Oleinik Line Method and its Application

Huashui Zhan
School of Applied Mathematics
Xiamen University of Technology
CHINA
2012111007@xmut.edu.cn

Abstract: The paper studies the existence of the solution to the equation $\partial_{\eta\eta}w + w\partial_{\xi}w - \partial_t w = f(\eta, \xi, t, w)$. The equation comes from mathematics finance. With the help of Fichera-Oleinik theory, we find the suitable boundary value conditions to assure the posedness of the equation. By modifying Oleinik's line method, we translate the mathematics finance equation to a system of ordinary differential equations, then by the uniformly estimates of the solution of the system, using Arzela Theorem, we can extract a convergent subsequence, which is convergent to the solution of the mathematics finance equation itself. Some constraint conditions of known functions f, w_0, w_1 and w_2, that appearing in the initial boundary value conditions, are imposed.

Key–Words: Mathematics finance, Oleinik's line method, Fichera-Oleinik theory, boundary condition, t-global solution.

1 An introduction of Oleinik's line method

Oleinik's line method is used in the study of the well-known Prandtl boundary layer system, which was proposed by Prandtl in 1904 (see [8]) and now becomes one of the fundamental parts of fluid dynamics. Many scholars have been carrying out research in this field, achievements are abundant in literature on theoretical, numerical experimental aspects of the theory, see [9-12] etc. Let us give some details. Assuming that the motion of a fluid occupying a two-dimensional region is characterized by the velocity vector $V = (u, v)$, where u, v are the projections of V onto the coordinate axes x, y, respectively, the Prandtl system for a non-stationary boundary layer arising in an axially symmetric incompressible flow past a solid body has the form as

$$\partial_t u + u\partial_x u + v\partial_y u = \partial_t U + U\partial_x U + \nu\partial_y^2 u, \quad (1)$$

$$\partial_x(ru) + \partial_y(rv) = 0, \quad (2)$$

in a domain $D = \{0 < t < T, 0 < x < X, 0 < y < \infty\}$, where ν=const>0 is the coefficient of kinematic viscosity; $U(t, x)$ is called the velocity at the outer edge of the boundary layer, $U(t, 0) = 0$, $U(t, x) > 0$ for $x > 0$; $r(x)$ is the distance from that point to the axis of a rotating body, $r(0) = 0$, $r(x) > 0$ for $x > 0$. If we introduce Crocco variables,

$$\tau = t, \xi = x, \eta = \frac{u(t, x)}{U(t, x)},$$

then Prandtl system is succeeded to be changed to a degenerate parabolic equation for $w(\tau, \xi, \eta) = \frac{\partial_y u}{U}$:

$$\nu w^2 w_{\eta\eta} - w_\tau - \eta U w_\xi + A w_\eta + B w = 0, \quad (3)$$

where A, B are two known functions derived from Prandtl system, one can refer to [7] for details. The main technique of Oleinik's line method lies in three aspects. Firstly, for any functions $f(\tau, \xi, \eta)$, Oleinik used the notation

$$f^{m,k}(\eta) \equiv f(mh, kh, \eta), h = const > 0.$$

Instead of equation (3), she considered the following system of ordinary differential equations

$$\begin{aligned}&\nu(w^{m-1,k} + h)^2 w_{\eta\eta}^{m,k} - \frac{w^{m,k} - w^{m-1,k}}{h}\\ &-\eta U^{m,k}\frac{w^{m,k} - w^{m,k-1}}{h} + A^{m,k}w_\eta^{m,k}\\ &+B^{m,k}w^{m,k} = 0,\end{aligned} \quad (4)$$

and proved that

$$w_\eta^{m,k}, \qquad \frac{w^{m,k} - w^{m,k-1}}{h},$$

$$\frac{w^{m,k} - w^{m-1,k}}{h}, \quad (1 - \eta + h)w_{\eta\eta}^{m,k}$$

are bounded in Ω for $mh \le T_1$ and $h \le h_0$, uniformly with respect to h. Secondly, by linear extension of the solution of the ordinary differential equations, she got the strong solution of the differential equation (3).

Thirdly, by Crocco inverse transformation, the solution of Prandtl system (1)-(2) was obtained.

In our paper, we will study an degenerate parabolic equation from mathematics finance by modifying Oleinik's line method.

