A toxoplasmosis spread model between cat and oocyst populations with independent stochastic perturbations

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Abstract: A toxoplasmosis spread model between cat and oocyst populations with independent stochastic perturbations is proposed, the existence of global positive solution is derived. By the method of stochastic Lyapunov functions, we study their asymptotic behavior in terms of the intensity of the stochastic perturbations and the reproductive number. When the perturbations about the susceptible and infective cats are sufficiently small, as well as magnitude of the reproductive number is less than one, the infective cats, recovered cats and population oocysts decay to zero whilst the susceptible components converge to a class of explicit stationary distributions regardless of the perturbations on the recovered cats and population oocysts. When all the perturbations are small and the reproductive number is larger than one, we construct a new class of stochastic Lyapunov functions to show the positive recurrence, and our results reveal some cycling phenomena of recurrent diseases. These results mean that stochastic system has the similar property with the corresponding deterministic system when the white noise is small. Finally, numerical simulations are carried out to support our findings.

Key–Words: Stochastic model, Brownian motion, Asymptotically stable in the large, Lyapunov function

1 Introduction

Toxoplasma gondii, often referred to as T.gondii, is a parasite that is able to infect a wide range of hosts, including all mammals and birds[1,2]. Up to one third of the words human population are estimated to carry a Toxoplasma infection[3]. The increasing prevalence of infection in human population is probably due to the increase in the number of cats[4]. Cats are the key to control T. gondii due to the fact they shed, via feces[5], millions of oocysts, which after sporulation in the environment might infect warm-blooded animals including human beings.

Mathematical modeling is often used to study the transmission dynamics of diseases in population from an epidemiological point of view[6-12]. Abraham et al.[9] presented an epidemiological model to study the transmission dynamics of toxoplasmosis in a cat population under a continuous vaccination schedule.

$$\begin{cases} \dot{S}(t) = \iota R(t) - \beta S(t)O(t) - \gamma S(t), \\ \dot{I}(t) = \beta S(t)O(t) - \alpha I(t), \\ \dot{R}(t) = \alpha I(t) - \iota R(t) + \gamma S(t), \\ \dot{O}(t) = kI(t) - \mu_0 O(t). \end{cases} \tag{1}$$

• The total population of cats is divided into three disjoint subpopulations: cats who may become infected (Susceptible $S(t)$), cats infected by T. Gondii (Infected $I(t)$), and cats who have been vaccinated or have immunity (Vaccinated(recovered) $R(t)$). $O(t)$: number of oocyst in the environment.

• β: the rate of a susceptible cat transits to the infected subpopulation.

• μ: the cat natural death rate.

• γ: the rate of a susceptible cat transits to the vaccinated subpopulation.

• α: the rate of an infected cat transits to the vaccinated subpopulation.

• μ_0: the death rate of oocysts.

• k: the rate of appearance of new oocysts in the environment per infected cat.

For model (1), authors assumed that the total cat population remains constant. But for many cases, taking into consideration that the size of population varies is more reasonable. Furthermore, we assume that the vaccination rate γ of a susceptible cat equals to zero. Then model (1) is revised as the follows

$$\begin{cases} \dot{S}(t) = \Lambda - \beta S(t)O(t) - \mu S(t), \\ \dot{I}(t) = \beta S(t)O(t) - (\alpha + \mu)I(t), \\ \dot{R}(t) = \alpha I(t) - \mu R(t), \\ \dot{O}(t) = kI(t) - \mu_0 O(t). \end{cases} \tag{2}$$

The basic reproduction number

$$R_0 = k\beta\Lambda/(\mu\mu_0(\alpha + \mu_1))$$

measures the average number of new infections generated by a single infected in a completely susceptible population. After a simple calculation, we find that the basic reproductive number R_0 controls completely the dynamics of the infection. In detail, system (1) has a disease-free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0)$, which is stable when $R_0 \leq 1$, whereas system (1) admits an epidemic equilibrium $E^* = (S^*, I^*, R^*, O^*)$, which is stable when $R_0 > 1$, where

$$\begin{array}{l} S^* = \frac{(\alpha+\mu)\mu_0}{\beta k}, \;\; I^* = \frac{\mu_0 O^*}{k} \\ R^* = \frac{\alpha\mu_0 O^*}{k\mu}, \;\; O^* = \frac{\Lambda k}{(\alpha+\mu)\mu_0} - \frac{\mu}{\beta}. \end{array} \tag{3}$$

Thus the basic reproduction number R_0 is often considered as the threshold quantity that determines when an infection can invade and persist in a new host population. The disease-free equilibrium corresponds to maximal levels of susceptible, no infected and no recovered cats or oocysts. The epidemic equilibrium corresponds to positive levels of all four components including susceptible, infected, recovered cats as well as oocysts.

In fact, epidemic models are inevitably affected by environmental white noise which is an important component in realism, because it can provide an additional degree of realism in comparison to their deterministic counterparts [13-20]. Many stochastic models for epidemic populations have been developed in Refs. Dalal et al.[13] have previously used the technique of parameter perturbation to examine the effect of environmental stochasticity in a model of AIDS and condom use. Yu et al.[15] proved the endemic equilibrium of the two-group SIR model with random perturbation is stochastic asymptotically stable. Meng [16] presented the stability conditions of the disease-free equilibrium of the SIR model without stochastic perturbation and with stochastic perturbation. Zhao et al. [17] investigated the extinction and persistence of the stochastic SIS epidemic model with vaccination. These results reveal the significant effect of the environmental noise on some epidemic models, because the stochastic models can provide some additional degree of realism compared to their deterministic counterparts [21].

