Planar Graph Characterization - Using γ - Stable Graphs

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Abstract: A graph G is said to be γ - stable if $\gamma(G_{xy}) = \gamma(G)$, for all $x, y \in V(G), x$ is not adjacent to y, where G_{xy} denotes the graph obtained by merging the vertices x, y. In this paper we have provided a necessary and sufficient condition for $\bar{G}$ to be γ - stable, where $\bar{G}$ denotes the complement of G. We have obtained a characterization of planar graphs when G and $\bar{G}$ are γ - stable graphs.

Key–Words: γ - stable graph, planar, nonplanar, dominating set.

1 Introduction

In graph theory, a planar graph is a graph that can be embedded in the plane, that is it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other [13].

Planarity of graph is related with various properties. Tilings appears in many fields such as architecture, crystal structure etc. In [7], Hao Li has obtained some properties of planar normal tilling. In [6], Gurami Tsitsiashvili and Marina Osipova have discussed the asymptotic analysis of connectivity probability in random planar graphs. Pulley blank has proved that deciding whether a graph is supereulerian within planar graphs is NP-complete[9]. Hong-Jian Lai, Yehong Shao and Huiya Yan have provided a survey on supereulerian graph [9]. In[11], C. H. C. Little and G. Sanjith have provide a new characterization of planar graphs that concerns the structure of the cocycle space of a graph.

Characterizing planar graphs based on graph properties is a common problem discussed by various authors. In [2], By Joseph Battle, Frank Harary and Yukihiro Kodama have proved that every planar graph with nine points has a nonplanar complement. In[1], Jin Akiyama and Frank Harary have characterized all graphs for which G and $\bar{G}$ are outerplanar.

In [4], Rosa I. Enciso and Ronald D. Dutton have classified planar graph based on the complement of G. They have proved the following result.

R$_1$. If G is a planar graph, $\gamma(\bar{G}) \leq 4$.

In this paper we have characterized planar graphs based on the property of $\bar{G}$.

2 Terminology

We consider only simple connected undirected graphs $G = (V, E)$. The open neighborhood of vertex $v \in V(G)$ is defined by $N(v) = \{u \in V(G) | uv \in E(G)\}$, while its closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. We say that H is a subgraph of G, if $V(H) \subseteq V(G)$ and $uv \in E(H)$ implies $uv \in E(G)$. If a subgraph H satisfies the added property that for every pair u, v of vertices, $uv \in E(H)$ if and only if $uv \in E(G)$, then H is called an induced subgraph of G and is denoted by $\langle H \rangle$. Two graphs are said to be homeomorphic if one graph can obtained from the other by the creation of edges in series(that is by insertion of vertices of degree two) or by the merger of edges in series. In the literature of graph theory, K_5 and $K_{3,3}$ are called Kuratowski´s graph.

An elementary contraction of a graph G is obtained by identifying two adjacent points u and v, that is by the removal of u and v and the addition of a new point w adjacent to those points to which u or v was adjacent. A graph G is contractible to a graph H if H can be obtained from G by a sequence of elementary contractions[5]. For properties related to graph theory we refer to F. Harary [5]. We indicate that u is adjacent to v by writing $u \perp v$ [3].

A set of vertices D in a graph $G = (V, E)$ is a dominating set if every vertex of $V - D$ is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set - abbreviated MDS. The cardinality of any MDS for G is called the domination number of G and it is denoted by $\gamma(G)$. A γ - set denotes a dominating set for G with minimum cardinality. The private neighborhood of $v \in D$ is defined by $pn[v, D] = N(v) - N(D - \{v\})$. For properties related to domination we refer to T. W. Haynes, S. T. Hedetniemi, and P. J. Slater [8].

3 Results and Discussions

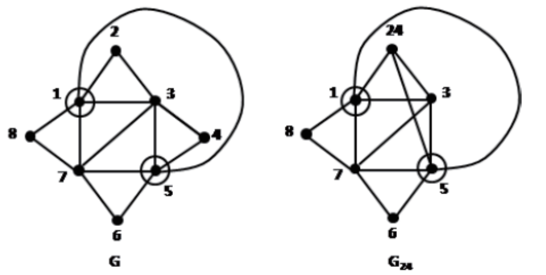
In [14], M. Yamuna and K. Karthika defined γ - stable graphs. A graph G is said to be γ - stable if $\gamma(G_{xy}) = \gamma(G)$, for all $x, y \in V(G), x$ is not adjacent to y, where G_{xy} denotes the graph obtained by merging the vertices x, y

They have proved the following results.

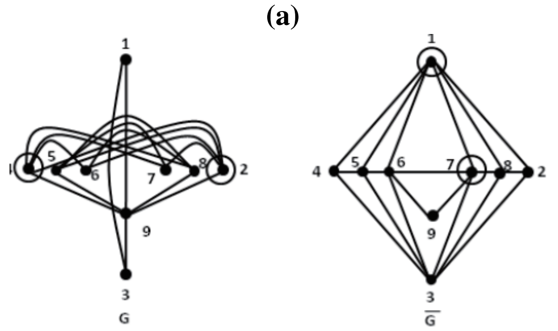
R$_2$. A graph G is γ - stable if and only if every γ-set D of G is a clique.

R$_3$. If G is γ - stable, then $pn[u, D] \geq 2$, for all $u \in V(G)$.

In all the figures encircled vertices denotes a γ-set.



(a)



(b)

Fig.1

In Fig. 1(a), G_{24} denote the graph obtained by merging the nonadjacent vertices 2 and 4, $\gamma(G) = \gamma(G_{24})$. This is true for all $x, y \in V(G)$, x is not adjacent to y, which implies G is γ - stable.

The graph in Fig. 1 (a) is self complement, that is $G, \bar{G}$ are γ - stable, G planar. In Fig.1 (b) $G, \bar{G}$ are γ - stable, G nonplanar. We observe that when G and $\bar{G}$ are γ - stable graphs G may or may not be planar.

In this paper we focus on obtaining conditions under which $\bar{G}$ is γ - stable and hence use it for characterizing planarity of γ - stable graphs.

Theorem 1 *For any graph G such that $\gamma(\bar{G}) = k, \bar{G}$ is γ - stable if and only if*

1. *there is at least one set of k - independent vertices $S \subseteq V(G)$ such that there is no $v \in V - S$, v adjacent to all vertices in S.*

2. *for all k - non - independent vertices S in G, there is at least one $v \in V - S, v$ adjacent to all vertices in S.*

Proof: Assume that $\bar{G}$ is a γ - stable graph such that $\gamma(\bar{G}) = k$. We know that any γ - set in $\bar{G}$ is a clique. Let $S = \{v_1, v_2, \cdots, v_k\}$ be a γ - set in $\bar{G}$. $\langle S \rangle$ is a clique in $\bar{G}$, which implies there is no adjacency between these vertices in G, that is S is independent with respect to G. Assume that there is a vertex $v \in V(G) - S(G)$ such that v adjacent to all vertices in S. This means that, in $\bar{G}$ vertex v is not adjacent to any vertex in S, a contradiction as S is a γ - set in $\bar{G}$. Hence condition 1 is true.

