Two iterative algorithms for k-strictly pseudo-contractive mappings in a CAT(0) space

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Abstract: In this paper, we prove the Δ-convergence of the cyclic algorithm for k-strictly pseudo-contractive mappings and give also the strong convergence theorem of the modified Halpern iteration for these mappings in a CAT(0) space. Our results extend and improve the corresponding recent results announced by many authors in the literature.

Key-Words: CAT(0) space, fixed point, strong convergence, Δ-convergence, k-strictly pseudo-contractive mapping, iterative algorithm.

1 Introduction

Let C be a nonempty subset of a Hilbert space X. Recall that a mapping $T:C\to C$ is said to be k-strictly pseudo-contractive if there exists a constant $k\in[0,1)$ such that

$$\|Tx-Ty\|^2\le\|x-y\|^2+k\|(I-T)x-(I-T)y\|^2$$

for all $x,y\in C$.

A point $x\in C$ is called a fixed point of T if $x=Tx$. We will denote the set of fixed points of T by $F(T)$. Note that the class of k-strictly pseudo-contractions includes the class of nonexpansive mappings T on C as a subclass. That is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive. The mapping T is also said to be pseudo-contractive if $k=1$ and T is said to be strongly pseudo-contractive if there exists a constant $\lambda\in(0,1)$ such that $T-\lambda I$ is pseudo-contractive. Clearly, the class of k-strictly pseudo-contractive mappings is the one between classes of nonexpansive mappings and pseudo-contractive mappings. Also we remark that the class of strongly pseudo-contractive mappings is independent from the class of k-strictly pseudo-contractive mappings.

Recently, many authors have been devoting the studies on the problems of finding fixed points for k-strictly pseudo-contractive mappings (see, e.g., [1]- [6]).

We define the concept of k-strictly pseudo-contractive mapping in a $CAT(0)$ space as follows.

Let C be a nonempty subset of a $CAT(0)$ space X. A mapping $T:C\to C$ is said to be k-strictly pseudo-contractive if there exists a constant $k\in[0,1)$ such that

$$d(Tx,Ty)^2\le d(x,y)^2+k\left(d\left(x,Tx\right)+d\left(y,Ty\right)\right)^2 \quad (1)$$

for all $x,y\in C$.

Acedo and Xu [7] introduced a cyclic algorithm in a Hilbert space. We modify this algorithm in a $CAT(0)$ space.

Let $x_0\in C$ and $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a,b\in(0,1)$. The cyclic algorithm generates a sequence $\{x_n\}$ in the following way:

$$\begin{cases} x_1 = \alpha_0 x_0 \oplus (1-\alpha_0)T_0 x_0, \\ x_2 = \alpha_1 x_1 \oplus (1-\alpha_1)T_1 x_1, \\ \vdots \\ x_N = \alpha_{N-1} x_{N-1} \oplus (1-\alpha_{N-1})T_{N-1} x_{N-1}, \\ x_{N+1} = \alpha_N x_N \oplus (1-\alpha_N)T_0 x_N, \\ \vdots \end{cases}$$

or, shortly,

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)T_{[n]} x_n, \ \forall n \geq 0, \quad (2)$$

where $T_{[n]} = T_i$, with $i = n \ (modN)$, $0 \leq i \leq N-1$. By taking $T_{[n]} = T$ for all n in (2), we obtain the Mann iteration in [8].

In this paper, motivated by the above results, we prove the demiclosedness principle for k-strictly pseudo-contractive mappings in a $CAT(0)$ space. Also we present the Δ-convergence of the cyclic algorithm and the strong convergence the modified Halpern iteration which is introduced for Hilbert space by Hu [9] for these mappings in a $CAT(0)$ space.

2 Preliminaries on CAT(0) space

A metric space X is a $CAT(0)$ space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a $CAT(0)$ space. Other examples include Pre-Hilbert spaces (see [10]), Euclidean buildings (see [11]), $\mathbb{R}$-trees (see [12]), the complex Hilbert ball with a hyperbolic metric (see [13]) and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [10].