However, to the best of our knowledge, the dynamics of a toxoplasmosis spread model between cat and oocyst populations with independent stochastic perturbations seem rare. In this paper, taking into account the effect of randomly fluctuating environment, we assume that fluctuations in the environment will manifest themselves mainly as fluctuations in the parameter. The symbol $B(t) = (B_1(t), B_2(t), B_3(t), B_4(t))$ denotes a 4-dimensional Wiener process. The non-negative constants $\sigma_1, \sigma_2, \sigma_3$ and σ_4 denote the intensities of the stochastic perturbations. The stochastic version corresponding to the deterministic model (2) takes the following form:

$$\begin{cases} dS(t) = & (\Lambda - \beta S(t)O(t) - \mu S(t))dt \\ & +\sigma_1 S(t)dB_1(t), \\ dI(t) = & (\beta S(t)O(t) - (\alpha + \mu)I(t))dt \\ & +\sigma_2 I(t)dB_2(t), \\ dR(t) = & (\alpha I(t) - \mu R(t))dt \\ & +\sigma_3 R(t)dB_3(t), \\ dO(t) = & (kI(t) - \mu_0 O(t))dt \\ & +\sigma_4 O(t)dB_4(t). \end{cases} \tag{4}$$

The paper is organized as follows. In section 2 we show the existence and uniqueness of a global positive solution of model (4). Since system (4) is constructed by adding stochastic perturbation in a deterministic system (2), it seems reasonable to investigate whether there are similar properties as in system (2). But there is neither a disease-free equilibrium E_0 nor an endemic equilibrium E^* for system (4). Hence in order to show the stability to some extent, we discuss the behavior around E_0 and E^* respectively, which will be shown in Sections 3 and 4. In Section 3, we will prove a stability result to the effect

$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t [(S(r) - \frac{\Lambda}{\mu})^2 + I(r)^2 + R(r) + O(r)]dr$$

is small, provided the diffusion coefficients are sufficiently small. While, in Section 4, we will show

$$\begin{array}{l} \limsup_{t\to\infty} \frac{1}{t} E\int_0^t \{[(S(r) - \dfrac{2\mu}{2\mu - \sigma_1^2} S^*)^2 \\ +[I(r) - \frac{2\mu\mu_0(\alpha+\beta) - \mu_0 p\alpha^2 - \mu q k_0^2}{2\mu\mu_0(\alpha+\beta) - \mu_0 p\alpha^2 - \mu q k_0^2 - \mu\mu_0\sigma_2^2} I^*]^2 \\ +[R(r) - \frac{\mu}{\mu - \sigma_3^2} R^*]^2 + [O(r) - \frac{\mu_0}{\mu_0 - \sigma_4^2} O^*]^2\} dr, \end{array}$$

is bounded by a constant which is relevant to the intensity of white noise. Finally, numerical simulations are present in Section 5 to illustrate our results.

The proof of the main theorems in this paper uses Lyapunov functions, together with graph theory. In this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, P)$ with the filtration $\{\mathcal{F}\}_{t\geq 0}$ be a complete probability space satisfying the usual satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $B_i(t)$ be the Brownian motions defined on this probability space, $i = 1,2,3,4$. Denote $R^n_+ = \{x \in R^n, x_i > 0 \text{ for all } 1 \leq i \leq n\}$ and $x(t) = (S(t), I(t), R(t), O(t))^T$. Here we show the following auxiliary statements which are introduced in Ref. [22].

In general, consider d-dimensional stochastic differential equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), \text{ for } t \geq t_0.$$

Denote by $C^{2,1}(R^d \times [t_0, \infty]; R_+)$ the family of all nonnegative functions $V(x,t)$ defined on $R^d \times [t_0, \infty]$ such that they are continuously twice differentiable in x and once in t. The differential operator L of Eq. (4) is defined [22] by formula

$$\begin{aligned} L = &\ \tfrac{\partial}{\partial t} + \sum f_i(x,t)\tfrac{\partial}{\partial x_i} \\ &+ \tfrac{1}{2}\sum [g^T(x,t)g(x,t)]_{ij}\tfrac{\partial^2}{\partial x_i \partial x_j}. \end{aligned}$$

If L acts on a function $V \in C^{2,1}(R^d \times [t_0, \infty]; R_+)$, then

$$\begin{aligned} LV(x,t) = &\ V_t(x,t) + V_x(x,t)f(x,t) \\ &+ \tfrac{1}{2}trace[g^T(x,t)V_{xx}(x,t)g(x,t)]. \end{aligned}$$

2 Global positive solution

In order to investigate the dynamical behavior, the first concerning thing is whether the solution is global existence. Moreover, for a model of population dynamics, whether the value of the solution is nonnegative is also considered. Hence in this section we show the solution of system (4) is global and nonnegative. As we have known, in order for a stochastic differential equation to have a unique global (i.e. no explosion in a finite time) solution for any given initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition [22]. However, the coefficients of system. (4) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of system (4) may explode at a finite time [22,23]. In this section, using the Lyapunov analysis method[22], we shall show the solution of system. (4) is positive and global.

Theorem 1 *For any given initial value $(S(0), I(0), R(0), O(0)) \in R^4_+$ there is a unique positive solution (S(t), I(t), R(t),O(t)) of model (4) on $t \geq 0$ and the solution will remain in R_+ with probability 1, namely (S(t), I(t), R(t),O(t))$\in R_+$ for all $t \geq 0$ almost surely.*

Proof. Since the coefficients of the equation is locally Lipschitz continuous, for any given initial value $(S(0), I(0), R(0), O(0)) \in R_+$ there is a unique local solution $(S(t), I(t), R(t), O(t)) \in R_+$ on $t\in [0, \tau_e]$, where τ_e is the explosion time [24]. To show this solution is global, we need to shoe that $\tau_e = \infty$ a.s. At first, we prove $S(t)$ and $I(t)$ do not explode to infinity in a finite time. Set $k_0 > 0$ be sufficiently large for $S(0) \in [\frac{1}{k_0}, k_0]$, $I(0) \in [\frac{1}{k_0}, k_0]$ and $O(0) \in [\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$, define the stopping time

$$\begin{aligned} \tau_k = &\ \inf\{t \in [0, \tau_e] : S(t) \in (\tfrac{1}{k}, k), I(t) \in (\tfrac{1}{k}, k) \\ &\quad or\ O(t) \in (\tfrac{1}{k}, k)\}. \end{aligned}$$

where throughout this paper we set $\inf \emptyset = \infty$ ($\emptyset$ denotes the empty set). Obviously, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, therefore $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = \infty$ a.s. is true, then $\tau_e = \infty$ a.s. and $(S(t), I(t)), O(t) \in R^3_+$ a.s. for $t \geq 0$. In other words, to complete the proof it is required to show that $\tau_\infty = \infty$ a.s. If this statement is false, then there exist a pair of constants $T > 0$ and $\varepsilon \in (0,1)$ such that $P\{\tau_\infty \leq T\} > \varepsilon$.

Thus there is an integer $k_1 \geq k_0$ such that $P\{\tau_\infty \leq T\} > \varepsilon$ for all $k \geq k_1$.