Let $S = \{u_1, u_2, \cdots, u_k\}$ be a set of non - independent vertices in G. Assume that there is no $v \in V - S$ such that v adjacent to all vertices in S. This means that in $\bar{G}$, every $v \in V - S$ is adjacent to at least one vertex in S, that is S is a γ - set with respect to $\bar{G}$. Since S is a non - independent set in G, there is at least one $u_i, u_j \in S(G)$ such that $u_i \perp u_j$. This means that there is no edge between u_i and u_j in $\bar{G}$, a contradiction as $\bar{G}$ is γ - stable and S is a γ - set with respect to $\bar{G}$. Hence condition 2 is satisfied.

Conversely, let G be a graph such that

i. $\gamma(\bar{G}) = k$

ii. condition 1 and 2 of the theorem satisfied.

We have to prove that $\bar{G}$ is γ - stable, that is we need to prove every γ - set of $\bar{G}$ is a clique [by R$_2$]. Let $S(G) = \{v_1, v_2, \cdots, v_k\}$ be a set of vertices that satisfies condition 1. Since S is independent in G, $\langle S \rangle$ is a clique with respect to $\bar{G}$. Since there is no $v \in V(G) - S(G)$, v adjacent to all vertices in $S(G)$, every $v \in V(\bar{G}) - S(\bar{G})$ is adjacent to at least one vertex in S, that is S is a γ - set in $\bar{G}$.

Let $S = \{u_1, u_2, \cdots, u_k\}$ be a set of vertices in G, that satisfies condition 2. Since S is non - independent in G, $\langle S \rangle$ cannot be a clique in $\bar{G}$. Since there is atleast one $v \in V(G) - S(G)$, v adjacent to all vertices in $S(G), v$ is not adjacent to any vertex in $S(\bar{G})$, that is S is not a γ - set with respect to $\bar{G}$. This is true for every set $S(\bar{G})$ such that

i. $\langle S(\bar{G}) \rangle$ is not a clique,

ii. $|S(\bar{G})| = k$,

that is there is no γ - set that is not a clique in $\bar{G}$.

From the above discussion we conclude that every γ - set of $\bar{G}$ is a clique. ☐

Theorem 2 *Let G and $\bar{G}$ be γ - stable graphs such that $\gamma(G) = k, \gamma(\bar{G}) = m, m > k$. Then every $m-1, m-2, \cdots, 2$ independent vertices in G are collectively adjacent to atleast two vertices in G.*

Proof: Let G and $\bar{G}$ be γ - stable graphs. Let $\gamma(\bar{G}) = D = \{x_1, x_2, \cdots, x_m\}$ and let $Z = \{v_1, v_2, \cdots, v_p\}$ be a set of independent vertices in G, where $p \in [m-1, m-2, \cdots, 2]$.

If possible assume that the vertices in Z are collectively adjacent to one vertex say $x \in V(G)$. In $\bar{G}, \langle v_1, v_2, \cdots, v_p\rangle$ is a clique. In $\bar{G}, Z$ dominates $V(\bar{G}) - \{x\}$, which implies $Z \cup \{x\}$ is a dominating set for $\bar{G}, |Z \cup \{x\}| \leq m, x$ selfish. $\langle Z \cup \{x\}\rangle$ is not a clique in $\bar{G}$.

If $|Z \cup \{x\}| < m$, then $\{Z \cup \{x\}\}$ dominates $\bar{G}$ such that $|Z \cup \{x\}| < m$, a contradiction as $\gamma(\bar{G}) = m$.

If $|Z \cup \{x\}| = m$, then since $\langle Z \cup \{x\}\rangle$ is not a clique, we get a contradiction to our assumption that $\bar{G}$ is a γ - stable graph.

So every $m-1, m-2, \cdots, 2$ independent vertices in G are collectively adjacent to atleast two vertices in G. □

4 Planar Characterization of γ - Stable Graphs

We recollect the following famous theorem on planar graphs.

R$_4$. A necessary and sufficient condition for a graph G to be planar is that G does not contain either of Kuratowski´s two graphs or any graph homeomorphic to either of them.

R$_5$. A graph is planar if and only if it does not have a subgraph contractible to K_5 or $K_{3,3}$.[5]

We shall prove that a γ - stable graph is planar or nonplanar using R_4 and R_5.

If $\gamma(G) = 1$, then $\bar{G}$ is disconnected and hence $\bar{G}$ is not a γ - stable graph. Also by R$_1$, if G is a planar graph, $\gamma(G) \leq 4$. So in the remaining part of this section we restrict our discussion to cases where $1 < \gamma(G) \leq 4, 1 < \gamma(\bar{G}) \leq 4$.

We shall use the following results of Theorem 1 and 2 frequently.

i. If$\gamma(\bar{G}) = k$, then every k - non - independent vertices in G are collectively adjacent to atleast one vertex in G.

ii. If $G, \bar{G}$ are γ - stable, $\gamma(G) = k, \gamma(\bar{G}) = m, m > k$, then every $m-1, m-2, \cdots, 2$ independent vertices in G are collectively adjacent to atleast two vertices in G.

In all figures, in the remaining part of the discussion,

i. $---$ represents the newly added edges in the current discussion.

ii. When we use edge contraction, a vertex receives a label of the contracted vertices. For example $y : bb_1x_1x_2$ means that the contracted edges are bb_1, b_1x_1, x_1x_2 and is assigned the new label as b.

iii. If $a, b, c, \cdots$ denote graphs in figures, then $a', b', c', \cdots$ denotes subgraphs of $a, b, c, \cdots$ respectively [$a', b', c', \cdots$ are either K_5 or $K_{3,3}$].

All cases and subcases are supported with graphs along with the discussion.

Theorem 3 *If G and $\bar{G}$ are γ - stable graphs such that $\gamma(G) \leq 4$ and $\gamma(\bar{G}) = 4$, then G is nonplanar.*

Proof: Case 1: $\gamma(G) = 2, \gamma(\bar{G}) = 4$.

Let $\gamma(G) = D = \{a, b\}$. Since G is γ - stable, $pn[u, D] \geq 2$, for all $u \in D$ [by R$_3$]. Let $\{a_1, a_2\} \in pn[a, D], \{b_1, b_2\} \in pn[b, D]$. $\langle a_1, a, b, b_1\rangle$ is non - independent in G, $\gamma(\bar{G}) = 4$. There is a vertex $x \in V(G)$ such that $x \perp (a_1, a, b, b_1)$ [Since $a_2 \in pn[a, D], b_2 \in pn[b, D], x \neq a_2, b_2$].

$\langle x, a, b, a_1\rangle$ is non - independent in G. So, there is one $y \in V(G)$ such that $y \perp (x, a, b, a_1)$ [Since $a_2 \in pn[a, D], \{b_1, b_2\} \in pn[b, D], x \neq a_2, b_1, b_2$].

Similarly $\langle x, a, b, y\rangle$ is non - independent, there is one $z \in V(G)$ such that $z \perp (x, a, b, y)$ [Since $\{a_1, a_2\} \in pn[a, D], \{b_1, b_2\} \in pn[b, D], x \neq a_1, a_2, b_1, b_2$]. $\langle a, b, x, y, z\rangle$ is K_5, which is a subgraph of G as seen in Fig. 2 implies G is nonplanar.

