New study of Classes of Hurwitz-Zeta Function Related to Integral Operator

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Abstract: By means of the Hadamard product, the present paper introduces new classes, $\Sigma_a^{t,*}(\alpha,\beta,\rho)$ and $\Sigma_a^t(\alpha,\beta,\rho)$ of Hurwitz-Lerch-Zeta function in the punctured disk $U^* = \{z : 0 < |z| < 1\}$. In addition, the study investigates a number of inclusion relationships, properties and derives some interesting properties depending on some integral properties.

Key–Words: Analytic function; Convex function; Meromorphic function; Hurwitz Zeta function; Linear operator; Hadamard product; Functions with positive real part; Integral operator.

1 Introduction

The theory of analytic univalent function is a classical problem of complex analysis which belongs to a beautiful part of geometric function theory (GFT). To our interest, GFT denotes the part of functions analysis devoted to estimations of different magnitudes related to conformal mapping of one region onto another.

Conformal mapping is a classical part of complex analysis being intimately connected with the theory of boundary value problems for harmonic functions, thus has numerous applications in mathematical physics and other branches of mathematics.

A large number of generalizations of the class of univalent function have been explored and properties such as distortion theorems and radii theorems are the main interests of solving problems. To date, various methods have been used such as method of differential subordinations, method of differential inequalities and methods of arising from the convolution theory.

These are rather some curiosity provoking problems which has recently attracted many other mathematicians to the derivation of new subclasses and new properties. Results from the theory of the geometric function are remarkable by their particular elegance and simplicity of formulations.

However, in searching for a new breakthrough in the field, new approach and new development are indeed needed, see [[1]].

One of the important studies in univalent functions is the integral operator. In this paper, we initiate the study of functions which are meromorphic in the punctured disk $U^* = \{z : 0 < |z| < 1\}$ with a Laurent expansion about the origin, see [2].

Also, we shall use the operator $L_a^t(\alpha,\beta) f(z)$ to introduce some new classes of meromorphic functions.

To begin with, a meromorphic function is a single-valued function that is analytic in all but possibly a discrete subset of its domain, and at those singularities it must go to infinity like a polynomial (i.e., these exceptional points must be poles and not essential singularities).
A simpler definition states that a meromorphic function $f(z)$ is a function of the form

$$f(z) = \frac{g(z)}{h(z)},$$

where $g(z)$ and $h(z)$ are entire functions with $h(z) \neq 0$ (see [3], p. 64). A meromorphic function therefore may only have finite-order, isolated poles and zeros and no essential singularities in its domain.

An equivalent definition of a meromorphic function is a complex analytic map to the Riemann sphere. For example the Gamma function is meromorphic in the whole complex plane C.

In the present paper, we introduce new classes, $\Sigma_a^{t,*}(\alpha,\beta,\rho)$ and $\Sigma_a^{t}(\alpha,\beta,\rho)$ of Hurwitz-Lerch-Zeta function defined by means of the Hadamard product.

The paper investigates a number of inclusion relationships of these classes. Moreover, some interesting properties are derived depending on some integral properties.

2 Preliminaries

Let Σ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n, \tag{1}$$

which are analytic in the punctured unit disk U^*. For $0\leq\beta$, we denote by $S^*(\beta)$ and $k(\beta)$, the subclasses of Σ consisting of all meromorphic functions which are, respectively, starlike of order β and convex of order β in U^*.

For functions $f_j(z)(j=1;2)$ defined by

$$f_j(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,j}z^n, \tag{2}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1*f_2)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,1}a_{n,2}z^n. \tag{3}$$

Let $P_n(\rho)$ be the class of functions $p(z)$ analytic in U^* satisfying the properties $p(0)=1$ and

$$\int_0^{2\pi}\left|\frac{\Re p(z)-\rho}{1-\rho}\right|d\theta\leq n\pi, \tag{4}$$

where $z=re^{i\theta}$, $n\geq 2$ and $0\leq\rho<1$. This class has been introduced in [4]. We note that $P_n(0)=P_n$ and $P_2(\rho)=P(\rho)$, see [[5] and [6]], the class of analytic functions with positive real part greater than ρ and $P_2(0)=P$, the class of functions with positive real part. From (4) we can write $p\in P_n(\rho)$ as

$$p(z)=\left(\frac{n}{4}+\frac{1}{2}\right)p_1(z)+\left(\frac{1}{2}-\frac{n}{4}\right)p_2(z), \tag{5}$$

where $p_i(z)\in P(\rho), i=1,2$ and $z\in U^*$.