Let us define a C^2 function $V : R^2_+ \to R_+$ as follows

$$V(S, I) = S + I - \log I,$$

the nonnegativity of this function can be seen from $I - \log I \geq 0$ for $I \geq 0$. Let $k \geq k_0$ and $T > 0$ be arbitrary. Applying Itö formula, we obtain

$$\begin{aligned} dV(S,I) &= V_S dS + V_I dI + \tfrac{1}{2}(V_{SS}(dS)^2 + V_{II}(dI)^2 \\ &= (\Lambda - \beta S(t)O(t) - \mu S(t))dt \\ &\quad + (1 - \tfrac{1}{I})(\beta S(t)O(t) - (\alpha + \mu)I(t))dt \\ &\quad + (1 - \tfrac{1}{I})\sigma_2 I(t)dB_2(t) \\ &\quad + \sigma_1 S(t)dB_1(t) + \tfrac{1}{2}\sigma_2^2 dt \\ &= [(\Lambda - \beta S(t)O(t) - \mu S(t)) \\ &\quad + (1 - \tfrac{1}{I})(\beta S(t)O(t) - (\alpha + \mu)I(t)) \\ &\quad + \tfrac{1}{2}\sigma_2^2]dt + \sigma_1 S(t)dB_1(t) \\ &\quad + (1 - \tfrac{1}{I})\sigma_2 I(t)dB_2(t) \\ &= LV(S,I)dt + \sigma_1 S(t)dB_1(t) \\ &\quad + (1 - \tfrac{1}{I})\sigma_2 I(t)dB_2(t). \end{aligned}$$

where $LV : R_+^2 \to R_+$ is defined by

$$\begin{aligned} LV(S,I) &= \Lambda - \beta S(t)O(t) - \mu S(t) \\ &\quad + (1-\tfrac{1}{I})(\beta S(t)O(t) - (\alpha+\mu)I(t)) \\ &\quad + \tfrac{1}{2}\sigma_2 \\ &\le \Lambda + \alpha + \mu + \tfrac{1}{2}\sigma_2^2 \\ &= K. \end{aligned}$$

Therefore,

$$\begin{aligned} dV(S,I) &\le Kdt + \sigma_1 S(t)dB_1(t) \\ &\quad + (1-\tfrac{1}{I})\sigma_2 I(t)dB_2(t). \end{aligned} \tag{5}$$

We can now integrate both sides of (5) from 0 to $\tau_k \wedge T$ and then take the expectations

$$\begin{aligned} EV(S(\tau_k\wedge T), I(\tau_k\wedge T)) &\le EV(S(0), I(0)) \\ &\quad + KE(\tau_k \wedge T). \end{aligned}$$

As a result

$$EV(S(\tau_k\wedge T), I(\tau_k\wedge T)) \le EV(S(0), I(0)) + KT. \tag{6}$$

Let $\Omega_k = \{\tau_k \le T\}$ for $k \ge k_1$ and, by (5), $P(\Omega_k) \ge \varepsilon$. Note that for every $\omega \in \Omega_k$, there is $S(\tau_k, \omega)$ or $I(\tau_k, \omega)$ equals either k or $\frac{1}{k}$ and therefore $V(S(\tau_k, I(\tau_k))$ is no less then either k or $\frac{1}{k}$ and therefore $V(S(\tau_k,\omega), I(\tau_k,\omega))$ is no less than either $k - \log k$ or $\frac{1}{k} - \log\frac{1}{k} = \frac{1}{k} + \log k$. Hence

$$V(S(\tau_k,\omega), I(\tau_k,\omega)) \ge [k - \log k] \wedge [\frac{1}{k} + \log k].$$

It then follows from (6) that

$$\begin{aligned} V(S(0), I(0)) + KT &\ge E[1_{\Omega_k}(\omega) V(S(\tau_k,\omega) I(\tau_k,\omega))] \\ &\ge \varepsilon[k - \log k] \wedge [\tfrac{1}{k} + \log k], \end{aligned}$$

where 1_{Ω_k} is the indicator function of Ω_k. Letting $k \to \infty$, we have that

$$V(S(0), I(0)) + KT \ge \infty$$

which is impossible, then we must have $\tau_\infty = \infty$. Therefore it implies $S(t)$and $I(t)$ will not explode in a finite time with probability one. On the other hand, by the last two equations of model (4), we can represent the solution $R(t)$ and $O(t)$ as follows

$$\begin{aligned} R(t) &= e^{-(\mu+\frac{1}{2}\sigma_3^2)t+\sigma_3 B_3(t)}[R(0) \\ &\quad + \alpha\int_0^t e^{-(\mu+\frac{1}{2}\sigma_3^2)s-\sigma_3 B_3(t)} I(s)ds] \\ &= R(0)e^{-(\mu+\frac{1}{2}\sigma_3^2)t+\sigma_3 B_3(t)} \\ &\quad + \alpha\int_0^t e^{-(\mu+\frac{1}{2}\sigma_3^2)(t-s)+\sigma_3(B_3(t)-B_3(s))} I(s)ds. \end{aligned}$$

Since $I(t)$ has been proved to be global and positive, $R(t)$ is also global and positive. The proof is complete. □

3 Asymptotic behavior around the disease-free equilibrium of the deterministic model

As mentioned in the introduction, for the deterministic system (2), there is a disease-free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0, 0)$ and it is globally stable if $R_0 \le 1$. While for the stochastic system (4), $E_0 = (\frac{\Lambda}{\mu}, 0, 0)$ is no longer the disease-free equilibrium, and the stochastic solutions do not converge to E_0. In this section, we will study the asymptotic behavior around E_0.