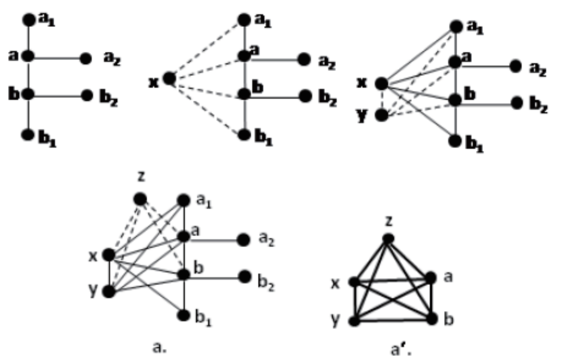


Fig. 2

Case 2: $\gamma(G) = 3, \gamma(\bar{G}) = 4$.

Let $\gamma(G) = D = \{a, b, c\}$ be a γ - set for G. Since G is γ - stable, $pn[u, D] \geq 2$, for all $u \in D$ [by R$_3$]. Let $\{a_1, a_2\} \in pn[a, D], \{b_1, b_2\} \in pn[b, D]$ and $\{c_1, c_2\} \in pn[c, D]$.

$\langle a_1, a, b, c\rangle$ is non - independent in $G, \gamma(\bar{G}) = 4$, there is a vertex $x \in V(G)$ such that $x \perp (a_1, a, b, c)$ [Since $a_2 \in pn[a, D], \{b_1, b_2\} \in pn[b, D]$ and $\{c_1, c_2\} \in pn[c, D], x \neq a_2, b_1, b_2, c_1, c_2]$.

$\langle x, a, b, c\rangle$ is non - independent in G, there is a vertex y such that $y \perp (x, a, b, c)$ [Since $\{a_1, a_2\} \in pn[a, D], \{b_1, b_2\} \in pn[b, D]$ and $\{c_1, c_2\} \in pn[c, D], y \neq a_1, a_2, b_1, b_2, c_1, c_2]$.

So, $\langle x, y, a, b, c\rangle$ is K_5, which is a subgraph of G as seen in Fig. 3 implies G is nonplanar.

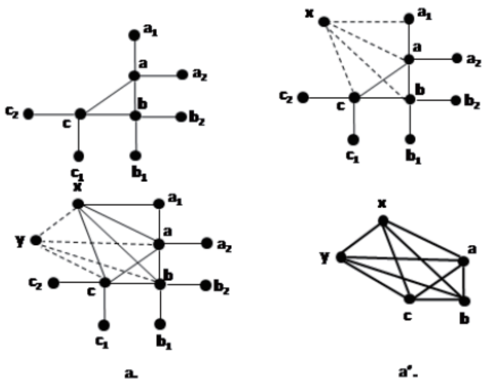


Fig.3

Case 3: $\gamma(G) = 4$ and $\gamma(\bar{G}) = 4$.

Let $\gamma(G) = \{a, b, c, d\}.\langle a, b, c, d\rangle$ is non - independent in G, $\gamma(\bar{G}) = 4$. There is a vertex $x \in V(G)$ such that $x \perp (a, b, c, d)$, that is $\langle a, b, c, d, x\rangle$ is K_5 as seen in Fig. 4 implies G is nonplanar.

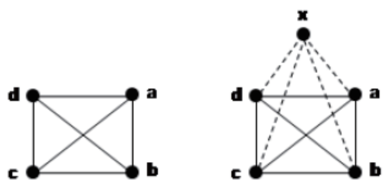


Fig.4

From case 1, 2 and 3 we conclude that, if G and $\bar{G}$ are γ - stable graphs such that $\gamma(G) \leq 4$ and $\gamma(\bar{G}) = 4$, then G is nonplanar. □

Theorem 4 *If G and $\bar{G}$ are γ - stable graphs such that $\gamma(G) \leq 4$ and $\gamma(\bar{G}) = 3$, then G is nonplanar.*

Proof: We prove the following claims to prove the theorem.

Claim 1: If $\gamma(G) = 2, \gamma(\bar{G}) = 3$, then G is nonplanar.

Proof: Let $\gamma(G) = D = \{a, b\}$. Since G is γ-stable, $pn[u, D] \geq 2$, for all $u \in D$ [by R$_3$]. Let $\{a_1, a_2\} \in pn[a, D], \{b_1, b_2\} \in pn[b, D]$. $\langle a, b, a_1\rangle$ is non - independent in $G, \gamma(\bar{G}) = 3$, there is a vertex $x_1 \in V(G)$ such that $x_1 \perp (a, b, a_1)$. [Since $a_2 \in pn[a, D], \{b_1, b_2\} \in pn[b, D], x_1 \neq a_2, b_1, b_2]$.

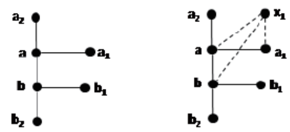


Fig.5

$\langle a, a_1, x_1\rangle$ is non - independent, there is one vertex adjacent to these vertices. Note that this common vertex cannot be b, b_1, b_2 [Since $a_1 \in pn[a, D], \{b_1, b_2\} \in pn[b, D]]$. So, the common vertex is either a_2 or any other vertex (say x_3).

Case 1 : $a, a_1, x_1 \perp a_2$

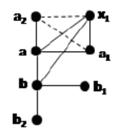


Fig.6

$\langle a_2, x_1, b_2\rangle$ is non-independent. The common adjacent vertex can be a_1, b_1, or any vertex say x_2.

Subcase 1: $a_2, x_1, b_2 \perp a_1$

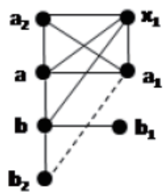


Fig.7

A. If $a_2 \perp b_2$, then contracting the edge $bb_2, \langle a, b, a_1, a_2, x_1\rangle$ is K_5 as seen in Fig. 8 implies G is nonplanar.

B. If a_2 is not adjacent to b_1, b_2, since a_2, b_1 and a_2, b_2 are independent in G, by Theorem 2, there is at least two vertices common to a_2, b_1 and a_2, b_2. These two common vertices can be x_1, a_1.

   a. $a_2, b_1 \perp x_1, a_1$ and $a_2, b_2 \perp x_1, a_1$. $\acute{a}$ is $K_{3,3}$ as seen in Fig. 9 implies G is nonplanar.

   b. a_2, b_1 and a_2, b_2 is not adjacent to a_1, x_1.This means that, a_2, b_1 and a_2, b_2 are adjacent to some vertex in G. Let a_2, b_1 be adjacent to some $y \in V(G)$.

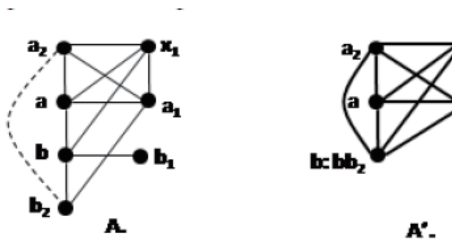


Fig.8

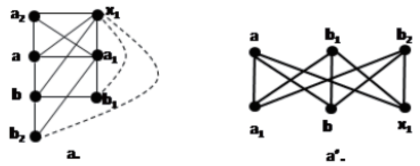


Fig.9

Contracting edges yb_1, b_1b and bb_2, $\langle a, b, a_1, a_2, x_1\rangle$ is K_5 as seen in Fig. 10 implies G is nonplanar.