2 Fichera-Oleinik theory

Let us remind of Fichera-Oleinik theory. Let $x \in \Omega \subset \mathbb{R}^N$, and Ω is a bounded domain. If we want to consider the boundary value problem of the following linear degenerate elliptic equation,

$$\frac{\partial}{\partial x_i}(a^{ij}(x)\frac{\partial u}{\partial x_j}) + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = 0, \quad (5)$$

Fichera-Oleinik theory tells us that only a part of boundary $\partial\Omega$ should be assigned the boundary value. In details, let $\{n_i\}$ be the unit interior normal vector of $\partial\Omega$ and denote that

$$\Sigma_2 = \left\{ x \in \partial\Omega : \begin{array}{l} a^{ij}n_i n_j = 0, \\ (b_i - a^{ij}_{x_j})n_i < 0 \end{array} \right\}, \quad (6)$$

$$\Sigma_3 = \{x \in \partial\Omega : a^{ij}n_i n_j > 0\}. \quad (7)$$

Then, to ensure the well-posedness of equation (5), Fichera-Oleinik theory tells us that the suitable boundary value condition is

$$u|_{\sum_2 \bigcup \sum_3} = g(x). \quad (8)$$

In particular, if the matrix (a^{ij}) is positive definite, (8) is just the usual Dirichlet boundary value condition.

Now, the reaction-diffusion equation

$$u_t = \triangle A(u), \quad (9)$$

if A^{-1} exists, in other words, equation (9) is weakly degenerate, let $v = A(u), u = A^{-1}(v)$. Then

$$\triangle v - (A^{-1}(v))_t = 0. \quad (10)$$

According to Fichera-Oleinik theory, we know that we can impose the Dirichlet boundary condition. For the boundary layer equation (3), if the domain $\Omega = \{0 < \tau < T, 0 < \xi < X, 0 < \eta < 1\}$, then comparing (3) with (5), according to Fichera-Oleinik theory, the initial-boundary value conditions for w have the form

$$\left\{ \begin{array}{l} w|_{\tau=0} = w_0(\xi,\eta), w|_{\eta=1} = 0, \\ (\nu w w_\eta - v_0 w + c(\tau,\xi))|_{\eta=0} = 0, \end{array} \right. \quad (11)$$

where ν is the viscous coefficient, v_0 and $c(\tau,\xi)$ are known functions, one can refer to [7] for the details.

But, if equation (9) is strongly degenerate, then A^{-1} is not existential, we can not deal with it as (10). We can consider a more general equation

$$\frac{\partial u}{\partial t} = \Delta A(u) + div(b(u)), \text{ in } Q_T = \Omega \times (0,T). \quad (12)$$

Rewrite equation (12) as

$$\frac{\partial u}{\partial t} = a(u)\Delta u + a'(u)|\nabla u|^2 + \mathrm{div}(b(u)), \quad (13)$$

the domain is a cylinder $\Omega \times (0,T)$. If we let $t = x_{N+1}$ and see the strongly degenerate parabolic equation (12) as the form of a "linear" degenerate elliptic equation as follows: when $i,j = 1,2,\cdots,N$, $a^{ii}(x,t) = a(u(x,t))$, $a^{ij}(x,t) = 0, i \neq j$, then

$$(\widetilde{a}^{rs})_{(N+1)\times(N+1)} = \left(\begin{array}{cc} a^{ij} & 0 \\ 0 & 0 \end{array} \right).$$

If $a(0) = 0$, which means that equation (13) is not only strongly degenerate in the interior of Ω, but also degenerate on the boundary $\partial\Omega$. Then $\sum_3$ is an empty set. While

$$\widetilde{b}_s(x,t) = \left\{ \begin{array}{ll} b_i'(u) + a'(u)\frac{\partial u}{\partial x_i}, & 1 \leq s \leq N, \\ -1, & s = N+1. \end{array} \right.$$

Under this observation, according to Fichera-Oleinik theory, the initial value condition

$$u(0,x) = u_0(x), \quad (14)$$

is always needed, but on the lateral boundary $\partial\Omega \times (0,T)$, by $a(0) = 0$, the partly boundary on which we should pose the boundary value is

$$\begin{array}{l} \Sigma_p = \left\{ x \in \partial\Omega : \begin{array}{l} (b_i'(0) + a'(0)\frac{\partial u}{\partial x_i}|_{x\in\partial\Omega} \\ -a'(0)\frac{\partial u}{\partial x_i}|_{x\in\partial\Omega})n_i < 0 \end{array} \right\} \\ = \{x \in \partial\Omega : b_i'(0)n_i < 0\}. \end{array} \quad (15)$$

where $\{n_i\}$ be the unit inner normal vector of $\partial\Omega$.

Though (15) seems reasonable and beautiful, whether the term $\frac{\partial u}{\partial x_i}|_{x\in\partial\Omega}$ has a explicit definition is unclearly, unless that the equation (13) has a classical solution. In fact, due to the strongly degenerate property of a, (13) generally only has weak solution. In our paper, we consider the solution of (13) in BV sense, and we can not define the trace of $\frac{\partial u}{\partial x_i}$ on $\partial\Omega$, which means that we can not define

$$\Sigma_p = \left\{ x \in \partial\Omega : \begin{array}{l} (b_i'(0) + a'(0)\frac{\partial u}{\partial x_i}|_{x\in\partial\Omega} \\ -a'(0)\frac{\partial u}{\partial x_i}|_{x\in\partial\Omega})n_i < 0 \end{array} \right\}.$$

Fortunately, only if $b_i(s)$ is derivable, then

$$\Sigma_p = \{x \in \partial\Omega : b_i'(0)n_i < 0\}. \quad (16)$$

has a definite sense. Recently, in [14], the author have shown that that Σ_p defined in (16) can be imposed the boundary value condition in some way.