Let us define the function $\tilde{\phi}(\alpha,\beta;z)$ by

$$\tilde{\phi}(\alpha,\beta;z)=\frac{1}{z}+\sum_{n=0}^{\infty}\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}z^n, \tag{6}$$

for $\beta\neq 0,-1,-2,...,$ and $\alpha\in C\setminus\{0\}$, where $(\lambda)_n$ denotes the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n:=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}\lambda(\lambda+1)...(\lambda+n-1)\\ 2\end{cases},$$

it being understood *conventionally* that $(0)_0:=1$ and assumed *tacitly* that the Γ-quotient exists. We note that

$$\tilde{\phi}(\alpha,\beta;z)=\frac{1}{z}{}_2F_1(1,\alpha,\beta;z)$$

where

$${}_2F_1(b,\alpha,\beta;z)=\sum_{n=0}^{\infty}\frac{(b)_n(\alpha)_n}{(\beta)_n}\frac{z^n}{n!}$$

is the well-known Gaussian hypergeometric function.

We recall here a general Hurwitz-Lerch-Zeta function, which is defined in [[7], [8]] by the following series:

$$\Phi(z,t,a)=\frac{1}{a^t}+\sum_{n=1}^{\infty}\frac{z^n}{(n+a)^t} \tag{7}$$

$a\in C\backslash Z_0^-$, $Z_0^-=\{0,-1,-2,...\};t\in C$ when $z\in U=U^*\cup\{0\};\Re(t)>1$ when $z\in\partial U$.

Important special cases of the function $\Phi(z,t,a)$ include, for example, the Reimann zeta function $\zeta(t)=\Phi(1,t,1)$, the Hurwitz zeta function $\zeta(t,a)=\Phi(1,t,a)$, the Lerch zeta function $l_t(\zeta)=\Phi\left(\exp^{2\pi i\xi},t,1\right),(\xi\in R,\Re(t)>1)$, the polylogarithm $L_t^i(z)=z\Phi(z,t,a)$ and so on. Recent results on $\Phi(z,t,a)$, can be found in the expositions [[9], [10]]. By making use of the following normalized function we define:

$$G_{t,a}(z)=(1+a)^t\left[\Phi(z,t,a)-a^t+\frac{1}{z(1+a)^t}\right]$$

$$=\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1+a}{n+a}\right)^tz^n, \tag{8}$$

$(z\in U^*)$.

Corresponding to the functions $G_{t,a}(z)$ and using the Hadamard product for $f(z) \in \Sigma$, we define a new linear operator $L_{t,a}(\alpha, \beta)$ on Σ by the follow series:

$$L_a^t(\alpha,\beta) f(z) = \phi(\alpha,\beta;z) * G_{t,a}(z)$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \left(\frac{1+a}{n+a}\right)^t a_n z^n. \tag{9}$$

$(z \in U^*)$.

The meromorphic functions with the generalized hypergeometric functions were considered recently by many others see for example [[11], [12], [13], [14], [15], and [16]]

It follows from (9) that

$$z\left(L_a^t(\alpha,\beta) f(z)\right)' = \alpha\left(L_a^t(\alpha+1,\beta) f(z)\right)$$

$$-(\alpha+1) L_a^t(\alpha,\beta) f(z). \tag{10}$$

Definition 1 *Let $f \in \Sigma$. Then $f \in \Sigma_a^{t,*}(\alpha,\beta,\rho)$, if and only if*

$$-(1-\alpha) z^2 \left(L_a^t(\alpha,\beta) f(z)\right)' - \alpha z^2 \left(L_a^t(\alpha,\beta) f(z)\right)' \in P_n(\rho)$$

where $\alpha > 0$, $n \geq 2$, $0 \leq \rho < 1$ and $z \in U^*$.