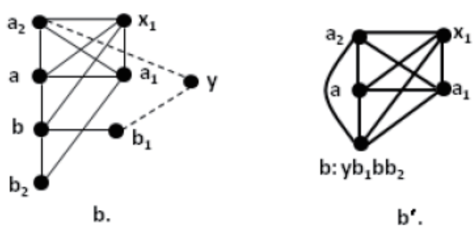


Fig.10

Similarly if there is any $y \in V(G) \perp a_2, b_2$, $\langle a, b, a_1, a_2, x_1\rangle$ is K_5 implies G is nonplanar.

Subcase 2: $a_2, x_1, b_2 \perp b_1$

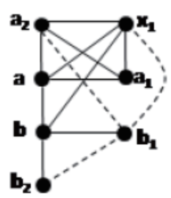


Fig.11

A. If $a_1 \perp b_2$, then contracting the edges b_1b and bb_2, $\langle a, b, a_1, a_2, x_1\rangle$ is K_5 as seen in Fig. 12 implies G is non planar.

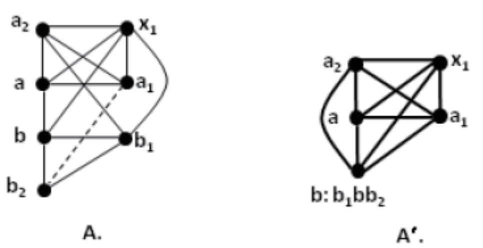


Fig.12

B. If a_1 is not adjacent to b_2, then by Theorem 2 a_1, b_2 are adjacent to at least two vertices. The different possible cases is discussed in a to c.

   a. If $a_1, b_2 \perp a_2, x_1$, then a' is $K_{3,3}$ as seen in Fig. 13 implies G is nonplanar.

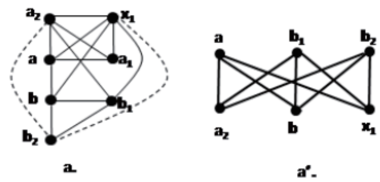


Fig.13

   b. If $a_1, b_2 \perp a_2, b_1$, then contracting the edges b_1b and bb_2, $\langle a, b, a_1, a_2, x_1\rangle$ is K_5 as seen in Fig. 14 implies G is nonplanar.
   c. If $a_1, b_2 \perp x_1, b_1$, then contracting the edges b_1b and bb_2, $\langle a, b, a_1, a_2, x_1\rangle$ is K_5 as seen in Fig. 14 implies G is nonplanar.

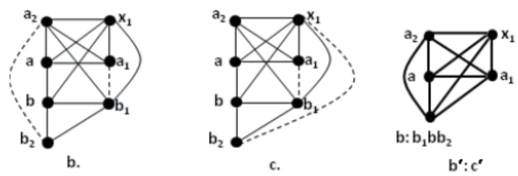


Fig.14

C. If a_1, b_2 are adjacent to some $y \in V(G)$, then contracting the edges b_1b, bb_2, b_2y, $\langle a, b, a_1, a_2, x_1\rangle$ is K_5 as seen in Fig. 15 implies G is nonplanar.

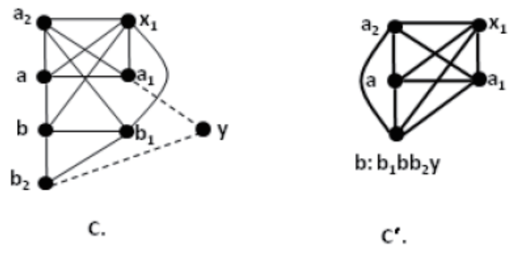


Fig.15

Subcase 3: a_2, x_1, b_2 are adjacent to some $x_2 \in V(G)$.

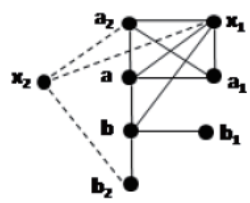


Fig.16

A. If $a_1 \perp b_2$, then contracting the edges x_2b_2, b_2b, $\langle a, b, a_1, a_2, x_1\rangle$ is K_5 as seen in Fig. 17 implies G is nonplanar.

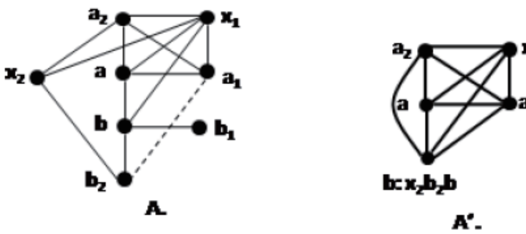


Fig.17

B. If a_1 is not adjacent to b_2, then by Theorem 2 a_1, b_2 are adjacent to at least two vertices. The different possible cases is discussed in a to g.

a. $a_1, b_2 \perp a_2, x_1$

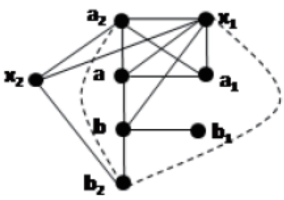


Fig.18

x_2 is dominated either by a or b.

a_1. If $x_2 \perp a$, then a_1' is $K_{3,3}$ implies G is nonplanar.

a_2. If $x_2 \perp b$, then contracting the edge ab we see that a_2' is $K_{3,3}$ implies G is nonplanar.

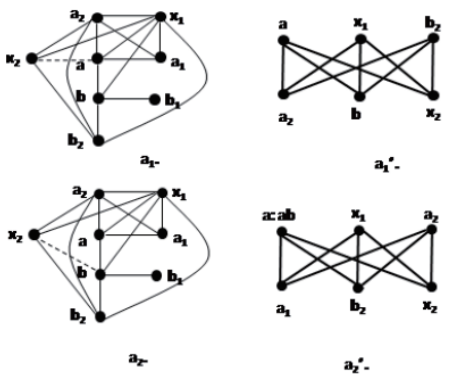


Fig.19

b. If $a_1, b_2 \perp a_2, b_1$, then contract the edges b_1b_2, b_2b.

c. If $a_1, b_2 \perp x_1, b_1$, then contract the edges x_2b_2, b_2b, bb_1.

d. If $a_1, b_2 \perp b_1, x_2$, then contract the edges x_2b_2, b_2b.

e. If $a_1, b_2 \perp a_2, x_2$, then contract the edges x_2b_2, b_2b.

f. If $a_1, b_2 \perp x_1, x_2$, then contract the edges x_2b_2, b_2b.

g. If a_1, b_2 are adjacent to some vertex $y \in V(G)$, then contract edges x_2b_2, b_2b, b_2y.

In all cases from b to g, $\langle a, b, a_1, a_2, x_1\rangle$ is K_5 as seen in Fig. 20 implies G is nonplanar.