Fixed point theory in a $CAT(0)$ space has been first studied by Kirk (see [14], [15]). He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space always has a fixed point. Since then the fixed point theory in a $CAT(0)$ space has been rapidly developed and many papers have appeared (see e.g., [16]-[19]). It is worth mentioning that fixed point theorems in a $CAT(0)$ space (specially in $\mathbb{R}$-trees) can be applied to graph theory, biology and computer science (see, e.g., [12], [20]- [23]).

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic and X is said to be a *uniquely geodesic* if there is exactly one geodesic joining x to y for each $x, y \in X$.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X,d) consist of three points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X,d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) = \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1,2,3\}$. Such a triangle always exists (see [10]).

A geodesic metric space is said to be a $CAT(0)$ space [10] if all geodesic triangles of appropriate size satisfy the following comparison axiom:

Let Δ be a geodesic triangle in X and $\overline{\Delta}$ be a comparison triangle for Δ. Then, Δ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$,

$$d(x,y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}).$$

If x, y_1, y_2 are points in a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies that

$$d(x, y_0)^2 \leq \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2.$$

This is the (CN) inequality of Bruhat and Tits [24]. In fact (see [10, p.163]), a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality. It is worth mentioning that the results in a $CAT(0)$ space can be applied to any $CAT(k)$ space with $k \le 0$ since any $CAT(k)$ space is a $CAT(k')$ space for every $k' \ge k$ (see [10, p.165]).

Let $x, y \in X$ and by Lemma 2.1 (iv) of [16] for each $t \in [0,1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x,z) = td(x,y), \;\; d(y,z) = (1-t)d(x,y). \qquad (3)$$

From now on, we will use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (3). We now collect some elementary facts about $CAT(0)$ spaces which will be used in sequel the proofs of our main results.

Lemma 1 *Let X be a $CAT(0)$ space. Then*
(i) (see [16, Lemma 2.4]) for each $x, y, z \in X$ and $t \in [0,1]$, one has

$$d((1-t)x \oplus ty, z) \le (1-t)d(x,z) + td(y,z),$$

(ii) (see [16, Lemma 2.5]) for each $x, y, z \in X$ and $t \in [0,1]$, one has

$$d\left((1-t)x \oplus ty, z\right)^2 \le (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2.$$

3 Demiclosedness principle for k-strictly pseudo-contractive mappings

In 1976 Lim [25] introduced a concept of convergence in a general metric space setting which is called Δ-convergence. Later, Kirk and Panyanak [26] used the concept of Δ-convergence introduced by Lim [25] to prove on the $CAT(0)$ space analogs of some Banach space results which involve weak convergence. Also, Dhompongsa and Panyanak [16] obtained the Δ-convergence theorems for the Picard, Mann and Ishikawa iterations in a $CAT(0)$ space for nonexpansive mappings under some appropriate conditions.

We now give the definition and collect some basic properties of the Δ-convergence.

Let X be a complete $CAT(0)$ space and $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\}.$$

The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}.$$

It is known that in a complete $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point (see [27, Proposition 7]).

Definition 1 *([25], [26]) A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and x is called the Δ-limit of $\{x_n\}$.*

Lemma 2

(i) Every bounded sequence in a complete $CAT(0)$ space always has a Δ-convergent subsequence. (see [26, p.3690])

(ii) Let C be a nonempty closed convex subset of a complete $CAT(0)$ space and let $\{x_n\}$ be a bounded sequence in C. Then the asymptotic center of $\{x_n\}$ is in C. (see [28, Proposition 2.1])

Lemma 3 *([16, Lemma 2.8]) If $\{x_n\}$ is a bounded sequence in a complete $CAT(0)$ space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ is convergent then $x = u$.*

Let C be a closed convex subset of a $CAT(0)$ space X and $\{x_n\}$ be a bounded sequence in C. We denote the notation

$$\{x_n\} \rightharpoonup w \Leftrightarrow \Phi(w) = \inf_{x\in C} \Phi(x) \qquad (4)$$

where $\Phi(x) = \limsup_{n\to\infty} d(x_n, x)$.