In the paper, we consider the initial-boundary value problem of the following equation: for $(\eta, \xi, t) \in \Omega \times (0, T)$,

$$\partial_{\eta\eta} w + w \partial_\xi w - \partial_t w = f(\eta, \xi, t, w), \quad (17)$$

where $\Omega = (0, R) \times (0, N) \subset \mathbb{R}^2$ is a rectangle, T is a positive constant. Since equation (17) lacks the second order partial derivative term $\partial_{\xi\xi} w$, it is a strongly degenerate parabolic equation. Now, we can use Fichera-Oleinik theory to determine the partly boundary condition to assure the posedness of the solutions of equation (17). If equation (17) is considered as a degenerate elliptic equation as (5), noticing that the domain is just a cube, $[0, R] \times [0, N] \times [0, T] = \overline{\Omega} \times [0, T]$, and (a^{ij}) in equation (17) has the special form

$$(a^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

according to Fichera-Oleinik theory, we can quote the following initial-boundary value conditions

$$w \mid_{t=0} = w_0(\eta, \xi, 0), \quad (18)$$

$$w \mid_{\xi=N} = w_2(\eta, N, t), \quad (19)$$

$$\begin{array}{l} w \mid_{\{\eta=0\} \times [0,T)} = w_1(0, \xi, t), \\ w \mid_{\{\eta=R\} \times [0,T)} = 0. \end{array} \quad (20)$$

We assume the functions w_0, w_1, w_2 are smooth adequately on $\overline{\Omega \times (0, T)}$. We will use some ideas of Oleinik's line method [7] to discuss the initial-boundary value problem (17)-(20).

3 The main result

Equation (17) arises in mathematics finance, arises when studying nonlinear physical phenomena such as the combined effects of diffusion and convection of matter (cf.[1]). From references [2-5], we know that there is a unique local classical solution to Cauchy problem of equation (17). As usual, the local classical solution w of equation (17) means that, for $t \leq T_0$, T_0 is small enough, all the partial derivatives of w appearing in equation (17) are continuous functions. The author also had also studied the boundary layer theory and the mathematics finance equation (17) for a long time, see [14]-[19] please.

Comparing equation (17) with equation (3), two equations are similar to each other. It is natural to conjecture that we are able to use Oleinik's line method to study equation (17). The essential difference lies in the term $+w\partial_\xi w$ of equation(17) and the term $-\eta U w_\xi$ of equation (3). This difference gives rise to many difficulties when we use Oleinik's line method. How to overcome these difficulties is the main inspiring technique in our paper. For example, instead of the linearized function

$$w^{m,k}(\eta) = w(\eta, kh, mh), h = const > 0,$$

used in [7], we use the following linearized function

$$w^{m,k}(\eta) = w(\eta, N - kh, mh), h = const > 0, \quad (21)$$

in our paper. For another example, when we use the maximum principle, not only the auxiliary function (45) in our paper is complete different from that used in [7], but also the undetermined constants α, β depends on step length h, whereas in [7], they are independent of step length h.

The main result of our paper is the following theorem.

Theorem 1 *Assume that there exist two constants K_0, K_1 such that,*

$$\begin{array}{l} K_0(R - \eta) \leq w_2(\eta, N, t) \mid_{\xi=N} \leq K_1(R - \eta), \\ K_1 \leq \frac{1}{2R}, \end{array} \quad (22)$$

$$0 \leq w_1(0, \xi, t) \leq \frac{1}{2}, 0 \leq w_0(\eta, \xi, 0) \leq \frac{1}{2}, \quad (23)$$

and the functions w_0, w_1, w_2 are smooth adequately on $\overline{\Omega \times (0, T)}$. Supposed that f is a C^1 function and there are positive constants c_1, c_2, such that when $u - v \geq 0$, f satisfies

$$c_2(u - v) \geq f(\eta, \xi, t, u) - f(\eta, \xi, t, v) \geq c_1(u - v). \quad (24)$$