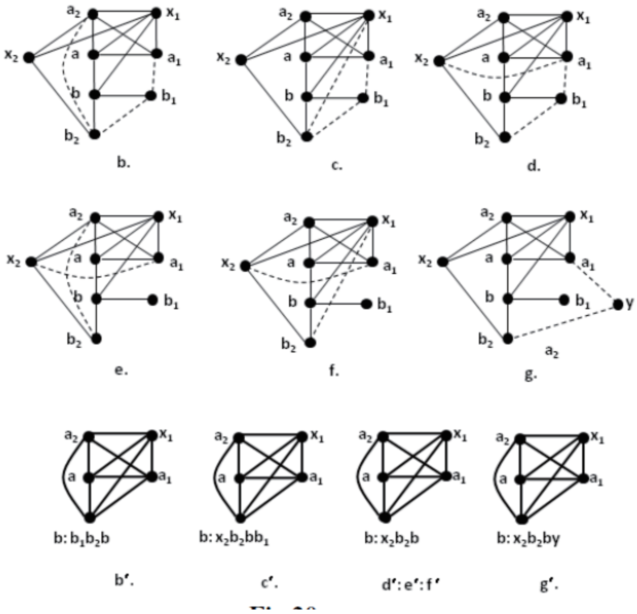


Fig.20

Case 2: $a, a_1, x_1 \perp x_3$

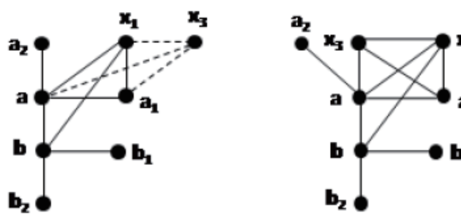


Fig.21

$\langle x_3, x_1, b_2\rangle$ is non-independent. So, x_3, x_1, b_2 are collectively adjacent to either a_1, a_2, b, b_1 or some $x_4 \in V(G)$.

Subcase 1: $x_3, x_1, b_2 \perp a_1$

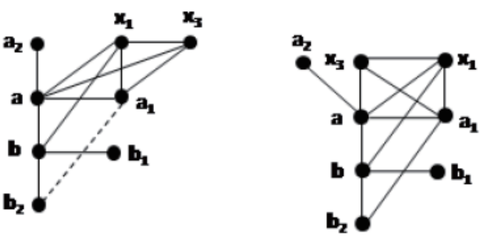


Fig.22

The graph in Fig. 22 is isomorphic to graph in Fig. 7 of subcase 1 of case 1. So, the discussion is analogues to subcase 1 of case 1.

Subcase 2: $x_3, x_1, b_2 \perp a_2$

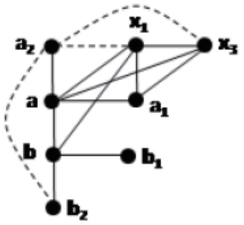


Fig.23

A. If $a_1 \perp b_1$, then contracting the edge a_2b_2, b_2b, bb_1, $\langle a, b, a_1, x_1, x_3 \rangle$ is K_5 as seen in Fig. 24 implies G is nonplanar.

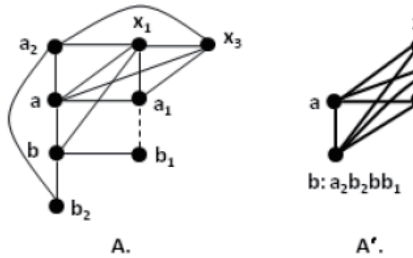


Fig.24

B. If a_1 is not adjacent to b_1, then by Theorem 2 a_1, b_1 are adjacent to at least two vertices. The different possible cases is discussed in a to g.

   a. If $a_1, b_1 \perp b_2, a_2$, then $\langle a, a_1, a_2, x_1, x_3 \rangle$ is K_5.

   b. If $a_1, b_1 \perp b_2, x_1$, then contract the edge a_2b_2.

   c. If $a_1, b_1 \perp b_2, x_3$, then contract the edge a_2b_2.

   d. If $a_1, b_1 \perp a_2, x_1$, then $\langle a, a_1, a_2, x_1, x_3 \rangle$ is K_5.

   e. If $a_1, b_1 \perp a_2, x_3$, $\langle a, a_1, a_2, x_1, x_3 \rangle$ is K_5.

   f. If $a_1, b_1 \perp y$, then contract the edges a_2b_2, b_2b, bb_1, b_1y.

   g. If $a_1, b_1 \perp x_1, x_3$, then contracting the edge a_2b_2, we see that g' is $K_{3,3}$.

In cases a to f, $\langle a, a_1, a_2, x_1, x_3 \rangle$ is K_5 and g' is $K_{3,3}$ as seen in Fig. 25 implies G is nonplanar.

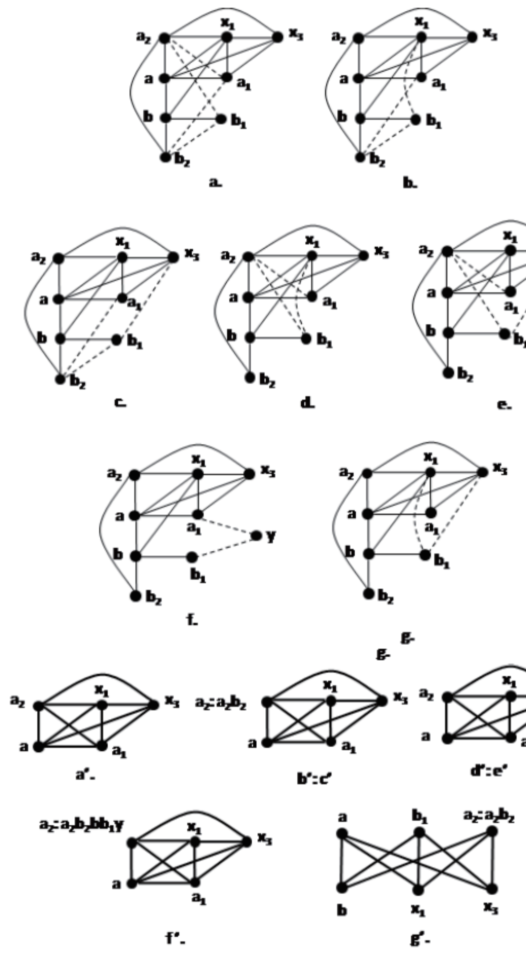


Fig.25

Subcase 3: $x_3, x_1, b_2 \perp b$.

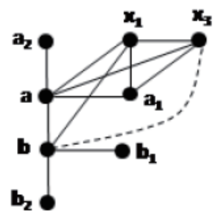


Fig.26

A. If $a_1 \perp b_1$, then contracting the edge bb_1, $\langle a, b, a_1, x_1, x_3 \rangle$ is K_5 as seen in Fig. 27 implies G is nonplanar.

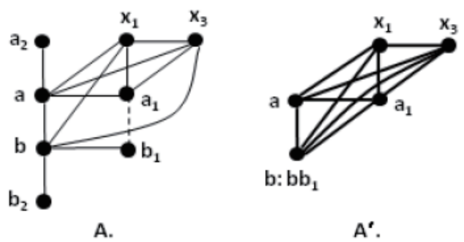


Fig.27

B. If a_1 is not adjacent to b_1, then by Theorem 2, a_1, b_1 are adjacent to at least two vertices. The different possible cases is discussed in a to g.

   a. If $a_1, b_1 \perp b_2, a_2$, then contract the edge bb_2.
   b. If $a_1, b_1 \perp b_2, x_1$, then contract the edge bb_2.
   c. If $a_1, b_1 \perp b_2, x_3$, then contract the edge bb_2.
   d. If $a_1, b_1 \perp a_2, x_1$, then contract the edges a_2b_1, b_1b.
   e. If $a_1, b_1 \perp a_2, x_3$, then contract the edges a_2b_1, b_1b.
   f. If $a_1, b_1 \perp y$, then contract the edges yb_1, b_1b.