Nanjaras and Panyanak [29] gave a connection between the "$\mapsto$" convergence and Δ-convergence.

Proposition 1 *([29, Proposition 3.12]) Let C be a closed convex subset of a $CAT(0)$ space X and $\{x_n\}$ be a bounded sequence in C. Then Δ-$\lim_{n\to\infty} x_n = p$ implies that $\{x_n\} \mapsto p$.*

The purpose of this section is to prove *demiclosedness principle for k-strictly pseudo-contractive mappings in a CAT(0) space* by using the convergence defined in (4).

Theorem 1 *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T: C \to C$ be a k-strictly pseudo-contractive mapping such that $k \in \left[0, \frac{1}{2}\right)$ and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a bounded sequence in C such that Δ-$\lim_{n\to\infty} x_n = w$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $Tw = w$.*

Proof By the hypothesis, Δ-$\lim_{n\to\infty} x_n = w$. From Proposition 1, we get $\{x_n\} \mapsto w$. Then we obtain $A(\{x_n\}) = \{w\}$ by Lemma 2 (ii) (see [29]). Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then we get

$$\Phi(x) = \limsup_{n\to\infty} d(x_n, x) = \limsup_{n\to\infty} d(Tx_n, x) \quad (5)$$

for all $x \in C$. In (5) by taking $x = Tw$, we have

$$\begin{aligned}
\Phi(Tw)^2 &= \limsup_{n\to\infty} d(Tx_n, Tw)^2 \\
&\le \limsup_{n\to\infty} \left\{ d(x_n, w)^2 + k(d(x_n, Tx_n) + d(w, Tw))^2 \right\} \\
&\le \limsup_{n\to\infty} d(x_n, w)^2 + k \limsup_{n\to\infty} \left(d(x_n, Tx_n) + d(w, Tw) \right)^2 \\
&= \Phi(w)^2 + kd(w, Tw)^2 \qquad (6)
\end{aligned}$$

The (CN) inequality implies that

$$d\left(x_n, \frac{w \oplus Tw}{2}\right)^2 \le \frac{1}{2} d(x_n, w)^2 + \frac{1}{2} d(x_n, Tw)^2 - \frac{1}{4} d(w, Tw)^2.$$

Letting $n \to \infty$ and taking superior limit on the both sides of the above inequality, we get

$$\Phi\left(\frac{w \oplus Tw}{2}\right)^2 \le \frac{1}{2} \Phi(w)^2 + \frac{1}{2} \Phi(Tw)^2 - \frac{1}{4} d(w, Tw)^2.$$

Since $A(\{x_n\}) = \{w\}$, we have

$$\Phi(w)^2 \le \Phi\left(\frac{w \oplus Tw}{2}\right)^2 \le \frac{1}{2} \Phi(w)^2 + \frac{1}{2} \Phi(Tw)^2 - \frac{1}{4} d(w, Tw)^2.$$

which implies that

$$d(w, Tw)^2 \le 2\Phi(Tw)^2 - 2\Phi(w)^2. \quad (7)$$

By (6) and (7), we get $(1-2k) d(w, Tw)^2 \le 0$. Since $k \in \left[0, \frac{1}{2}\right)$, then we have $Tw = w$ as desired.

Now, we prove the Δ-convergence of the cyclic algorithm for k-strictly pseudo-contractive mappings in a $CAT(0)$ space.