Definition 2 *Let $f \in \Sigma$. Then $f \in \Sigma_a^{t}(\alpha,\beta,\rho)$, if and only if*

$$(1-\alpha) z^2 \left(L_a^t(\alpha,\beta) f(z)\right)' - \alpha z \left(L_a^t(\alpha,\beta) f(z)\right)' \in P_n(\rho)$$

where $\alpha > 0$, $n \geq 2$, $0 \leq \rho < 1$ and $z \in U^*$.

Lemma 3 *[17] If $p(z)$ is analytic in U^* with $p(0) = 1$, and if α is a complex number satisfying $\Re(\alpha) \geq 0$, then*

$$\Re\{p(z) + \alpha z p'(z)\} > \beta \qquad (0 \leq \beta < 1).$$

Implies

$$\Re\, p(z) > \beta + (1-\beta)(2\gamma - 1),$$

where γ is given by

$$\gamma = \gamma(\alpha) = \int_0^1 \left(1 + t^{\Re\alpha}\right)^{-1} dt,$$

which is an increasing function of $\Re(\alpha)$ and $\frac{1}{2} \leq \gamma < 1$. The estimate is sharp in the sense that the bound cannot be improved.

Lemma 4 *[18] If $p(z)$ is analytic in U^*, $p(0) = 1$ and $\Re p(z) > \frac{1}{2}, z \in U^*$, then for any function F analytic in U^*, the function $p * F$ takes values in the convex hull of the image of U^* under F.*

Lemma 5 *[19] Let $p(z) = 1 + b_1 z + b_2 z_2 + ... \in P(\rho)$. Then*

$$\Re\, p(z) \geq 2\rho - 1 + \frac{2(1-\rho)}{1+|z|}.$$

3 Main results

Theorem 6 *Let $f \in \Sigma_a^{t,*}(\alpha,\beta,\rho)$. Then*

$$-z^2\left(L_a^t(\alpha,\beta) f(z)\right)' \in P_n(\rho_1)$$

where ρ_1 is given by

$$\rho_1 = \rho + (1-\rho)(2\gamma - 1), \tag{11}$$

and

$$\gamma = \int_0^1 \left(1 + t^{\Re\alpha}\right)^{-1} dt.$$

Proof: Let

$$-z^2\left(L_a^t(\alpha,\beta) f(z)\right)' =$$

$$p(z) = \left(\frac{n}{4} + \frac{1}{2}\right) p_1(z) + \left(\frac{1}{2} - \frac{n}{4}\right) p_2(z). \tag{12}$$

Then $p(z)$ is analytic in U^* with $p(0) = 1$. Applying the identity (9) in (12) and differentiating the resulting equation with respect to z, we have

$$-(1-\alpha) z^2 \left(L_a^t(\alpha,\beta) f(z)\right)' - \alpha z^2 \left(L_a^t(\alpha,\beta) f(z)\right)'$$

$$= \left[p(z) + \alpha z p'(z)\right].$$

Since $f \in \Sigma_a^{t,*}(\alpha,\beta,\rho)$, so $\{p(z) + \alpha z p'(z)\} \in P_n(\rho)$ for $z \in U^*$.
This implies that

$$\Re\{p_i(z) + \alpha z p_i'(z)\} > \rho, \quad i = 1, 2.$$

Using Lemma 3, we see that $\Re\{p_i(z)\} > \rho_1$, where ρ_1 is given by (11).

Consequently $p \in P_n(\rho_1)$ for $z \in U^*$, and the proof is complete.

Theorem 7 *Let $f \in \Sigma_a^{t,*}(0,\beta,\rho)$ and let*

$$F_\delta(f(z)) = \frac{\delta}{z^{\delta+1}} \int_0^z t^\delta f(t) dt \tag{13}$$

$(\delta > 0,\ z \in U^*)$*. Then*

$$-z^2 \left(L_a^t(\alpha,\beta) F(f(z))\right)' \in P_n(\rho_2)$$

where ρ_2 is given by

$$\rho_2 = \rho + (1-\rho)(2\gamma_1 - 1) \tag{14}$$

and

$$\gamma_1 = \int_0^1 \left(1 + t^{\Re\left(\frac{1}{\delta}\right)}\right)^{-1} dt.$$