In case a to f, $\langle a, b, a_1, x_1, x_3 \rangle$ is K_5 implies G is nonplanar.

If $a_1 \perp b_2$, then contracting the edge b_2b, $\langle a, b, a_1, x_1, x_3 \rangle$ is K_5 implies G is nonplanar. If a_1 is not adjacent to b_2, then a_1, b_1 and a_1, b_2 are pair of independent vertices. By Theorem 2, these vertices are collectively adjacent to at least two vertices. In all possible combinations except the case when a_1, b_1, b_2 are collectively adjacent to x_1, x_3, $\langle a, b, a_1, x_1, x_3 \rangle$ is K_5 implies G is nonplanar as in cases a to f.

g. If $a_1, b_1, b_2 \perp x_1, x_3$, then g' is $K_{3,3}$ implies G is nonplanar.

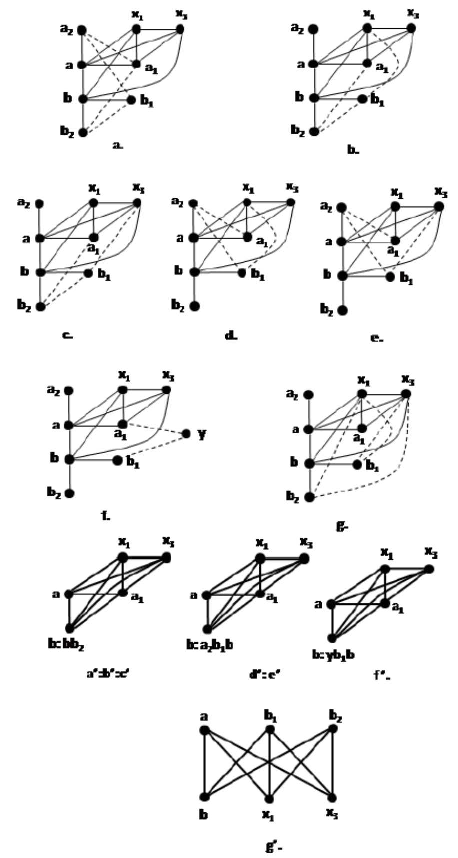


Fig.28

Subcase 4: $x_3, x_1, b_2 \perp b_1$.

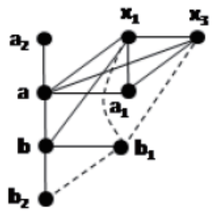


Fig.29

The graph in Fig. 29 is isomorphic to graph in Fig. 11 of subcase 2 of case 1. So, the discussion is analogues to subcase 2 of case 1.

Subcase 5: $x_3, x_1, b_2 \perp x_4$.

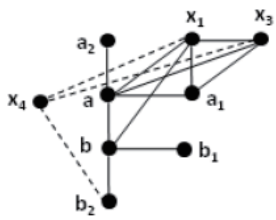


Fig.30

The graph in Fig. 30 is isomorphic to graph in Fig. 16 of subcase 3 of case 1. So, the discussion is analogues to subcase 3 of case 1.

By case 1 and case 2, we conclude that G is nonplanar.

Claim 2 If G and $\bar{G}$ are γ - stable graphs such that $\gamma(G) = \gamma(\bar{G}) = 3$, then G is nonplanar.

Proof: Let $\gamma(G) = D = \{a, b, c\}$. Since G is γ - stable, $pn[u, D] \geq 2$, for all $u \in D$ [by R_3]. Let $\{a_1, a_2\} \in pn[a, D], \{b_1, b_2\} \in pn[b, D]$ and $\{c_1, c_2\} \in pn[c, D]$. Since $\langle a, b, c \rangle$ is non - independent, $\gamma(\bar{G}) = 3$, there is a vertex $x_1 \in V(G)$ such that $x_1 \perp (a, b, c)$ [Since $a_1 \in pn[a, D], b_1 \in pn[b, D]$ and $c_1 \in pn[c, D], x_1 \neq a_1, b_1, c_1$]. Since $\langle a_1, a, x_1 \rangle$ is non - independent, let $x_2 \perp (a_1, a, x_1)$ [Since $a_1 \in pn[a, D], b_1 \in pn[b, D]$ and $c_1 \in pn[c, D], x_2 \neq b, b_1, c_1$].

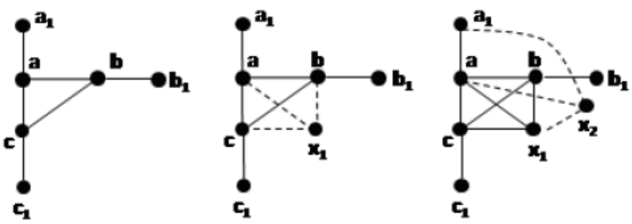


Fig.31

Since $\langle x_1, x_2, b_1\rangle$ is non - independent, there is atleast one vertex adjacent to them. The common vertex could be b, a_1, c_1 or any $x_j \in V(G)$.

Case 1: $x_1, x_2, b_1 \perp b$

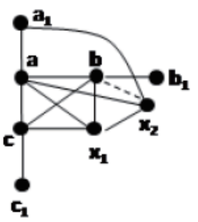


Fig.32

$\langle x_1, x_2, c_1\rangle$ is non - independent. x_1, x_2, c_1 are collectively adjacent to either c, a_1, b_1 or some $x_i \in V(G)$.

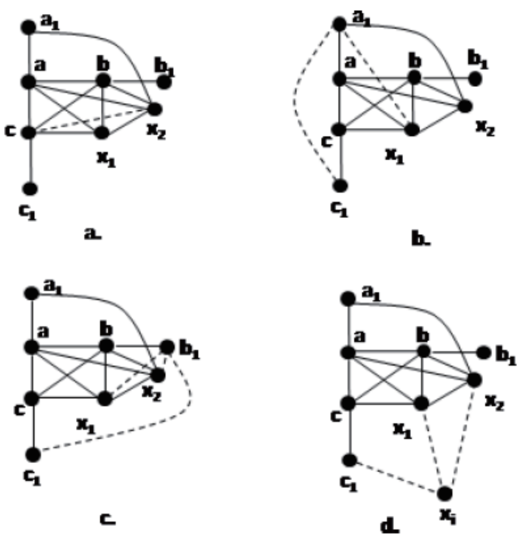
a. If $x_1, x_2, c_1 \perp c$, then $\langle a, b, c, x_1, x_2\rangle$ is K_5.

b. If $x_1, x_2, c_1 \perp a_1$, then contract the edges cc_1 and c_1a_1.

c. If $x_1, x_2, c_1 \perp b_1$, then contract the edges b_1x_2 and cc_1.

d. If $x_1, x_2, c_1 \perp$ some $x_i \in V(G)$, then contract the edges x_ic_1 and c_1c.

In all cases, $\langle a, b, c, x_1, x_2\rangle$ is K_5 as seen in Fig. 33 implies G is nonplanar.