Then the initial-boundary value problem (17)-(20) admits a solution with the following properties: w, w_η are continuous, $w_\xi, w_t, w_{\eta\eta}$ are bounded; equation (17) holds almost everywhere in $\Omega \times (0, T)$, provided that $\xi \leq N$, N is suitably small. Moreover,

$$c_3(R - \eta) \leq w \leq c_4(R - \eta) \leq \frac{1}{2}, \quad (25)$$

$$|w_t| \leq c(R - \eta), \quad |w_\xi| \leq c(R - \eta), \quad (26)$$

where c and c_i are the constants independent of T, and the constant c may be different from one to the another. If we assume that

$$\left\{ \begin{array}{l} K_0(R - \eta) \leq w_{2\eta}(\eta, N, t) \mid_{\xi=N} \leq K_1(R - \eta), \\ K_0(R - \eta) \leq w_{2\xi}(\eta, N, t) \mid_{\xi=N} \leq K_1(R - \eta), \\ K_0(R - \eta) \leq w_{2t}(\eta, N, t) \mid_{\xi=N} \leq K_1(R - \eta). \end{array} \right. \quad (27)$$

$$\begin{cases} 0 \le w_{1\eta}(0,\xi,t) \le \frac{1}{2}, 0 \le w_{0\eta}(\eta,\xi,0) \le \frac{1}{2}, \\ 0 \le w_{1\xi}(0,\xi,t) \le \frac{1}{2}, 0 \le w_{0\xi}(\eta,\xi,0) \le \frac{1}{2}, \\ 0 \le w_{1t}(0,\xi,t) \le \frac{1}{2}, 0 \le w_{0t}(\eta,\xi,0) \le \frac{1}{2}. \end{cases} \tag{28}$$

and moreover,

$$|w_\xi| \le c_5, \tag{29}$$

where $c_1 - c_5 > 0$ *and* $K_1 \le \frac{1}{2R}$*, then the solution* w *is in the classical sense.*

If we notice that the solution w in Theorem 1 is locally in ξ, but globally in t, we can call this solution as t-global solution.

At the end of the paper, base on Theorem 1, the stability of the solution of (17)-(20) is discussed.

Remark 2 *According the definition of the utility function, one knows that it always has* $0 \le w \le 1$*. Actually, the conditions (21),(22) can be generalized to that*

$$0 \le w_0 < 1, 0 \le w_1 < 1, 0 \le w_2 < 1,$$

and a part of the conclusion (24) can be made stronger as

$$0 \le c_3(R-\eta) \le w \le c_4(R-\eta) < 1.$$

However, the optimal conclusion should be $0 \le w \le 1$ *itself, and we are not able to get this conclusion for the time being.*

Remark 3 *The uniqueness of the global weak solutions to the problem (17)-(20) is able to be proved as that in [13]. Thus the* t*-global classical solution of (17)-(20) is unique.*

Remark 4 *In Theorem 1, it is provided that* $\xi \le N$*,* N *is suitably small. Recently, if* N *is not small enough, the author had shown that in [14], the classical solution blows up in finite time.*

4 The modifying line method

Consider the initial-boundary value problem (17)-(20). Suppose the functions w_0, w_1, w_2 are smooth adequately on $\overline{\Omega \times (0,T)}$, and f is a C^1 function satisfying (24).

For any functions, we use the following notation

$$\begin{array}{l} w^{m,k}(\eta) = w(\eta, N-kh, mh), \\ h = const > 0. \end{array} \tag{30}$$

Instead of the system (17)-(20), let us consider the following system of ordinary differential equations.

$$\begin{array}{l} w_{\eta\eta}^{m,k} - \dfrac{w^{m,k} - w^{m-1,k}}{h} \\ -w^{m,k-1}\dfrac{w^{m,k} - w^{m,k-1}}{h} \\ -f(\eta, N-kh, mh, w^{m,k}) = 0, \end{array} \tag{31}$$

$$\begin{array}{l} w^{m,k}\mid_{\eta=R} = 0, \\ w^{m,k}\mid_{\eta=0} = w_1(0, N-kh, mh), \end{array} \tag{32}$$

where

$$\begin{array}{l} w^{0,k}(\eta) = w_0(\eta, N-kh), \\ w^{m,0}(\eta) = w_2(\eta, mh, N), \end{array} \tag{33}$$

$$m = 1, \cdots, [T/h]; \; k = 1, \cdots, [N/h]. \tag{34}$$

The solution of (31)-(32) are defined in classical sense, its existence is clearly. We will prove that

$$w_{\eta\eta}^{m,k}, w_\eta^{m,k}, \frac{w^{m,k} - w^{m-1,k}}{h}, \frac{w^{m,k} - w^{m,k-1}}{h}$$

are uniformly bounded for any m, k.