Theorem 2 *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $N \ge 1$ be an integer. Let, for each $0 \le i \le N-1$, $T_i : C \to C$ be k_i-strictly pseudo-contractive mappings for some $0 \le k_i < \frac{1}{2}$. Let $k = \max\{k_i; 0 \le i \le N-1\}$, $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0,1)$ and $k < a$. Let $F = \cap_{i=0}^{N-1} F(T_i) \neq \emptyset$. For $x_0 \in C$, let $\{x_n\}$ be a sequence defined by (2). Then the sequence $\{x_n\}$ is Δ-convergent to a common fixed point of the family $\{T_i\}_{i=0}^{N-1}$.*

Proof Let $p \in F$. Using (1), (2) and Lemma 1, we have

$$\begin{aligned}
d(x_{n+1}, p)^2 &= d(\alpha_n x_n \oplus (1-\alpha_n) T_{[n]} x_n, p)^2 \\
&\le \alpha_n d(x_n, p)^2 + (1-\alpha_n) d(T_{[n]} x_n, p)^2 \\
&\quad - \alpha_n (1-\alpha_n) d(x_n, T_{[n]} x_n)^2 \\
&\le \alpha_n d(x_n, p)^2 + (1-\alpha_n) \left\{ d(x_n, p)^2 + kd(x_n, T_{[n]} x_n)^2 \right\} \\
&\quad - \alpha_n (1-\alpha_n) d(x_n, T_{[n]} x_n)^2 \\
&= d(x_n, p)^2 - (1-\alpha_n)(\alpha_n - k) d(x_n, T_{[n]} x_n)^2 \qquad (8) \\
&\le d(x_n, p)^2.
\end{aligned}$$

This inequality guarentees that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$. By (8), we also have

$$d(x_n, T_{[n]}x_n)^2 \le \frac{1}{(1-\alpha_n)(\alpha_n - k)}\left[d(x_n, p)^2 - d(x_{n+1}, p)^2\right]$$

$$\le \frac{1}{(1-b)(a-k)}\left[d(x_n, p)^2 - d(x_{n+1}, p)^2\right]$$

Since $\lim_{n\to\infty} d(x_n, p)$ exists, we obtain $\lim_{n\to\infty} d(x_n, T_{[n]}x_n) = 0$. To show that the sequence $\{x_n\}$ is Δ-convergent to a common fixed point of the family $\{T_i\}_{i=0}^{N-1}$, we prove that

$$\omega_w(x_n) = \bigcup_{\{u_n\}\subseteq\{x_n\}} A(\{u_n\}) \subseteq F$$

and $\omega_w(x_n)$ consists of exactly one point. Let $u \in \omega_w(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_{n\to\infty} v_n = v \in C$. By Theorem 1, we have $v \in F$ and by Lemma 3, we have $u = v \in F$. This shows that $\omega_w(x_n) \subseteq F$. Now we prove that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F$. Finally, since $\{d(x_n, v)\}$ is convergent, we have $x = v \in F$ by Lemma 3. This completes the proof.

4 The strong convergence theorem for the modified Halpern iteration

In [9], Hu introduced a modified Halpern iteration. We modify this iteration in $CAT(0)$ spaces as follows.

For an arbitary initial value $x_0 \in C$ and a fixed anchor $u \in C$, the sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = \alpha_n u \oplus (1-\alpha_n) y_n, \\ y_n = \dfrac{\beta_n}{1-\alpha_n} x_n \oplus \dfrac{\gamma_n}{1-\alpha_n} T x_n, \ \forall n \ge 0, \end{cases} \tag{9}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three real sequences in $(0,1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$. Clearly, the iterative sequence (9) is a natural generalization of the well known iterations.

(i) If we take $\beta_n = 0$ for all n in (9), then the sequence (9) reduces to the Halpern's iteration in [30].

(ii) If we take $\alpha_n = 0$ for all n in (9), then the sequence (9) reduces to the Mann iteration in [8].

In this section, we prove the strong convergence of the modified Halpern's iteration in a $CAT(0)$ space.

Recall that a continous linear functional μ on ℓ_∞, the Banach space of bounded real sequences, is called a Banach limit if $\|\mu\| = \mu(1,1,\ldots) = 1$ and $\mu(a_n) = \mu(a_{n+1})$ for all $\{a_n\}_{n=1}^{\infty} \subset \ell_\infty$.

