\widetilde{r}(\xi)}$$
$$+\frac{[2\widetilde{\varphi}(\xi)\widetilde{\phi}(\xi)\widetilde{\Delta}(\xi,\xi_2)+\widetilde{r}(\xi)\widetilde{\phi}'(\xi)]^2}{4\widetilde{\phi}(\xi)\widetilde{\Delta}(\xi,\xi_2)\widetilde{r}(\xi)}$$
$$\xi\geq\xi_2. \quad (29)$$

Substituting ξ with s in (29), an integration for (29) with respect to s from ξ_2 to ξ yields

$$\int_{\xi_2}^{\xi}\{\widetilde{\phi}(s)\widetilde{q}(s)-\widetilde{\phi}(s)\widetilde{\varphi}'(s)+\frac{\widetilde{\phi}(s)\widetilde{\Delta}(s,\xi_2)\widetilde{\varphi}^2(s)}{\widetilde{r}(s)}$$
$$-\frac{[2\widetilde{\varphi}(s)\widetilde{\phi}(s)\widetilde{\Delta}(s,\xi_2)+\widetilde{r}(s)\widetilde{\phi}'(s)]^2}{4\widetilde{\phi}(s)\widetilde{\Delta}(s,\xi_2)\widetilde{r}(s)}\}ds$$
$$\leq\omega(\xi_2)-\omega(\xi)\leq\omega(\xi_2)<\infty,$$

which contradicts to (25). So the proof is complete.

Remark 12 *From the results established above one can see that using the method summarized in Remark 9, oscillatory criteria for fractional differential equations of higher order can also be established.*

5 Applications

Example 1. Consider the following functional fractional differential equation:

$$D_t^\alpha(\sqrt{\frac{t^\alpha}{\Gamma(1+\alpha)}}(D_t^\alpha x(t))^{\frac{5}{3}})+\frac{\Gamma(1+\alpha)}{t^\alpha}D_t^\alpha x(t)$$
$$+(\frac{t^\alpha}{\Gamma(1+\alpha)})^{-\frac{13}{6}}x^{\frac{5}{3}}(t)e^{x^2(t)}=0,$$
$$t\geq 2,\ 0<\alpha<1. \quad (30)$$

In Eq. (1), if we set $t_0=2$, $\gamma=\frac{5}{3}$,

$$r(t)=\sqrt{\frac{t^\alpha}{\Gamma(1+\alpha)}},\quad p(t)=\frac{\Gamma(1+\alpha)}{t^\alpha},$$

$$q(t)=(\frac{t^\alpha}{\Gamma(1+\alpha)})^{-\frac{13}{6}},\quad f(x)=x^{\frac{5}{3}}e^{x^2},$$

then we obtain (2). So $\xi_0=\frac{2^\alpha}{\Gamma(1+\alpha)}$,

$$\widetilde{r}(\xi)=r(t)=\sqrt{\frac{t^\alpha}{\Gamma(1+\alpha)}}=\sqrt{\xi},$$

$$\widetilde{p}(\xi)=p(t)=\frac{\Gamma(1+\alpha)}{t^\alpha}=\xi^{-1},$$

$$\widetilde{q}(\xi)=q(t)=(\frac{t^\alpha}{\Gamma(1+\alpha)})^{-\frac{13}{6}}=\xi^{-\frac{13}{6}}$$

and $f(x)/x^{\frac{5}{3}}\geq 1$, which implies $K=1$. Furthermore, since

$$A(\xi)=\exp(\int_{\xi_0}^{\xi}\tau^{-\frac{3}{2}}d\tau)=\exp(2\xi_0^{-\frac{1}{2}}-2\xi^{-\frac{1}{2}})$$

so $1 \leq A(\xi) \leq \exp(2\xi_0^{-\frac{1}{2}})$, and in (6),

$$\int_{\xi_0}^{\infty} \frac{1}{[A(s)\widetilde{r}(s)]^{\frac{1}{\gamma}}} ds \geq \exp(-\frac{6}{5}\xi_0^{-\frac{1}{2}}) \int_{\xi_0}^{\infty} s^{-\frac{3}{10}} ds = \infty.$$

On the other hands, in (9), letting $\widetilde{\zeta}(\xi) = \xi^{\frac{7}{6}},\ \widetilde{\rho}(\xi) = 0$, we obtain

$$\begin{aligned}&\int_{\xi_0}^{\infty} \{KA(s)\widetilde{\zeta}(s)\widetilde{q}(s) - \widetilde{\zeta}(s)\widetilde{\rho}'(s) + \frac{\widetilde{\rho}^{1+\frac{1}{\gamma}}(s)\widetilde{\zeta}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}} \\ &- \frac{\{(\gamma+1)\widetilde{\zeta}(s)\widetilde{\rho}^{\frac{1}{\gamma}}(s) + \widetilde{\zeta}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\widetilde{\zeta}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\} ds \\ &= \int_{\xi_0}^{\infty} A(s)[1 - (\frac{8}{3})^{-\frac{8}{3}}]\frac{1}{s} ds \\ &\geq \int_{\xi_0}^{\infty} [1 - (\frac{8}{3})^{-\frac{8}{3}}]\frac{1}{s} ds = \infty.\end{aligned}$$