Theorem 2 *If $R_0 < 1$ and the following conditions are satisfied*

$$\sigma_1^2 < 2\mu, \sigma_2^2 < 2(\alpha+\mu) \tag{7}$$

then for any given initial value $(S(0), I(0), R(0), O(0)) \in R_+$,the solution of model (4) has the property

$$\limsup_{t\to\infty} \tfrac{1}{t}E\int_0^t [(S(r)-\frac{\Lambda}{\mu})^2 + I(r)^2 + R(r) + O(r)]dr \le \tfrac{\sigma_1^2\Lambda^2}{K_1},$$

where

$$K_1 = \min\{2(\mu-\sigma_1^2), 2(\alpha+\mu)-\sigma_2^2, c_2, c_3\}.$$

Proof. First, change the variables $u = S - \frac{\Lambda}{\mu}$, $v = I$, $w = R$, $x = O$, then system (4) can be written as

$$\begin{cases} du(t) &= (-\mu u - \beta\mu x - \beta\frac{\Lambda}{\mu}x)dt \\ &\quad + \sigma_1(u+\frac{\Lambda}{\mu})dB_1(t), \\ dv(t) &= (\beta\mu x + \beta\frac{\Lambda}{\mu}x - (\alpha+\mu)v)dt \\ &\quad + \sigma_2 v dB_2(t), \\ dw(t) &= (\alpha v - \mu w)dt + \sigma_3 w dB_3(t), \\ dx(t) &= (kv - \mu_0 x)dt + \sigma_4 x dB_4(t). \end{cases}$$

and $u \in R, v > 0, w > 0, x > 0$. Define a function

$$V(u,v,w) = (u+v)^2 + c_1(u+v) + c_2 w + c_3 x,$$

where c_1, c_2, c_3 are three positive constants to be defined later. Applying Itô formula, we obtain

$$\begin{aligned} dV &= LVdt + [2(u+v)+c_1]\sigma_1(u+\tfrac{\Lambda}{\mu})dB_1(t) \\ &\quad + (2(u+v)+c_1)\sigma_2 v dB_2(t) \\ &\quad + c_2\sigma_3 w dB_3(t) + c_3\sigma_4 x dB_4(t). \end{aligned}$$

where

$$\begin{aligned} LV = &\ 2(u+v))[-\mu u-(\alpha+\mu)v] \\ &+c_1(-\mu u-\beta\mu x-\beta\tfrac{\Lambda}{\mu}x)+c_2(\alpha v-\mu w) \\ &+c_3(kv-\mu_0 x)\sigma_1^2(u+\tfrac{\Lambda}{\mu})^2+\sigma_2^2 v^2, \\ = &\ (-2\mu-\sigma_1^2)u^2-[2(\alpha+\mu)-\sigma_2^2]v^2 \\ &+(2\sigma_1^2\tfrac{\Lambda}{\mu}-c_1)u+\sigma_1^2(\tfrac{\Lambda}{\mu})^2-c_2\mu w \\ &-c_3\mu_0 x+[c_2\alpha+c_3k-c_1(\alpha+\mu)]v. \end{aligned}$$

We choose c_1 such that $2\sigma_1^2\frac{\Lambda}{\mu}-c_1=0$, i.e $c_1=2\sigma_1^2\frac{\Lambda}{\mu}$. Besides, we can find appropriate c_2, c_3 such that $[c_2\alpha+c_3k-c_1(\alpha+\mu)]v\le 0$. Hence we can obtain

$$\begin{aligned} LV \le &\ (-2\mu-\sigma_1^2)u^2-[2(\alpha+\mu)-\sigma_2^2]v^2 \\ &+\sigma_1^2(\tfrac{\Lambda}{\mu})^2-c_2\mu w-c_3\mu_0 x. \end{aligned}$$

Therefore

$$\begin{aligned} dV \le &\ (-2\mu-\sigma_1^2)u^2-[2(\alpha+\mu)-\sigma_2^2]v^2 \\ &+\sigma_1^2(\tfrac{\Lambda}{\mu})^2-c_2\mu w-c_3\mu_0 x \\ &+[2(u+v)+c_1]\sigma_1(u+\tfrac{\Lambda}{\mu})dB_1(t) \\ &+(2(u+v)+c_1)\sigma_2 v dB_2(t) \\ &+c_2\sigma_3 w dB_3(t)+c_3\sigma_4 x dB_4(t). \end{aligned} \tag{8}$$

Integrating both sides of (8)from 0 to t, and then taking expectation, yields

$$\begin{aligned} 0 &\le E[V(u(t),v(t),w(t),x(t))] \\ &\le E[V(u(0),v(0),w(0),x(0))] \\ &\quad + E\textstyle\int_0^t[(-2\mu-\sigma_1^2)u(s)^2 \\ &\quad -(2(\alpha+\mu)-\sigma_2^2)v(s)^2-c_2\mu w(s) \\ &\quad -c_3\mu_0 x(s)+\sigma_1^2(\tfrac{\Lambda}{\mu})^2]ds \end{aligned}$$

which implies

$$\begin{aligned} E\int_0^t &\ [(-2\mu-\sigma_1^2)u(s)^2-(2(\alpha+\mu)-\sigma_2^2)v(s)^2 \\ &-c_2\mu w(s)-c_3\mu_0 x(s)]ds \\ &\le E[V(u(0),v(0),w(0),x(0))]+\sigma_1^2(\frac{\Lambda}{\mu})^2 t. \end{aligned}$$

Therefore

$$\begin{aligned} &\limsup_{t\to\infty}\tfrac{1}{t}E\int_0^t[(-2\mu-\sigma_1^2)u(s)^2-(2(\alpha+\mu) \\ &\qquad -\sigma_2^2)v(s)^2-c_2\mu w(s)-c_3\mu_0 x(s)]ds \\ &\le \sigma_1^2(\tfrac{\Lambda}{\mu})^2. \end{aligned}$$

Let

$$K_1=\min\{2\mu-\sigma_1^2, 2(\alpha+\mu)-\sigma_2^2, c_2, c_3\}.$$

Then

$$\begin{aligned} \limsup_{t\to\infty}\tfrac{1}{t}E\textstyle\int_0^t &\ [(S(r)-\tfrac{\Lambda}{\mu})^2+I(r)^2 \\ &+R(r)+O(r)]dr \le \tfrac{\sigma_1^2\Lambda^2}{K_1}. \end{aligned}$$

This completes the proof. □

Remark 3 *This theorem reveals the solution will oscillate around the disease-free equilibrium, and the intensity is relevant to the values of σ_1 and σ_2. The weaker the values are, the smaller the fluctuation is. In other words, if the stochastic perturbations become small, the solution of system (4) will be close to the disease-free equilibrium of system (2). Besides, if $\sigma_1=0$, then E_0 is also the disease-free equilibrium of system (4). From the proof of Theorem 2, we can obtain*

$$\begin{aligned} LV \le &\ (-2\mu-\sigma_1^2)u^2-[2(\alpha+\mu)-\sigma_2^2]v^2 \\ &+\sigma_1^2(\tfrac{\Lambda}{\mu})^2-c_2\mu w-c_3\mu_0 x, \end{aligned}$$

which is negative-definite, therefore E_0 is stochastically asymptotically stable in the large.