Lemma 4 *(see [31, Proposition 2]) Let* $\{a_1, a_2, \ldots\} \in \ell_\infty$ *be such that* $\mu(a_n) \le 0$ *for all Banach limits* μ *and* $\limsup_{n\to\infty}(a_{n+1} - a_n) \le 0$. *Then,* $\limsup_{n\to\infty} a_n \le 0$.

Lemma 5 *Let* C *be a nonempty closed convex subset of a complete* $CAT(0)$ *space* X, $T: C \to C$ *be a k-strictly pseudo-contractive mapping with* $k \in [0,1)$ *and* $S: C \to C$ *be a mapping defined by* $Sz = kz \oplus (1-k)Tz$, *for* $z \in C$. *Let* $u \in C$ *be fixed. For each* $t \in [0,1]$, *the mapping* $S_t: C \to C$ *defined by*

$$S_t z = tu \oplus (1-t) Sz = tu \oplus (1-t)\left(kz \oplus (1-k) Tz\right)$$

for $z \in C$, *has a unique fixed point* $z_t \in C$, *that is,*

$$z_t = S_t(z_t) = tu \oplus (1-t) S(z_t). \tag{10}$$

Proof As it has been proven in [32], if T is a k-strictly pseudo-contractive mapping with $k \in [0,1)$, S is a nonexpansive mapping such that $F(S) = F(T)$. Then, from Lemma 2.1 in [17], the mapping S_t has a unique fixed point $z_t \in C$.

Lemma 6 *Let* X, C, T *and* S *be as in Lemma 5. Then,* $F(T) \ne \varnothing$ *if and only if* $\{z_t\}$ *given by (10)* *remains bounded as* $t \to 0$. *In this case, the following statements hold:*

(1) $\{z_t\}$ *converges to the unique fixed point* z *of* T *which is nearest to* u,

(2) $d^2(u,z) \le \mu d^2(u,x_n)$ *for all Banach limits* μ *and all bounded sequences* $\{x_n\}$ *with* $\lim_{n\to\infty} d(x_n,Tx_n)=0.$

Proof If $F(T) \ne \emptyset,$ then we have $F(S)=F(T)\ne\emptyset.$ Also, if $\lim_{n\to\infty} d(x_n,Tx_n)=0$, we obtain that

$$d(x_n,Sx_n)=d(x_n,kx_n\oplus(1-k)Tx_n) \le (1-k)d(x_n,Tx_n)\to 0 \text{ as } n\to\infty.$$

Thus, from Lemma 2.2 in [17], the rest of the proof of this lemma can be seen.

The following lemma can be found in [33].

Lemma 7 *(see [33, Lemma 2.1]) Let* $\{a_n\}$ *be a sequence of non-negative real numbers satisfying the condition*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\sigma_n,\ \forall n\ge 0,$$

where $\{\gamma_n\}$ *and* $\{\sigma_n\}$ *are sequences of real numbers such that*

(1) $\{\gamma_n\}\subset[0,1]$ *and* $\sum_{n=1}^{\infty}\gamma_n=\infty,$

(2) either $\limsup_{n\to\infty}\sigma_n\le 0$ *or* $\sum_{n=1}^{\infty}|\gamma_n\sigma_n|<\infty.$

Then, $\lim_{n\to\infty} a_n = 0.$

We are now ready to prove our main result.

Theorem 3 *Let* C *be a nonempty closed convex subset of a complete* $CAT(0)$ *space* X *and* $T:C\to C$ *be a* k*-strictly pseudo-contractive mapping such that* $0\le k<\frac{\beta_n}{1-\alpha_n}<1$ *and* $F(T)\ne\emptyset$. *Let* $\{x_n\}$ *be a sequence defined by (9). Suppose that* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *satisfy the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n=0$,

(C2) $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(C3) $\lim_{n\to\infty}\beta_n\ne k$ *and* $\lim_{n\to\infty}\gamma_n\ne 0$.

Then the sequence $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof We divide the proof into three steps. In the first step we show that $\{x_n\},\{y_n\}$ and $\{Tx_n\}$ are bounded sequences. In the second step we show that $\lim_{n\to\infty} d(x_n,Tx_n)=0.$ Finally, we show that $\{x_n\}$ converges to a fixed point $z\in F(T)$ which is nearest to u.