Proof: Setting

$$-z^2 \left(L_a^t(\alpha,\beta) F(f(z))\right)' = p(z) =$$

$$\left(\frac{n}{4} + \frac{1}{2}\right) p_1(z) + \left(\frac{1}{2} - \frac{n}{4}\right) p_2(z). \tag{15}$$

Then $p(z)$ is analytic in U^* with $p(0) = 1$. Using the following operator identity:

$$z\left(L_a^t(\alpha,\beta) F(f(z))\right)' = \delta\left(L_a^t(\alpha,\beta) F(f(z))\right)$$

$$-(\delta+1)\left(L_a^t(\alpha,\beta) F(f(z))\right) \tag{16}$$

in (15), and differentiating the resulting equation with respect to z, we find that

$$-z^2 \left(L_a^t(\alpha,\beta) f(z)\right)' =$$

$$\left\{p(z) + \frac{1}{\delta} z p'(z)\right\} \in P_n(\rho)$$

Using Lemma 3, we see that $-z^2 \left(L_a^t(\alpha,\beta) F(f(z))\right)' \in P_n(\rho_2)$ for $z \in U^*$, where ρ_2 is given by (14).

Hence, the proof is complete.

Theorem 8 *Let $\phi(z) \in \Sigma$ satisfy the inequality:*

$$\Re(z\phi(z)) > \frac{1}{2}, \qquad z \in U^*. \tag{17}$$

If $f \in \Sigma_a^t(\alpha,\beta,\rho)$. Then $\phi * f \in \Sigma_a^t(\alpha,\beta,\rho)$.

Proof: Let $G = \phi * f$ and $h \in P_n(\rho)$. Then

$$(1-\alpha) z \left(L_a^t(\alpha,\beta) G(z)\right) + \alpha z \left(L_a^t(\alpha,\beta) G(z)\right)$$

$$= (1-\alpha) z \left(L_a^t(\alpha,\beta)(\phi * f)(z)\right) +$$

$$\alpha z \left(L_a^t(\alpha,\beta)(\phi * f)(z)\right) = z\phi(z) * h(z)$$

$$= \left(\frac{n}{4} + \frac{1}{2}\right)\left[(1-\rho)\left\{z\phi(z) * h_1(z) + \rho\right\}\right]$$

$$+ \left(\frac{1}{2} - \frac{n}{4}\right)\left[(1-\rho)\left\{z\phi(z) * h_2(z) + \rho\right\}\right]$$

$h_1, h_2 \in P$. Since $\Re(z\phi(z)) > \frac{1}{2}$ and by using Lemma 4, we can conclude that $G = \phi * f \in \Sigma_a^t(\alpha,\beta,\rho)$.

Theorem 9 *Let $\phi(z) \in \Sigma$ satisfy the inequality (17), and $f \in \Sigma_a^{t,*}(0,\beta,\rho)$. Then $\phi * f \in \Sigma_a^{t,*}(0,\beta,\rho)$.*

Proof: We have

$$-z^2 \left(L_a^t(\alpha,\beta)(\phi * f)(z)\right)' =$$

$$-z^2 \left(L_a^t(\alpha,\beta) f(z)\right)' * z\phi(z), \qquad z \in U^*.$$

Now the remaining part of Theorem 9 follows by employing the techniques that we used in proving Theorem 8 above.

Theorem 10 *Let $f \in \Sigma_a^t(\alpha, \beta, \rho_3)$ and $g \in \Sigma_a^t(\alpha, \beta, \rho_4)$ and let $F = f * g$. Then $F \in \Sigma_n(\alpha,\beta,\rho_5)$, where*

$$\rho_5 = 1 - 4(1-\rho_3)(1-\rho_4)\Upsilon(z) \tag{18}$$

Where

$$\Upsilon(z) = \left[1 - \frac{1}{\alpha}\int_0^1 \frac{u^{\left(\frac{1}{(1-\alpha)}\right)-1}}{1+u} du\right].$$

This result is sharp.