Lemma 5 *Under the conditions of (22)-(24), for small enough* h*, there is a suitably small positive number* N_0 *such that the problem (31)-(32) admits a unique solution for* $kh \le N_0$*. The solution satisfies the following estimate*

$$V_0(\eta, N_0 - kh) \le w^{m,k} \le V_1(\eta, N_0 - kh), \tag{35}$$

where V_0, V_1 *are continuous functions, positive in* $(0,R)$*,* $V_1 \le \frac{1}{2}$ *and satisfy*

$$V_0 \equiv K_0(R-\eta), V_1 \equiv K_1(R-\eta), \tag{36}$$

in a neighborhood of $\eta = R$*, where the constant* $K_0 \le K_1 \le \frac{1}{2R}$*.*

Proof: Let $Q^{m,k}$ be the difference of two solutions $w_1^{m,k}$, $w_2^{m,k}$. Then $Q^{m,k}$ can attain neither a positive maximum nor a negative minimum at $\eta = 0$, or R. By $0 \le w^{m,k-1} \le \frac{1}{2}$, and

$$\begin{array}{rcl} 0 & = & Q_{\eta\eta}^{m,k} - \dfrac{1}{h}Q^{m,k} - w^{m,k-1}\dfrac{1}{h}Q^{m,k} \\ & & +f(\eta, N-kh, mh, w_1^{m,k}) \\ & & -f(\eta, N-kh, mh, w_2^{m,k}), \end{array} \tag{37}$$

$Q^{m,k}$ can attain neither a positive maximum nor a negative minimum in interior of $(0,R)$ by (24), provided that $h \le h_0$, h_0 small enough. Consequently, under our assumption, equation (31) cannot have more than one solution. Therefore, we shall prove (35) for m and k, under the assumption of that the solution $w^{m,k-1}$ of equation (31) admits the following priori estimate

$$V_1(\eta, N-(k-1)h) \ge w^{m,k-1}$$

$$\geq V_0(\eta, N-(k-1)h), \quad (38)$$

where $m = 0, 1, \cdots ,$.

For a series of functions $\{u^{m,k}\}$, we introduce the operator

$$\begin{aligned} L_{m,k}(u) &= u^{m,k}_{\eta\eta} - \frac{1}{h}(u^{m,k} - u^{m-1,k}) \\ &\quad -w^{m,k-1}\frac{1}{h}(u^{m,k} - u^{m,k-1}) \\ &\quad -f(\eta, N-kh, mh, u^{m,k}). \end{aligned}$$

In order to prove the priori estimate (35), we shall show that there exists the function V_1 with the properties specified in Lemma 5 and satisfies

$$\begin{aligned} 0 \geq & L_{m,k}(V_1) = V^{m,k}_{1\eta\eta} - \frac{1}{h}(V^{m,k}_1 - V^{m-1,k}_1) \\ & -w^{m,k-1}\frac{1}{h}(V^{m,k}_1 - V^{m,k-1}_1) \\ & -f(\eta, N-kh, mh, V^{m,k}_1), \end{aligned} \quad (39)$$

$$V_1(0, mh) \geq w_1(0, N-kh, mh), \quad (40)$$

$V_1(R, mh) = K_1(R-\eta)$ in a neighborhood of $\eta = R$, where $K_1 \leq \frac{1}{2R}$.

Now, (35) can be proved by induction with respect to k. Indeed, let $q^{m,k} = V^{m,k}_1 - w^{m,k}$. Then $q^{m,k}(0) \geq 0$ and we shall show that $q^{m,k}$ can not attain its negative minimum in the interior of $(0, R)$. Otherwise, at this negative minimum point, we have

$$\begin{aligned} 0 \geq & L_{m,k}(V_1) - L_{m,k}(w) \\ = & q^{m,k}_{\eta\eta} - \frac{1}{h}(q^{m,k} - q^{m-1,k}) \\ & -w^{m,k-1}\frac{1}{h}(q^{m,k} - q^{m,k-1}) \\ & +f(\eta, N-kh, mh, V^{m,k}_1) \\ & -f(\eta, N-kh, mh, w^{m,k}) \\ \geq & q^{m,k}_{\eta\eta} - \frac{1}{h}(q^{m,k} - q^{m-1,k}) \\ & -w^{m,k-1}\frac{1}{h}(q^{m,k} - q^{m,k-1}) + c_2 q^{m,k}, \end{aligned} \quad (41)$$

if we choose h is small enough, then the right hand side of (41) is positive. This is a contradiction. Thus the inequality on the right hand side of (35) is proved.

So, it remains to show that, by (38), there is a positive N_0 such that for $kh \leq N_0$, we can construct the function V_1 to satisfy (35) and (36). Let $\varphi_1(s)$ be a smooth function such that when $\eta > \frac{R}{2}$,

$$\varphi_1(s) = R - \eta,$$

when $\frac{1}{4}R \leq s \leq \frac{1}{2}R$,

$$\frac{R}{2} \leq \varphi_1 \leq R,$$

when $\eta < \frac{1}{4}R$,

$$\varphi_1(s) = R.$$

Set

$$V_1(\eta, N-kh) = M\varphi_1(\eta)\varphi_2(\beta_1\eta)e^{\beta_2 kh}, \quad (42)$$

where φ_2 is a smooth function such that when $0 \leq s \leq R$,

$$\varphi_2(s) = 4 - e^{\frac{1}{R}s},$$

when $s \geq 2R$,

$$\varphi_2(s) = 1,$$

when $R \leq s \leq 2R$

$$1 \leq \varphi_2(s) \leq 4 - e.$$

The constant M is chosen from the condition $V_1 \leq \frac{1}{2}$. The positive constants β_1, β_2 will be specified shortly.