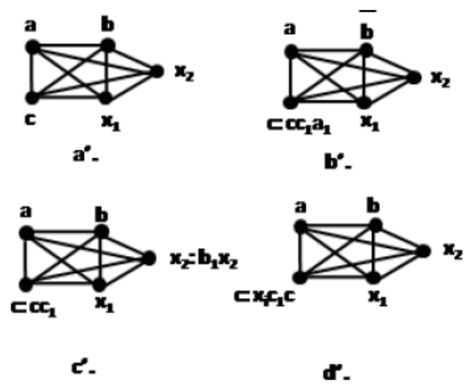


Fig.33

Case 2: $x_1, x_2, b_1 \perp a_1$

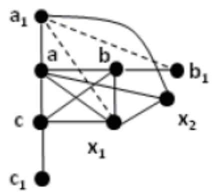


Fig.34

$\langle x_1, x_2, c_1\rangle$ is non-independent. x_1, x_2, c_1 are collectively adjacent to either c, a_1, b_1 or some $x_i \in V(G)$.

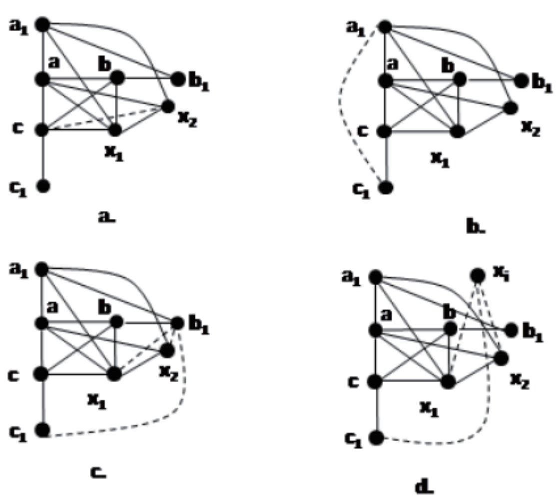
a. If $x_1, x_2, c_1 \perp c$, then contract the edges bb_1 and b_1a_1.

b. If $x_1, x_2, c_1 \perp a_1$, then contract the edges cc_1, x_2a_1 and bb_1.

c. If $x_1, x_2, c_1 \perp b_1$, then contract the edges b_1a_1, a_1x_2 and cc_1.

d. If $x_1, x_2, c_1 \perp$ some $x_i \in V(G)$, then contract the edges a_1b_1, b_1b and x_ic_1, c_1c.

In all cases $\langle a, b, c, x_1, x_2\rangle$ is K_5 as seen in Fig. 35 implies G is nonplanar.



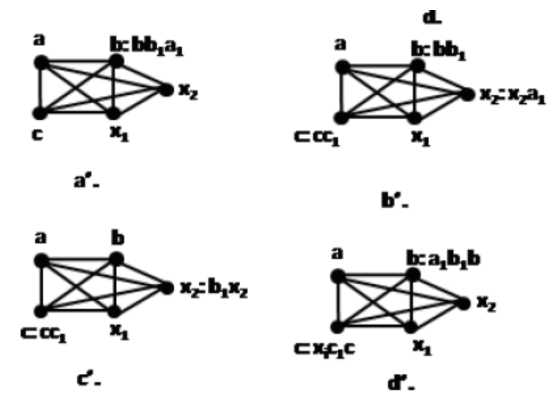


Fig.35

Case 3: $x_1, x_2, b_1 \perp c_1$

Contracting the edges bb_1 and a_1x_2, $a\acute{}$ is $K_{3,3}$ as seen in Fig. 36 implies G is nonplanar.

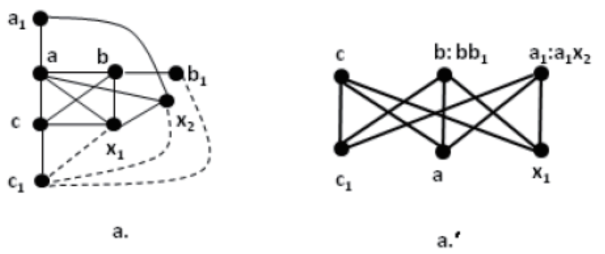


Fig.36

Case 4: x_1, x_2, b_1 adjacent to some $x_j \in V(G), x_j \neq b$.

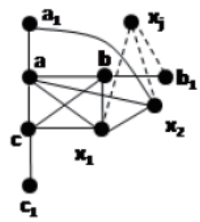


Fig.37

Subcase 1: Since $\langle x_1, x_2, c_1 \rangle$ is non - independent, x_1, x_2, c_1 adjacent to either c, a_1, b_1 or $x_j \in V(G)$.

a. If $x_1, x_2, c_1 \perp c$, then contract the edges x_jb_1, b_1b.

b. If $x_1, x_2, c_1 \perp a_1$, then contract the edges x_jb_1, b_1b and a_1c_1, c_1c.

c. If $x_1, x_2, c_1 \perp b_1$, then contract the edges b_1x_2 and c_1c.

d. If $x_1, x_2, c_1 \perp x_j$, then contract the edges bb_1, x_2x_j and c_1c.

In all cases, $\langle a, b, c, x_1, x_2 \rangle$ is K_5 as seen in Fig. 38 implies G is nonplanar.

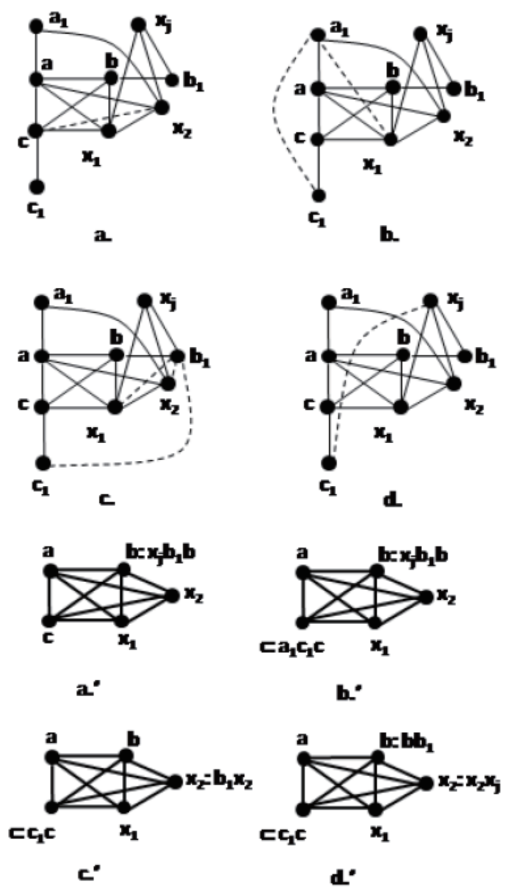


Fig.38

Subcase 2: $x_1, x_2, c_1 \perp$ some $x_k \in V(G), x_k \neq c$.

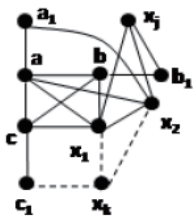


Fig.39

a. If $x_j = x_k$, then x_j is adjacent to b_1, x_2, x_1, c_1. Contract the edges x_2x_j, bb_1, cc_1.

b. If x_k and x_j are distinct, then contract the edges bb_1, b_1x_j and x_kc_1, c_1c.

In both the cases $\langle a, b, c, x_1, x_2 \rangle$ is K_5 as seen in Fig. 40 implies G is nonplanar.