4 Asymptotic behavior around the endemic equilibrium of the deterministic model

In this section, we assume $R_0>1$. Then there is the endemic equilibrium E^* for system (2) but not the endemic equilibrium E^* for system (4), because system (4) does not have the endemic equilibrium. Similarly, we also expect to find out whether or not the solution goes around E^*. The following result gives a positive answer.

Theorem 4 *If $R_0>1$ and the following conditions are satisfied*

$$\sigma_1^2<2\mu, \sigma_3^2<\mu, \sigma_4^2<\mu_0, \tag{9}$$

then for any given initial value $(S(0), I(0), R(0), O(0))\in R_+$, the solution of model (4) has the property

$$\begin{aligned} &\limsup_{t\to\infty}\tfrac{1}{t}E\int_0^t\{[(S(r)-\frac{2\mu}{2\mu-\sigma_1^2}S^*)^2 \\ &+[I(r)-\tfrac{2\mu\mu_0(\alpha+\beta)-\mu_0 p\alpha^2-\mu q k_0^2}{2\mu\mu_0(\alpha+\beta)-\mu_0 p\alpha^2-\mu q k_0^2-\mu\mu_0\sigma_2^2}I^*]^2 \\ &+[R(r)-\tfrac{\mu}{\mu-\sigma_3^2}R^*]^2+[O(r)-\tfrac{\mu_0}{\mu_0-\sigma_4^2}O^*]^2\}dr \\ &\le \tfrac{K_\sigma}{M}, \end{aligned}$$

where p and q are positive constants and satisfy

$$2\mu\mu_0(\alpha+\mu)-\mu_0 p\alpha^2-\mu qk^2-\mu\mu_0\sigma_2^2>0,$$

and

$$\begin{aligned} K_\sigma = & \tfrac{\mu}{2\mu-\sigma_1^2}S^{*2} \\ & +\tfrac{\sigma_2^2[2\mu\mu_0(\alpha+\mu)-\mu_0 p\alpha^2-\mu qk^2]}{4\mu\mu_0(\alpha+\mu)-2\mu_0 p\alpha^2-2\mu qk^2-2\mu\mu_0\sigma_2^2}I^{*2} \\ & +\tfrac{p\mu}{2(\mu-\sigma_3^2)}R^{*2}+\tfrac{q\mu_0}{2(\mu_0-\sigma_4^2)}O^{*2}, \end{aligned}$$

$$\begin{aligned} M = & \min\{\mu-\tfrac{1}{2\sigma_1^2}, \\ & 2\mu\mu_0(\alpha+\mu)-\mu_0 p\alpha^2-\mu qk^2-\mu\mu_0\sigma_2^2, \\ & \tfrac{\mu}{\mu-\sigma_3^2},\tfrac{\mu_0}{\mu_0-\sigma_4^2}\}. \end{aligned}$$

Proof. If $R_0>1$ then system (4) has a unique positive epidemic equilibrium as (3).

Define a C^2 function $V: R_+^4\to R_+$ by

$$\begin{aligned} V(S,I,R,O) = & \tfrac{1}{2}(S-S^*+I-I^*)^2 \\ & +w_1(S+I)+\tfrac{p}{2}(R-R^*)^2 \\ & +\tfrac{q}{2}(O-O^*)^2, \end{aligned} \qquad (10)$$

where $w_1>0, p>0, q>0$ are positive constants to be chosen later. For simplicity, we divide (10) into two functions: $V(x)=V_1+V_2$, where

$$V_1=\frac{1}{2}(S-S^*+I-I^*)^2+w_1(S+I),$$

$$V_2=\frac{1}{2}p(R-R^*)^2+\frac{1}{2}q(O-O^*)^2.$$

Applying Itô formula, we obtain

$$\begin{aligned} dV_1 = & LV_1dt+(S-S^*+I-I^*+w_1) \\ & \times(\sigma_1 S(t)dB_1(t)+\sigma_2 I(t)dB_2 t), \end{aligned}$$

$$\begin{aligned} dV_2 = & LV_2dt+p(R-R^*)RdB_3(t) \\ & +q(O-O^*)OdB_4(t). \end{aligned}$$

In detail

$$\begin{aligned} LV_1 &= (S-S^*+I-I^*)[\Lambda-\mu S-(\alpha+\mu)I] \\ &\quad +w_1[\Lambda-\mu S-(\alpha+\mu)I] \\ &\quad +\tfrac{1}{2}(\sigma_1^2S^2+\sigma_2^2I^2) \\ &= (S-S^*+I-I^*)[-\mu S+\mu S^* \\ &\quad +(\alpha+\mu)I^*-(\alpha+\mu)I] \\ &\quad +w_1[\Lambda-\mu S-(\alpha+\mu)I] \\ &\quad +\tfrac{1}{2}(\sigma_1^2S^2+\sigma_2^2I^2) \\ &= -\mu(S-S^*)^2-(\alpha+\mu)(I-I^*)^2 \\ &\quad -(\alpha+2\mu)(S-S^*)(I-I^*) \\ &\quad +w_1[\Lambda-\mu S-(\alpha+\mu)I] \\ &\quad +\tfrac{1}{2}(\sigma_1^2S^2+\sigma_2^2I^2) \\ &= -\mu(S-S^*)^2-(\alpha+\mu)(I-I^*)^2 \\ &\quad +w_1\Lambda+[(\alpha+2\mu)I^*-w_1\mu]S \\ &\quad +[(\alpha+2\mu)S^*-(\alpha+\mu)w_1]I \\ &\quad +\tfrac{1}{2}(\sigma_1^2S^2+\sigma_2^2I^2) \\ &\quad -(\alpha+2\mu)(SI+S^*I^*), \end{aligned} \qquad (11)$$

and

$$\begin{aligned} LV_2 &= p(R-R^*)(\alpha I-\mu R)+\tfrac{1}{2}p\sigma_3^2R^2 \\ &\quad +q(O-O^*)(KI-\mu_0 O)+\tfrac{1}{2}q\sigma_4^2O^2 \\ &= p(R-R^*)[\alpha(I-I^*)-\mu(R-R^*)] \\ &\quad +\tfrac{1}{2}p\sigma_3^2R^2+q(O-O^*)[K(I-I^*) \\ &\quad -\mu_0(O-O^*)]+\tfrac{1}{2}q\sigma_4^2O^2 \\ &\le \tfrac{p\alpha^2}{2\mu}(I-I^*)^2+(\tfrac{p\mu}{2}-p\mu)(R-R^*)^2 \\ &\quad +\tfrac{1}{2}p\sigma_3^2R^2+\tfrac{qk^2}{2\mu_0}(I-I^*)^2 \\ &\quad +(\tfrac{q\mu_0}{2}-q\mu_0)(O-O^*)^2+\tfrac{1}{2}q\sigma_4^2O^2 \\ &= (\tfrac{p\alpha^2}{2\mu}+\tfrac{qk^2}{2\mu_0})(I-I^*)^2-\tfrac{p\mu}{2}(R-R^*)^2 \\ &\quad -\tfrac{q\mu_0}{2}(O-O^*)^2+\tfrac{1}{2}p\sigma_3^2R^2+\tfrac{1}{2}q\sigma_4^2O^2. \end{aligned} \qquad (12)$$