Proof: Since $f \in \Sigma_a^t(\alpha,\beta,\rho_3)$ and $g \in \Sigma_a^t(\alpha,\beta,\rho_4)$, it follows that

$$S(z) = (1-\alpha)\, z\left(L_a^t(\alpha,\beta)\, f(z)\right)$$

$$+\alpha z\left(L_a^t(\alpha,\beta)\, f(z)\right) \in P_n(\rho_3)$$

and

$$S^*(z) = (1-\alpha)\, z\left(L_a^t(\alpha,\beta)\, g(z)\right)$$

$$+\alpha z\left(L_a^t(\alpha,\beta)\, g(z)\right) \in P_n(\rho_4)$$

and so using identity (9) in the above equation, we have

$$L_a^t(\alpha,\beta)\, f(z) = \frac{1}{\alpha} z^{-1-\frac{1}{\alpha}} \int_0^z t^{\frac{1}{\alpha}-1} S(t)\, dt. \quad (19)$$

$$L_a^t(\alpha,\beta)\, g(z) = \frac{1}{\alpha} z^{-1-\frac{1}{\alpha}} \int_0^z t^{\frac{1}{\alpha}-1} S^*(t)\, dt. \quad (20)$$

Using (19) and (20), we have

$$L_a^t(\alpha,\beta)\, f(z) = \frac{1}{\alpha} z^{-1-\frac{1}{\alpha}} \int_0^z t^{\frac{1}{\alpha}-1} Q(t)\, dt, \quad (21)$$

where

$$Q(z) = \left(\frac{n}{4}+\frac{1}{2}\right) q_1(z) + \left(\frac{1}{2}-\frac{n}{4}\right) q_2(z) =$$

$$\frac{1}{\alpha} z^{-1-\frac{1}{\alpha}} \int_0^z t^{\frac{1}{\alpha}-1} (S * S^*)(t)\, dt. \quad (22)$$

Now

$$S(z) = \left(\frac{n}{4}+\frac{1}{2}\right) s_1(z) + \left(\frac{1}{2}-\frac{n}{4}\right) s_2(z)$$

and

$$S^*(z) = \left(\frac{n}{4}+\frac{1}{2}\right) s_1^*(z) + \left(\frac{1}{2}-\frac{n}{4}\right) s_2^*(z) \quad (23)$$

where $s_i \in P(\rho_3)$ and $s_i^* \in P(\rho_4)$, $i = 1, 2$. Since

$$P_i^*(z) = \frac{s_i^*(z) - \rho_4}{2(1-\rho_4)} + \frac{1}{2} \in P\left(\frac{1}{2}\right), \qquad i = 1, 2$$

we obtain that $(s_i * p_i^*)(z) \in P(\rho_3)$, by using the Herglots formula. Thus

$$(s_i * s_i^*)(z) \in P(\rho_5)$$

with

$$\rho_5 = 1 - 2(1-\rho_3)(1-\rho_4) \quad (24)$$

Using (21), (22), (23), (24) and Lemma 5, we have

$$\Re\, q_i(z) = \frac{1}{\alpha} \int_0^1 u^{\frac{1}{\alpha}-1} \Re\left\{(s_i * s_i^*)(uz)\right\} du$$

$$\geq \frac{1}{\alpha} \int_0^1 u^{\frac{1}{\alpha}-1} \left(2\rho_5 - 1 + \frac{2(1-\rho_5)}{1+u|z|}\right) du$$

$$\geq \frac{1}{\alpha} \int_0^1 u^{\frac{1}{\alpha}-1} \left(2\rho_5 - 1 + \frac{2(1-\rho_5)}{1+u}\right) du$$

$$= 1 - 4(1-\rho_3)(1-\rho_4) \left[1 - \frac{1}{\alpha} \int_0^1 \frac{u^{\left(\frac{1}{(1-\alpha)}\right)-1}}{1+u} du\right].$$