First step: Take any $p\in F(T)$, then, from Lemma 1 and (9), we have

$$\begin{aligned} &d(y_n,p)^2\\ &\le \frac{\beta_n}{1-\alpha_n}d(x_n,p)^2+\frac{\gamma_n}{1-\alpha_n}d(Tx_n,p)^2-\frac{\beta_n\gamma_n}{(1-\alpha_n)^2}d(x_n,Tx_n)^2\\ &\le \frac{\beta_n}{1-\alpha_n}d(x_n,p)^2+\frac{\gamma_n}{1-\alpha_n}\left(d(x_n,p)^2+kd(x_n,Tx_n)^2\right)\\ &\quad -\frac{\beta_n\gamma_n}{(1-\alpha_n)^2}d(x_n,Tx_n)^2\\ &= d(x_n,p)^2-\frac{\gamma_n}{1-\alpha_n}\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2\\ &\le d(x_n,p)^2. \end{aligned}$$

Also, we obtain

$$\begin{aligned} &d(x_{n+1},p)^2\\ &\le \alpha_n d(u,p)^2+(1-\alpha_n)d(y_n,p)^2-\alpha_n(1-\alpha_n)d(u,y_n)^2\\ &\le \alpha_n d(u,p)^2\\ &\quad +(1-\alpha_n)\left\{d(x_n,p)^2-\frac{\gamma_n}{1-\alpha_n}\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2\right\}\\ &\quad -\alpha_n(1-\alpha_n)d(u,y_n)^2\\ &= \alpha_n d(u,p)^2+(1-\alpha_n)d(x_n,p)^2-\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2\\ &\quad -\alpha_n(1-\alpha_n)d(u,y_n)^2 \qquad (11)\\ &\le \alpha_n d(u,p)^2+(1-\alpha_n)d(x_n,p)^2\\ &\le \max\left\{d(u,p)^2,d(x_n,p)^2\right\} \end{aligned}$$

By induction,

$$d(x_{n+1},p)^2\le\max\left\{d(u,p)^2,d(x_0,p)^2\right\}$$

This proves the boundedness of the sequence $\{x_n\}$, which leads to the boundedness of $\{Tx_n\}$ and $\{y_n\}$.
Second step: In fact, we have from (11) (for some appropriate constant $M>0$) that

$$\begin{aligned} &d(x_{n+1},p)^2 \\ &\le \alpha_n d(u,p)^2+(1-\alpha_n)d(x_n,p)^2 \\ &\quad -\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2 \\ &=\alpha_n(d(u,p)^2-d(x_n,p)^2)+d(x_n,p)^2 \\ &\quad -\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2 \\ &\le \alpha_n M+d(x_n,p)^2-\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2, \end{aligned}$$

which implies that

$$\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2-\alpha_n M\le d(x_n,p)^2-d(x_{n+1},p)^2. \tag{12}$$

If $\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2-\alpha_n M\le 0$, then

$$d(x_n,Tx_n)^2\le\frac{\alpha_n}{\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)}M,$$

and hence the desired result is obtained by the conditions (C1) and (C3).

If $\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2-\alpha_n M>0$, then following (12), we have

$$\begin{aligned} &\sum_{n=0}^{m}\left[\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2-\alpha_n M\right] \\ &\le d(x_0,p)^2-d(x_{m+1},p)^2 \\ &\le d(x_0,p)^2. \end{aligned}$$

That is

$$\sum_{n=0}^{\infty}\left[\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2-\alpha_n M\right]<\infty.$$

Thus

$$\lim_{n\to\infty}\left[\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2-\alpha_n M\right]=0.$$