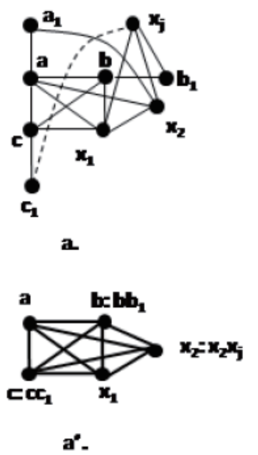


Fig.40

From case 1, 2, 3 and 4 we see that if G and $\bar{G}$ are γ - stable graphs satisfying the conditions of the theorem, then G is nonplanar.

Claim 3: If $\gamma(G) = 4, \gamma(\bar{G}) = 3$, then G is nonplanar.

Proof: Let $D = \gamma(G) = \{a, b, c, d\}$. Since G is γ - stable, we know that $pn[u, D] \geq 2$, for all $u \in D$, [by R$_3$]. Let $\{a_1, a_2\} \in pn[a, D]$, $\{b_1, b_2\} \in pn[b, D]$, $\{c_1, c_2\} \in pn[c, D]$, $\{d_1, d_2\} \in pn[d, D]$. $\langle a_1, a, b\rangle$ is non - independent. Let $x_1 \perp a_1, a, b$ [Since $a_1 \in pn[a, D], b_1 \in pn[b, D], c_1 \in pn[c, D], d_1 \in pn[d, D], x_1 \neq b_1, c_1, d_1, c, d]$.

$\langle c, c_1, x_1\rangle$ is non - independent. Let $x_2 \perp c, c_1, x_1$ [Since $a_1 \in pn[a, D], b_1 \in pn[b, D], c_1 \in pn[c, D], d_1 \in pn[d, D], x_2 \neq a_1, b_1, d_1, a, b, d]$.

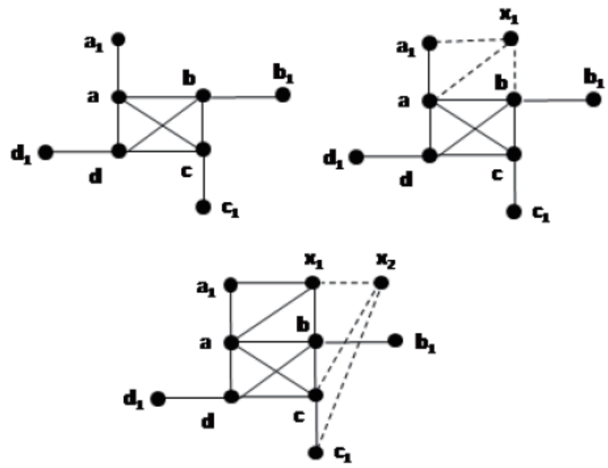


Fig.41

$\langle d, d_1, x_1\rangle$ is non - independent. The common adjacent vertex is either x_2 or any vertex $x_3 \in V(G)$ [Note that d, d_1, x_1 cannot be collectively adjacent to a, b, c, a_1, b_1, c_1].

a. If $d, d_1, x_1 \perp x_2$, then contract the edge x_2x_1.

b. If $d, d_1, x_1 \perp x_3$, then contract the edges x_3x_1 and x_1x_2 .

In both the cases, $\langle a, b, c, d, x_1\rangle$ is K_5 as seen in the Fig. 42 implies G is nonplanar.

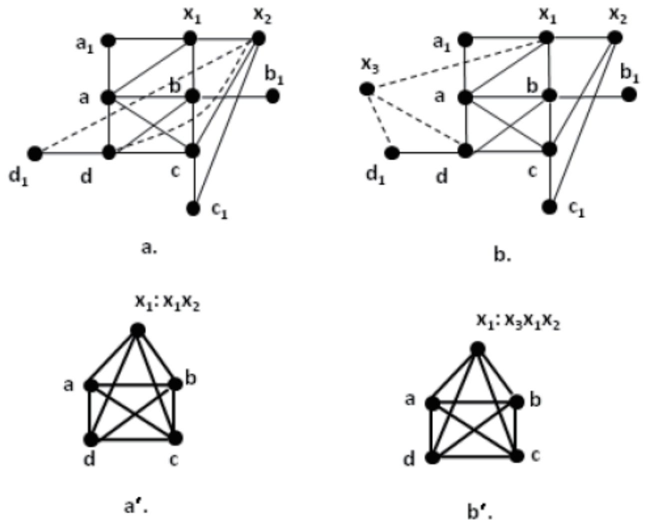


Fig.42

From the above discussion, we conclude that G is nonplanar.

Remark 5 *If G and $\bar{G}$ are γ-stable graphs such that $\gamma(G) \leq 4$ and $\gamma(\bar{G}) = 2$, then G need not be nonplanar.*

Example 6

Case 1: $\gamma(G) = 2$ and $\gamma(\bar{G}) = 2$.

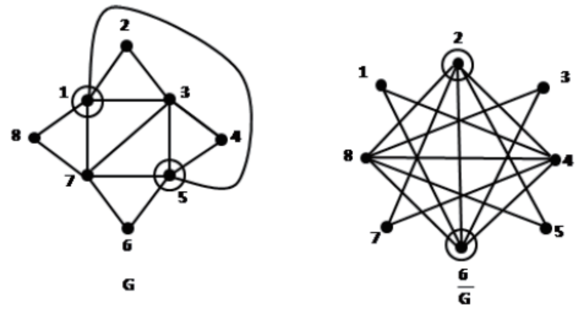


Fig.43

Case 2: $\gamma(G) = 3$ and $\gamma(\bar{G}) = 2$.

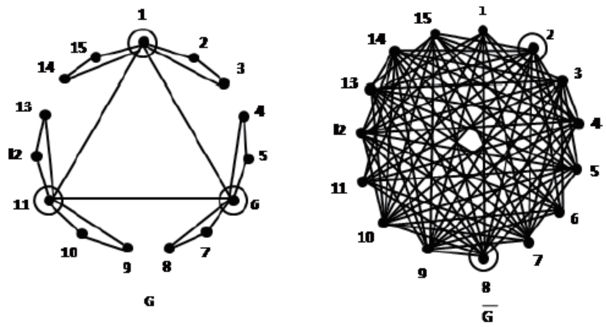


Fig.44

Case 3 $\gamma(G) = 4$ and $\gamma(\bar{G}) = 2$.

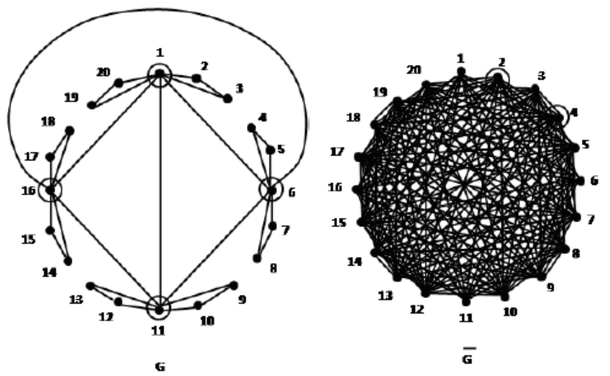


Fig.45

In Fig. 43, 44 and 45 G and $\bar{G}$ are γ - stable graphs, G planar.

5 Conclusion

From the above discussion we conclude that, if $G, \bar{G}$ are γ - stable graphs, then

1. G is nonplanar, if $2 < \gamma(\bar{G}) \leq 4$.

2. G need not be nonplanar, if$\gamma(\bar{G}) = 2$.

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