From this we conclude that $F \in \Sigma_a^t(\alpha,\beta,\rho_5)$, where ρ_5 is given by (18). We discuss the sharpness as follows: We take

$$S(z) = \left(\tfrac{n}{4}+\tfrac{1}{2}\right) \tfrac{1+(1-\rho_3)z}{1-z} + \left(\tfrac{1}{2}-\tfrac{n}{4}\right) \tfrac{1-(1-\rho_3)z}{1+z}$$

and

$$S^*(z) = \left(\tfrac{n}{4}+\tfrac{1}{2}\right) \tfrac{1+(1-\rho_4)z}{1-z} + \left(\tfrac{1}{2}-\tfrac{n}{4}\right) \tfrac{1-(1-\rho_4)z}{1+z}.$$

Since

$$\left(\frac{1+(1-\rho_3)z}{1-z}\right) * \left(\frac{1+(1-\rho_4)z}{1-z}\right) =$$

$$1 - 4(1-\rho_3)(1-\rho_4) + \frac{4(1-\rho_3)(1-\rho_4)}{1-z},$$

it follows from (22) that

$$q_i(z) =$$

$$\tfrac{1}{\alpha} \int_0^1 u^{\frac{1}{\alpha}-1} \left\{1 - 4(1-\rho_3)(1-\rho_4) + \tfrac{4(1-\rho_3)(1-\rho_4)}{1-z}\right\} du$$

$$\to 1 - 4(1-\rho_3)(1-\rho_4) \left\{1 - \frac{1}{\alpha} \int_0^1 \frac{u^{\left(\frac{1}{(1-\alpha)}\right)-1}}{1+u} du\right\}$$

as $z \to -1$. This completes the proof.

Theorem 11 *Let $f \in \Sigma_a^{t,*}(0, \beta, \rho)$ for $z \in U^*$. Then $f \in \Sigma_a^{t,*}(\alpha, \beta, \rho)$ for $|z| < r_\alpha$, where*

$$r_\alpha = \frac{1}{\alpha + \sqrt{1+\alpha^2}}. \tag{25}$$

Proof: Set

$$-z^2 \left(L_a^t(\alpha, \beta) f(z) \right)' = (1-\rho) h(z) + \rho$$

$h \in P_n$

Now proceeding as in Theorem 6, we have

$$- (1-\alpha) z^2 \left(L_a^t(\alpha,\, \beta) f(z) \right)' - \alpha z^2 \left(L_a^t(\alpha,\, \beta) f(z) \right)' - \rho =$$

$$(1-\rho) \left\{ h(z) + \alpha z h'(z) \right\} =$$

$$(1-\rho) \left[\left(\frac{n}{4} + \frac{1}{2} \right) \left\{ h_1(z) + \alpha z h_1'(z) \right\} + \right.$$

$$\left. - \left(\frac{n}{4} - \frac{1}{2} \right) \left\{ h_2(z) + \alpha z h_2'(z) \right\} \right] \tag{26}$$

where we have used (6) and $h_1, h_2 \in P, z \in U^*$.

Using the following well known estimate [20]

$$\left| z h_i'(z) \right| \le \frac{2r}{1-r^2} \Re \left\{ h_i(z) \right\}.$$

$(|z| = r < 1,\ i = 1, 2)$,
we have

$$\Re \left\{ h_i(z) + \alpha z h_i'(z) \right\} \ge$$

$$\Re \left\{ h_i(z) + \alpha \left| z h_i'(z) \right| \right\} \ge \Re h_i(z) \left\{ 1 - \frac{2\alpha r}{1-r^2} \right\}$$

The right hand side of this inequality is positive if $r < \Re(\alpha)$, where $\Re(\alpha)$ is given by (25). Consequently it follows from (26) that $f \in \Sigma_a^{t,*}(\alpha, \beta, \rho)$ for$|z| < \Re(\alpha)$. Sharpness of this result follows by taking $h_i(z) = \frac{1+z}{1-z}$ in (26), $i = 1, 2$.

4 Conclusion

The first study of univalent functions conducted by P. Koebe was published in 1907. Throughout the last century and until today, geometric functions theories have developed greatly.

With time, the univalent functions realized conformal representations, having applications in areas such as fluid mechanics, electrotechnics, nuclear physics and others.

In a series of possible research areas and speculations, this paper contributes to the development of one types of geometric function theory. This effort opens the door for further research on Hurwitz Zeta function, Hadamard product, univalent functions with positive real part and integral operator.

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