Clearly,

$$\begin{aligned} & L(V_1) \\ & = Me^{\beta_2 kh}(\varphi_1(\eta)\varphi_2(\beta_1\eta))_{\eta\eta} \\ & -\frac{w^{m,k-1}}{h}M\varphi_1(\eta)\varphi_2(\beta_1\eta)(e^{\beta_2 kh} - e^{\beta_2(k-1)h}) \\ & -f(\eta, N-kh, mh, V^{m,k}_1). \end{aligned} \quad (43)$$

For a given small positive number δ, if $R - \eta \leq \delta$, we can choose β_1 such that $\beta_1\eta \geq 2R$, then according to the definitions of φ_1 and φ_2, we have $\varphi_1(\eta)\varphi_2(\beta_1\eta) = R - \eta$. Now, if we choose β_2 large enough, $kh \leq N_0$ small enough, then

$$\begin{aligned} & L(V_1) \\ \leq & -\frac{w^{m,k-1}}{h}M(R-\eta)(e^{\beta_2 kh} - e^{\beta_2(k-1)h}) + cV^{m,k}_1 \\ \leq & -\frac{w^{m,k-1}}{h}M(R-\eta)(e^{\beta_2 kh} - e^{\beta_2(k-1)h}) \\ & +c(M(R-\eta)e^{\beta_2 kh}) \\ \leq & M(R-\eta)[-\beta_2 e^{\beta_2 h'} + c] \leq 0, \end{aligned}$$

where $0 < h' < h$.

If $R - \eta > \delta$, noticing that

$$| (\varphi_1(\eta)\varphi_2(\beta_1\eta))_{\eta\eta} | \leq c,$$

then

$$\begin{aligned} & L(V_1) \\ \leq & -\frac{w^{m,k-1}}{h}M\varphi_1(\eta)\varphi_2(\beta_1\eta)(e^{\beta_2 kh} - e^{\beta_2(k-1)h}) \\ & +cMe^{\beta_2 kh} + cV^{m,k}_1 \\ \leq & -\frac{w^{m,k-1}}{h}M\varphi_1\varphi_2(e^{\beta_2 kh} - e^{\beta_2(k-1)h}) \\ & +cMe^{\beta_2 kh} + c(M\varphi_1\varphi_2 e^{\beta_2 kh}) \\ \leq & M(\varphi_1\varphi_2)[-\beta_2 e^{\beta_2 h'} + c] + cMe^{\beta_2 kh} \leq 0. \end{aligned}$$

At the same time, setting

$$V_0(\eta, N-kh) = \mu\varphi(\alpha_1\eta)\varphi_1(\eta)e^{\alpha_2 kh}, \qquad (44)$$

where μ is small enough such that $V_0(0, mh) \le w_1(0, kh, mh)$, $\varphi(s)$ is a smooth function such that when $0 \le s \le R$,

$$\varphi(s) = e^{\frac{1}{R}s},$$

when $R \le s \le \frac{3}{2}R$,

$$1 \le k \le e,$$

when $s > \frac{3}{2}R$,

$$\varphi(s) = 1.$$

also by choosing α_1, α_2 large enough, $kh \le N_0$ small enough, we have $L(V_0) \ge 0$. Thus we can prove the inequality on the left hand side of (35) similarly. □

Lemma 6 *Assume that the conditions of Lemma 5 are fulfilled, then*

$$w_\eta^{m,k}, \frac{1}{h}(w^{m,k} - w^{m-1,k}), \frac{1}{h}(w^{m,k} - w^{m,k-1}), w_{\eta\eta}^{m,k}$$

are bounded for $kh \le N_0$ and $h \le h_0$, uniformly with respect to h.

Proof: Let $\Phi^{m,k}(\eta)$ be the functions defined as follows: for $k \ge 1, m \ge 1$,

$$\Phi^{m,k}(\eta) = (\frac{w^{m,k} - w^{m-1,k}}{h})^2 + (\frac{w^{m,k} - w^{m,k-1}}{h})^2. \qquad (45)$$

Also, for $m \ge 1$, we define

$$\Phi^{m,0}(\eta) = (\frac{w^{m,0} - w^{m,-1}}{h})^2.$$

To the end, let $w^{m,-1} = w^{m,-1}(\eta, kh)$ be a bounded function such that

$$\begin{aligned} &\frac{w^{m,0} - w^{m,-1}}{h} \\ = \ & w_{\eta\eta}^{m,0} - w^{m,-1}\frac{w^{m,0} - w^{m-1,0}}{h} \\ & -f(\cdot, w^{m,0}). \end{aligned} \qquad (46)$$