Choose $w_1=\max\{\frac{(\alpha+2\mu)I^*}{\mu},\frac{(\alpha+2\mu)S^*}{\alpha+\mu}\}$. As a result

$$(\alpha+2\mu)I^*-w_1\mu S\le 0,$$

and

$$(\alpha+2\mu)S^*-(\alpha+\mu)w_1\le 0.$$

Taking (11) and (12) together, we get

$$\begin{aligned} LV = \; & LV_1 + LV_2 \le -\mu(S-S^*)^2 \\ & -(\alpha+\mu+\tfrac{p\alpha^2}{2\mu}+\tfrac{qk^2}{2\mu_0})(I-I^*)^2 \\ & +\tfrac{1}{2}(\sigma_1^2 S^2+\sigma_2^2 I^2+p\sigma_3^2 R^2+\sigma_4^2 O^2) \\ & -\tfrac{p\mu}{2}(R-R^*)^2-\tfrac{q\mu_0}{2}(O-O^*)^2 \\ & = -(\mu-\tfrac{1}{2\sigma_1^2})(S-\tfrac{2\mu}{2\mu-\sigma_1^2}S^*)^2 \\ & \quad -(2\mu\mu_0(\alpha+\beta)-\mu_0 p\alpha^2 \\ & \quad -\mu qk_0^2-\mu\mu_0\sigma_2^2) \\ & \quad \times[I-\tfrac{2\mu\mu_0(\alpha+\beta)-\mu_0p\alpha^2-\mu qk_0^2}{2\mu\mu_0(\alpha+\beta)-\mu_0p\alpha^2-\mu qk_0^2-\mu\mu_0\sigma_2^2}I^*]^2 \\ & \quad -(\tfrac{p\mu}{2}-\tfrac{p\sigma_3^2}{2})(R-\tfrac{\mu}{\mu-\sigma_3^2}R^*)^2 \\ & \quad -(\tfrac{p\mu}{2}-\tfrac{q\sigma_4^2}{2})(O-\tfrac{\mu_0}{\mu_0-\sigma_4^2}O^*)^2 \\ & \quad +\tfrac{\mu}{2\mu-\sigma_1^2}S^{*2} \\ & \quad +\tfrac{\sigma_2^2[2\mu\mu_0(\alpha+\mu)-\mu_0p\alpha^2-\mu qk^2]}{4\mu\mu_0(\alpha+\mu)-2\mu_0p\alpha^2-2\mu qk^2-2\mu\mu_0\sigma_2^2}I^{*2} \\ & \quad +\tfrac{p\mu}{2(\mu-\sigma_3^2)}R^{*2}+\tfrac{q\mu_0}{2(\mu_0-\sigma_4^2)}I^{*2}. \end{aligned} \tag{13}$$

Note that p and q are positive constants and satisfy

$$2\mu\mu_0(\alpha+\mu)-\mu_0p\alpha^2-\mu qk^2-\mu\mu_0\sigma_2^2>0.$$

Besides, the condition (9) implies

$$\mu-\frac{1}{2}\sigma_1^2>0, \frac{p\mu}{2}-\frac{p\sigma_3^2}{2}>0, \frac{p\mu}{2}-\frac{q\sigma_4^2}{2}>0.$$

Thus

$$\begin{aligned} dV = \; & LVdt+(S-S^*+I-I^*+w_1) \\ & \times(\sigma_1 S(t)dB_1(t)+\sigma_2 I(t)dB_2 t) \\ & +p(R-R^*)RdB_3(t) \\ & \quad +q(O-O^*)OdB_4(t). \end{aligned} \tag{14}$$

Integrating both sides of (14) from 0 to t, then taking expectations, and considering inequality (13), yields

$$\begin{aligned} 0 \; & \le E[V(S(t),I(t),R(t),O(t))] \\ & \le E[V(S(0),I(0),R(0),O(0))] \\ & \quad + E\int_0^t\{-(\mu-\frac{1}{2\sigma_1^2})[S(r)-\frac{2\mu}{2\mu-\sigma_1^2}S^*]^2 \\ & \quad -(2\mu\mu_0(\alpha+\beta)-\mu_0p\alpha^2-\mu qk_0^2-\mu\mu_0\sigma_2^2) \\ & \quad \times[I(r)-\tfrac{2\mu\mu_0(\alpha+\beta)-\mu_0p\alpha^2-\mu qk_0^2}{2\mu\mu_0(\alpha+\beta)-\mu_0p\alpha^2-\mu qk_0^2-\mu\mu_0\sigma_2^2}I^*]^2 \\ & \quad -(\tfrac{p\mu}{2}-\tfrac{p\sigma_3^2}{2})[R(r)-\tfrac{\mu}{\mu-\sigma_3^2}R^*]^2 \\ & \quad -(\tfrac{p\mu}{2}-\tfrac{q\sigma_4^2}{2})[O(r)-\tfrac{\mu_0}{\mu_0-\sigma_4^2}O^*]^2\}dr+K_\sigma t, \end{aligned} \tag{15}$$

where K_σ is defined in Theorem 4. (15) implies that

$$\begin{aligned} & E\int_0^t\{(\mu-\frac{1}{2\sigma_1^2})[S(r)-\frac{2\mu}{2\mu-\sigma_1^2}S^*]^2 \\ & \quad +(2\mu\mu_0(\alpha+\beta)-\mu_0p\alpha^2-\mu qk_0^2-\mu\mu_0\sigma_2^2) \\ & \quad \times[I(r)-\tfrac{2\mu\mu_0(\alpha+\beta)-\mu_0p\alpha^2-\mu qk_0^2}{2\mu\mu_0(\alpha+\beta)-\mu_0p\alpha^2-\mu qk_0^2-\mu\mu_0\sigma_2^2}I^*]^2 \\ & \quad +(\tfrac{p\mu}{2}-\tfrac{p\sigma_3^2}{2})[R(r)-\tfrac{\mu}{\mu-\sigma_3^2}R^*]^2 \\ & \quad +(\tfrac{p\mu}{2}-\tfrac{q\sigma_4^2}{2})[O(r)-\tfrac{\mu_0}{\mu_0-\sigma_4^2}O^*]^2\}dr \\ & \le E[V(S(0),I(0),R(0),O(0))]+K_\sigma t. \end{aligned}$$