Therefore, Eq.(30) is oscillatory by Theorem 3. □

Example 2. Consider the following functional fractional differential equation:

$$\begin{aligned}&D_t^{\alpha}(\frac{t^{\alpha}}{\Gamma(1+\alpha)}(D_t^{\alpha}x(t))^{\frac{1}{3}}) + \frac{[\Gamma(1+\alpha)]^2}{t^{2\alpha}} D_t^{\alpha}x(t) \\ &+ \frac{t^{\alpha}}{\Gamma(1+\alpha)} x^{\frac{1}{3}}(t)[1 + x^2(t)] = 0, \\ &t \geq 5,\ 0 < \alpha < 1. \end{aligned} \tag{31}$$

In fact, if we set in Eq. (1) $t_0 = 5,\ \gamma = \frac{1}{3}$,

$$r(t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)},\quad p(t) = \frac{[\Gamma(1+\alpha)]^2}{t^{2\alpha}},$$

$$q(t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)},\quad f(x) = x^{\frac{1}{3}}(1 + x^2)$$

then we obtain (19). So $\xi_0 = \frac{5^{\alpha}}{\Gamma(1+\alpha)}$,

$$\widetilde{r}(\xi) = r(t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)} = \xi$$

$$\widetilde{p}(\xi) = p(t) = \frac{[\Gamma(1+\alpha)]^2}{t^{2\alpha}} = \xi^{-2}$$

$$\widetilde{q}(\xi) = q(t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)} = \xi$$

and $f(x)/x^{\frac{1}{3}} \geq 1$, which implies $K = 1$. Furthermore, since

$$A(\xi) = \exp(\int_{\xi_0}^{\xi} \tau^{-3} d\tau) = \exp(\frac{1}{2}\xi_0^{-\frac{1}{2}} - \frac{1}{2}\xi^{-\frac{1}{2}})$$

so $1 \leq A(\xi) \leq \exp(\frac{1}{2}\xi_0^{-\frac{1}{2}})$, and in (6),

$$\int_{\xi_0}^{\infty} \frac{1}{[A(s)\widetilde{r}(s)]^{\frac{1}{\gamma}}} ds \leq \int_{\xi_0}^{\infty} s^{-3} ds = \frac{1}{2}\xi_0^{-2} = \frac{1}{2}[\frac{5^{\alpha}}{\Gamma(1+\alpha)}]^{-2} < \infty.$$

So (6) does not hold. On the other hand, in (9) and (16), letting $\widetilde{\zeta}(\xi) = \xi,\ \widetilde{\rho}(\xi) = 0$, we obtain

$$\begin{aligned}&\int_{\xi_0}^{\infty} \{KA(s)\widetilde{\zeta}(s)\widetilde{q}(s) - \widetilde{\zeta}(s)\widetilde{\rho}'(s) + \frac{\widetilde{\rho}^{1+\frac{1}{\gamma}}(s)\widetilde{\zeta}(s)}{[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}} \\ &- \frac{\{(\gamma+1)\widetilde{\zeta}(s)\widetilde{\rho}^{\frac{1}{\gamma}}(s) + \widetilde{\zeta}'(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}\}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\widetilde{\zeta}^{\gamma}(s)[\widetilde{r}(s)A(s)]^{\frac{1}{\gamma}}}\} ds \\ &= \int_{\xi_0}^{\infty} A(s)[s^2 - (\frac{4}{3})^{-\frac{4}{3}} s^{\frac{2}{3}}] ds \\ &\geq \int_{\xi_0}^{\infty} [s^2 - (\frac{4}{3})^{-\frac{4}{3}} s^{\frac{2}{3}}] ds = \infty,\end{aligned}$$

and

$$\begin{aligned}&\int_{T}^{\infty} [\frac{1}{A(\tau)\widetilde{r}(\tau)} \int_{T}^{\tau} A(s)\widetilde{q}(s) ds]^{\frac{1}{\gamma}} d\tau \\ &\geq \exp(\frac{3}{2}\xi_0^{-\frac{1}{2}}) \int_{T}^{\infty} [\frac{1}{\widetilde{r}(\tau)} \int_{T}^{\tau} \widetilde{q}(s) ds]^{\frac{1}{\gamma}} d\tau \\ &= \exp(\frac{3}{2}\xi_0^{-\frac{1}{2}}) \int_{T}^{\infty} [\frac{1}{\tau} \int_{T}^{\tau} s ds]^{\frac{1}{\gamma}} d\tau \\ &= \exp(\frac{3}{2}\xi_0^{-\frac{1}{2}}) \int_{T}^{\infty} [\frac{\tau^2 - T^2}{2\tau}]^3 d\tau = \infty.\end{aligned}$$

Therefore, (9) and (16) hold, and then Eq. (31) is oscillatory according to Theorem 7.

Remark 13 *The oscillatory results for the two examples above can not be obtained by the oscillatory criteria established in [13-17].*

Remark 14 *It is worthy to note that the method used in Section 3 to establish oscillatory criteria for the functional fractional differential equation (1) can be used not only in the analysis of oscillation, but also in the analysis of asymptotic properties of solutions of functional fractional differential equations.*

6 Conclusions

In this paper, by use of certain generalized Riccati transformation functions, inequality and integration average technique, some new oscillatory criteria for a functional fractional differential equation with damping have been established. Using a similar analytical method, some oscillatory criteria for a higher order fractional differential equation have also been established. These results are of new forms so far in the literature. We note that the approach in establishing the main theorems above can be generalized to research oscillation of fractional differential equations with more complicated forms, which are expected to further research.

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