Then we get

$$\lim_{n\to\infty}d(x_n,Tx_n)=0. \tag{13}$$

Third step: Using the condition (C1) and (13), we obtain

$$\begin{aligned} &d(x_{n+1},x_n)\le d(x_{n+1},Tx_n)+d(Tx_n,x_n) \\ &\le \alpha_n d(u,Tx_n)+(1-\alpha_n)d(y_n,Tx_n)+d(Tx_n,x_n) \\ &\le \alpha_n d(u,Tx_n)+(1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n}d(x_n,Tx_n)\right)+d(Tx_n,x_n) \\ &=\alpha_n d(u,Tx_n)+(\beta_n+1)d(x_n,Tx_n) \\ &\to 0,\ \text{as } n\to\infty. \end{aligned}$$

Also, from (13), we have

$$d(x_n,y_n)\le\frac{\gamma_n}{1-\alpha_n}d(x_n,Tx_n)\to 0,\ \text{as } n\to\infty. \tag{14}$$

Let $z=\lim_{t\to 0}z_t$, where z_t is given by (10) in Lemma 5. Then, z is the point of $F(T)$ which is nearest to u. By Lemma 6 (2), we have $\mu\left(d(u,z)^2-d(u,x_n)^2\right)\le 0$ for all Banach limits μ. Moreover, since $\lim_{n\to\infty}d(x_{n+1},x_n)=0$,

$$\limsup_{n\to\infty}\left[\left(d(u,z)^2-d(u,x_{n+1})^2\right)-\left(d(u,z)^2-d(u,x_n)^2\right)\right]=0.$$

If we take $a_n=d(u,z)^2-d(u,x_n)^2$ in Lemma 4, then we obtain

$$\limsup_{n\to\infty}\left(d(u,z)^2-d(u,x_n)^2\right)\le 0. \tag{15}$$

It follows from the condition (C1) and (14) that

$$\begin{aligned} &\limsup_{n\to\infty}\left(d(u,z)^2-(1-\alpha_n)d(u,y_n)^2\right) \\ &=\limsup_{n\to\infty}\left(d(u,z)^2-d(u,x_n)^2\right). \end{aligned} \tag{16}$$

By (15) and (16), we have

$$\limsup_{n\to\infty}\left(d(u,z)^2-(1-\alpha_n)d(u,y_n)^2\right)\leq 0. \qquad (17)$$

We observe that

$$\begin{aligned}
&d(x_{n+1},z)^2\\
&\leq \alpha_n d(u,z)^2+(1-\alpha_n)d(y_n,z)^2-\alpha_n(1-\alpha_n)d(u,y_n)^2\\
&\leq \alpha_n d(u,z)^2+(1-\alpha_n)d(x_n,z)^2-\alpha_n(1-\alpha_n)d(u,y_n)^2\\
&=(1-\alpha_n)d(x_n,z)^2+\alpha_n\left[d(u,z)^2-(1-\alpha_n)d(u,y_n)^2\right].
\end{aligned}$$

It follows from the condition (C2) and (17), using Lemma 7, that $\lim_{n\to\infty}d(x_n,z)=0$. This completes the proof of Theorem 3.

We obtain the following corollary as a direct consequence of Theorem 3.

Corollary 1 *Let X, C and T be as Theorem 3. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ satisfying the conditions (C1) and (C2). For a constant $\delta\in(k,1)$, an arbitary initial value $x_0\in C$ and a fixed anchor $u\in C$, let the sequence $\{x_n\}$ be defined by*

$$x_{n+1}=\alpha_n u\oplus(1-\alpha_n)(\delta x_n\oplus(1-\delta)Tx_n),\ \forall n\geq 0. \,(18)$$

Then the sequence $\{x_n\}$ is strongly convergent to a fixed point of T.

Proof If, in proof of Theorem 3, we take $\beta_n=(1-\alpha_n)\delta$ and $\gamma_n=(1-\alpha_n)(1-\delta)$, then we get the desired conclusion.

Remark 1 *The results in this section contain the strong convergence theorems of the iterative sequences (9) and (18) for nonexpansive mappings in a $CAT(0)$ space. Also, our results contain the corresponding theorems proved for these iterative sequences in a Hilbert space.*

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