Dividing both sides by t and letting $t\to\infty$, one gets

$$\begin{aligned} & \limsup_{t\to\infty}\frac{1}{t}E\int_0^t\{(\mu-\frac{1}{2\sigma_1^2})[S(r)-\frac{2\mu}{2\mu-\sigma_1^2}S^*]^2 \\ & \quad +(2\mu\mu_0(\alpha+\beta)-\mu_0p\alpha^2-\mu qk_0^2-\mu\mu_0\sigma_2^2) \\ & \quad \times[I(r)-\tfrac{2\mu\mu_0(\alpha+\beta)-\mu_0p\alpha^2-\mu qk_0^2}{2\mu\mu_0(\alpha+\beta)-\mu_0p\alpha^2-\mu qk_0^2-\mu\mu_0\sigma_2^2}I^*]^2 \\ & \quad +(\tfrac{p\mu}{2}-\tfrac{p\sigma_3^2}{2})[R(r)-\tfrac{\mu}{\mu-\sigma_3^2}R^*]^2 \\ & \quad +(\tfrac{p\mu}{2}-\tfrac{q\sigma_4^2}{2})[O(r)-\tfrac{\mu_0}{\mu_0-\sigma_4^2}O^*]^2\}dr \\ & \le K_\sigma. \end{aligned}$$

Set

$$\begin{aligned} M = \; & \min\{\mu-\tfrac{1}{2\sigma_1^2}, \\ & 2\mu\mu_0(\alpha+\mu)-\mu_0p\alpha^2-\mu qk^2-\mu\mu_0\sigma_2^2, \\ & \tfrac{\mu}{\mu-\sigma_3^2},\tfrac{\mu_0}{\mu_0-\sigma_4^2}\}, \end{aligned}$$

then it is easy to obtain

$$\begin{aligned} & \limsup_{t\to\infty}\tfrac{1}{t}E\int_0^t\{[(S(r)-\tfrac{2\mu}{2\mu-\sigma_1^2}S^*)^2 \\ & +[I(r)-\tfrac{2\mu\mu_0(\alpha+\beta)-\mu_0p\alpha^2-\mu qk_0^2}{2\mu\mu_0(\alpha+\beta)-\mu_0p\alpha^2-\mu qk_0^2-\mu\mu_0\sigma_2^2}I^*]^2 \\ & +[R(r)-\tfrac{\mu}{\mu-\sigma_3^2}R^*]^2+[O(r)-\tfrac{\mu_0}{\mu_0-\sigma_4^2}O^*]^2\}dr \\ & \quad \le \tfrac{K_\sigma}{M}. \end{aligned}$$

This completes the proof. □

Remark 5 *Theorem 4 shows that the solution of model (4) fluctuates around the certain level which is relevant to*

$$\begin{aligned} P^* \quad & (\tfrac{2\mu}{2\mu-\sigma_1^2}S^*,\tfrac{2\mu\mu_0(\alpha+\beta)-\mu_0p\alpha^2-\mu qk_0^2}{2\mu\mu_0(\alpha+\beta)-\mu_0p\alpha^2-\mu qk_0^2-\mu\mu_0\sigma_2^2}I^*, \\ & \tfrac{\mu}{\mu-\sigma_3^2}R^*,\tfrac{\mu_0}{\mu_0-\sigma_4^2}O^*) \end{aligned}$$

and σ_i for $i=1,2,3,4$. With the value of σ_i decreasing, P^* will be closing to E^*, and the difference between X and P^* also decreases, where X denotes the solution of system (2).

5 Numerical Simulations and Conclusion

In order to conform the results above, we numerically simulate the solution of system (4). Using Milsteins Higher Order Method [25], we get the discretization equation:

$$\begin{cases} S_{k+1} = & S_k + (\Lambda - \beta S_k O_k - \mu S_k)\Delta t \\ & +\sigma_1 S_k \sqrt{\Delta t}\varepsilon_{1,k} + \frac{1}{2} S_k \Delta t(\varepsilon_{1,k}^2 - 1), \\ I_{k+1} = & I_k + (\beta S_k O_k - (\alpha+\mu) I_k)\Delta t \\ & +\sigma_2 I_k \sqrt{\Delta t}\varepsilon_{2,k} + \frac{1}{2} I_k \Delta t(\varepsilon_{2,k}^2 - 1), \\ R_{k+1} = & R_k + (\alpha I_k - \mu R_k)\Delta t \\ & +\sigma_3 R_k \sqrt{\Delta t}\varepsilon_{3,k} + \frac{1}{2} R_k \Delta t(\varepsilon_{3,k}^2 - 1), \\ O_{k+1} = & O_k + (k I_k - \mu_0 O_k)\Delta t \\ & +\sigma_4 O_k \sqrt{\Delta t}\varepsilon_{4,k} + \frac{1}{2} O_k \Delta t(\varepsilon_{4,k}^2 - 1). \end{cases}$$

where time increment $\Delta t > 0$, and $\varepsilon_{1,k}$, $\varepsilon_{2,k}$, $\varepsilon_{3,k}$, $\varepsilon_{4,k}$, are $N(0,1)$-distributed independent random variables.