Clearly, by that w_2 has the bounded first order derivatives and $w_{2\eta\eta}$ is bounded, $\frac{w^{m,0} - w^{m,-1}}{h}$ is uniformly bounded with respect to h, then

$$| \Phi^{m,0} | \le c. \qquad (47)$$

Denote

$$r^{m,k} = \frac{w^{m,k} - w^{m,k-1}}{h}, \quad \rho^{m,k} = \frac{w^{m,k} - w^{m-1,k}}{h}.$$

Let us write out the differential equations which hold for $\Phi^{m,k}(\eta)$ on the interval $0 \le \eta < R$. We find the equations for $\Phi^{m,k}$ with $k = 0, m \ge 1$ by taking only the first equation. In order to derive the equation for $\Phi^{m,k}(\eta)$ with $k = 1$, we utilize the relation (46) which defines the values of $w^{m,-1}$.

If $m \ge 1, k \ge 1$ from equation (31) for $w^{m,k}$, we subtract equation (31) for $w^{m-1,k}$ and multiply the result by $\frac{2\rho^{m,k}}{h}$ to get the first equation; from equation (31) for $w^{m,k}$ we subtract equation (31) for $w^{m,k-1}$ and multiply the result by $\frac{2r^{m,k}}{h}$ to get the second equation. Taking the sum of the two equations for $\Phi^{m,k}, m = 1, 2, \cdots; k = 1, 2, \cdots, [N/h]$. In details,

$$\begin{aligned} &((31)^{m,k} - (31)^{m-1,k})\frac{2\rho^{m,k}}{h} \\ = \ & 2\rho^{m,k}\rho_{\eta\eta}^{m,k} - \frac{2\rho^{m,k}}{h}(\rho^{m,k} - \rho^{m-1,k}) \\ & -\frac{2\rho^{m,k}}{h}(w^{m,k-1}r^{m,k} - w^{m-1,k-1}r^{m-1,k}) \\ & -\frac{2\rho^{m,k}}{h}[f(\eta, N-kh, mh, w^{m,k}) \\ & -f(\eta, N-kh, (m-1)h, w^{m-1,k})] \\ = \ & 2\rho^{m,k}\rho_{\eta\eta}^{m,k} - \frac{2\rho^{m,k}}{h}(\rho^{m,k} - \rho^{m-1,k}) \\ & -\frac{2\rho^{m,k}}{h}[f(\eta, N-kh, mh, w^{m,k}) \\ & -f(\eta, N-kh, (m-1)h, w^{m-1,k})] \\ & -\frac{2\rho^{m,k}}{h}[w^{m,k-1}(\rho^{m,k} + r^{m-1,k} - \rho^{m,k-1}) \\ & -w^{m-1,k-1}(\rho^{m,k-1} - \rho^{m-1,k-1} + r^{m-2,k})] = 0. \end{aligned}$$

$$\begin{aligned} &((31)^{m,k} - (31)^{m,k-1})\frac{2r^{m,k}}{h} \\ = \ & 2r^{m,k}r_{\eta\eta}^{m,k} - \frac{2r^{m,k}}{h}(\rho^{m,k} - \rho^{m,k-1}) \\ & -\frac{2r^{m,k}}{h}(w^{m,k-1}r^{m,k} - w^{m,k-2}r^{m,k-1}) \\ & -\frac{2r^{m,k}}{h}[f(\eta, N-kh, mh, w^{m,k}) \\ & -f(\eta, N-(k-1)h, mh, w^{m,k-1})] \\ = \ & 2r^{m,k}r_{\eta\eta}^{m,k} - \frac{2r^{m,k}}{h}[f(\eta, N-kh, mh, w^{m,k}) \\ & -f(\eta, N-(k-1)h, mh, w^{m,k-1})] \\ & -\frac{2r^{m,k}}{h}(w^{m,k-1}r^{m,k} - w^{m,k-2}r^{m,k-1}) \\ & -\frac{2r^{m,k}}{h}(r^{m,k} + \rho^{m,k-1} - r^{m-1,k} - r^{m,k-1}) = 0. \end{aligned}$$

$$\begin{aligned}\Phi_\eta^{m,k} &= 2\rho^{m,k}\rho_\eta^{m,k}+2r^{m,k}r_\eta^{m,k};\\ \Phi_{\eta\eta}^{m,k} &= 2\rho^{m,k}\rho_{\eta\eta}^{m,k}+2r^{m,k}r_{\eta\eta}^{m,k}\\ &\quad +2(\rho_\eta^{m,k})^2+2(r_\eta^{m,k})^2;\end{aligned}$$