Firstly, we show the effect of white noise on the disease-free equilibrium. As we have can see in section 1, deterministic system (2) has a disease-free equilibrium E_0, and it is globally stable if $R_0 = \frac{k\beta\Lambda}{\mu\mu_0(\alpha+\mu)} \leq 1$. By Theorem 2 and Remark 3, we showed that the expectations of $S(t), I(t), R(t)$ and $O(t)$ are bounded in time average when condition (7) is also satisfied. Obviously, the boundedness is proportional to σ_1, moreover, the smaller σ_1 is, the less the boundedness is. In addition, if σ_1 is decreasing to zero, E_0 is stochastically asymptotically stable in the large. The following numerical simulations of the strong solution of (4) confirm the above results we have shown in Fig.1, we always choose initial values $S(0) = 50$, $I(0) = 1$, $R(0) = 20$, $O(t) = 1$ and parameters $\Lambda = 1.154, \beta = 0.52/54, k = 1/20, \mu = 0.6/52, \mu_0 = 7/100, \alpha = 0.5$ with different intensities of white noise which satisfies condition (7). The dash line in the figure represents solutions of the deterministic system (2) . The red line in the figure represents solutions of the stochastic system (4) whose intensities of white noise value in group (a) and the blue line represents solutions of the stochastic system (4) whose intensities of white noise value in group (b) . As we can see, the curves of system (4) always fluctuate around the curves of system (2). In Fig.1, comparisons of different intensities of white noise suggest that the fluctuations reduce as the noise level decreases.

Secondly, in Fig. 2, we choose parameters satisfying $R_0 > 1$ and condition (9) in Theorem 4. Similarly as said above, the solution of system (4) also fluctuates around the solution of system (2), which supports the conclusion of Theorem 4. In detail, in Fig.2 parameters are the same except for the decreasing intensities. From the figures, with intensities decreasing, the fluctuation is weaker.

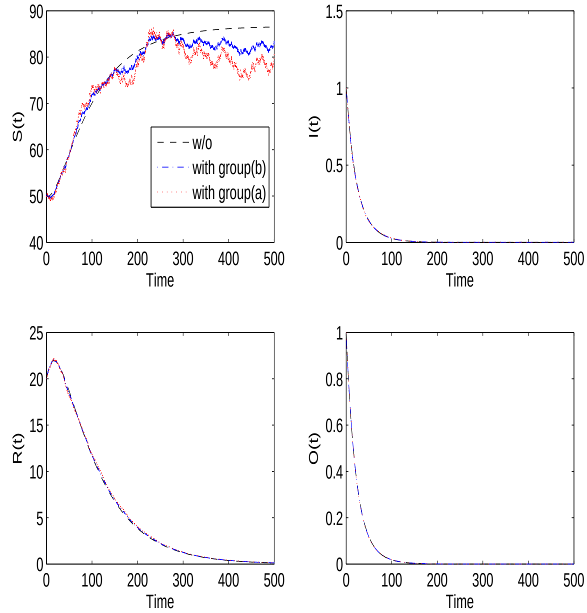

Figure 1: The stochastically asymptotical stability of disease-free equilibrium. (a): $\sigma_1 = 0.008, \sigma_2 = 0.006, \sigma_3 = 0.004, \sigma_4 = 0.002$. (b):$\sigma_1 = 0.004, \sigma_2 = 0.003, \sigma_3 = 0.002, \sigma_4 = 0.001$.

Finally, we show compare the effects of σ_1, σ_2, σ_3 and σ_4 on system (2). In Fig.3 Group (a) the intensities of white noise are much larger than those in Group (b) but σ_1 keeps the same, we can see the fluctuation does not substantial increase. That means the intensities of σ_2, σ_3 and σ_4 have little effect on the fluctuation. Fig.3 shows the fluctuation for $S(t)$ almost cannot be seen for σ_1. Then we can believe the solution is stochastically asymptotically stable in the large.

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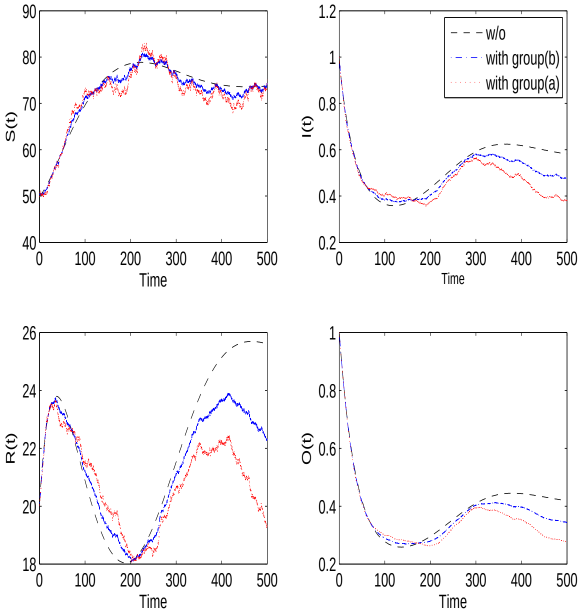

Figure 2: Asymptotic behavior around the endemic equilibrium of the deterministic model. (a): $\sigma_1 = 0.008, \sigma_2 = 0.006, \sigma_3 = 0.004, \sigma_4 = 0.002$. (b):$\sigma_1 = 0.004, \sigma_2 = 0.003, \sigma_3 = 0.002, \sigma_4 = 0.001$.

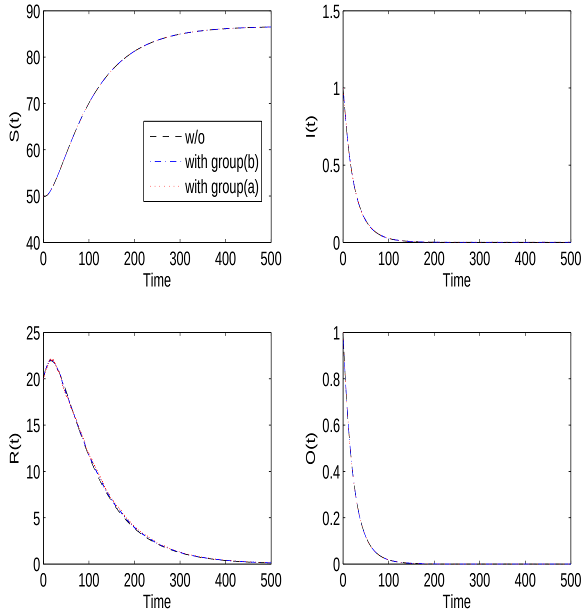

Figure 3: The comparison of the effects of σ_1, σ_2, σ_3 and σ_4 on system (2). (a): $\sigma_1 = 0, \sigma_2 = 0.003, \sigma_3 = 0.004, \sigma_4 = 0.002$. (b):$\sigma_1 = 0, \sigma_2 = 0.03, \sigma_3 = 0.02, \sigma_4 = 0.01$.

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