$$\begin{aligned}&-\frac{1}{h}(\Phi^{m,k}-\Phi^{m-1,k})\\ = &-\frac{1}{h}[(\rho^{m,k})^2-(\rho^{m-1,k})^2\\ &\quad +(r^{m,k})^2-(r^{m-1,k})^2];\end{aligned}$$

$$\begin{aligned}&-\frac{w^{m,k-1}}{h}(\Phi^{m,k}-\Phi^{m,k-1})\\ = &-\frac{w^{m,k-1}}{h}[(\rho^{m,k})^2-(\rho^{m,k-1})^2\\ &\quad +(r^{m,k})^2-(r^{m,k-1})^2];\end{aligned}$$

$$\begin{aligned}&\Phi_{\eta\eta}^{m,k}-\frac{1}{h}(\Phi^{m,k}-\Phi^{m-1,k})\\ &\quad -\frac{w^{m,k-1}}{h}(\Phi^{m,k}-\Phi^{m,k-1})\\ &=2(\rho_\eta^{m,k})^2+2(r_\eta^{m,k})^2\\ &-\frac{1}{h}[(\rho^{m,k})^2+(r^{m,k})^2]\\ &+\frac{1}{h}[(\rho^{m-1,k})^2+(r^{m-1,k})^2]\\ &-\frac{w^{m,k-1}}{h}[(\rho^{m,k})^2-(\rho^{m,k-1})^2\\ &\quad +(r^{m,k})^2-(r^{m,k-1})^2]\\ &+\frac{2\rho^{m,k}}{h}(\rho^{m,k}-\rho^{m-1,k})\\ &+\frac{2\rho^{m,k}}{h}[w^{m,k-1}(\rho^{m,k}+r^{m-1,k}-\rho^{m,k-1})\\ &-w^{m-1,k-1}(\rho^{m,k-1}-\rho^{m-1,k-1}+r^{m-2,k})]\\ &+\frac{2\rho^{m,k}}{h}[f(\eta,N-kh,mh,w^{m,k})\\ &-f(\eta,N-kh,(m-1)h,w^{m-1,k})]\\ &+\frac{2r^{m,k}}{h}(r^{m,k}+\rho^{m,k-1}-r^{m-1,k}-r^{m,k-1})\\ &+\frac{2r^{m,k}}{h}(w^{m,k-1}r^{m,k}-w^{m,k-2}r^{m,k-1})\\ &+\frac{2r^{m,k}}{h}[f(\eta,N-kh,mh,w^{m,k})\\ &-f(\eta,N-(k-1)h,mh,w^{m,k-1})].\end{aligned}$$

In what follows, let us suppose that $\rho^{m,k}\geq 0$ and $r^{m,k}\geq 0$. In the other cases, for example, $\rho^{m,k}\geq 0$ and $r^{m,k}\leq 0$, we can discuss the problem in a similar way, the only difference is the choice of the constant d_1 in the process of the proof.

Now, by (24), we have

$$\begin{aligned}&\Phi_{\eta\eta}^{m,k}-\frac{1}{h}(\Phi^{m,k}-\Phi^{m-1,k})\\ &-\frac{w^{m,k-1}}{h}(\Phi^{m,k}-\Phi^{m,k-1})\\ &\geq 2(\rho_\eta^{m,k})^2+2(r_\eta^{m,k})^2-\frac{1}{h}[(\rho^{m,k})^2+(r^{m,k})^2]\\ &+\frac{1}{h}[(\rho^{m-1,k})^2+(r^{m-1,k})^2]\\ &-\frac{w^{m,k-1}}{h}[(\rho^{m,k})^2-(\rho^{m,k-1})^2\\ &\quad +(r^{m,k})^2-(r^{m,k-1})^2]\\ &+\frac{2\rho^{m,k}}{h}(\rho^{m,k}-\rho^{m-1,k})\\ &+\frac{2\rho^{m,k}}{h}[w^{m,k-1}(\rho^{m,k}+r^{m-1,k}-\rho^{m,k-1})\\ &-w^{m-1,k-1}(\rho^{m,k-1}-\rho^{m-1,k-1}+r^{m-2,k})]\\ &+\frac{2r^{m,k}}{h}(r^{m,k}+\rho^{m,k-1}-r^{m-1,k}-r^{m,k-1})\\ &+\frac{2r^{m,k}}{h}(w^{m,k-1}r^{m,k}-w^{m,k-2}r^{m,k-1})\\ &-2d_1(\rho^{m,k})^2-2c_2\rho^{m,k}-2d_1(r^{m,k})^2-2c_2r^{m,k},\end{aligned}$$

where $d_1=c_1$ if $w^{m,k}\leq w^{m-1,k}$, $d_1=c_2$ if $w^{m,k}\geq w^{m-1,k}$. Then

$$\begin{aligned}&\Phi_{\eta\eta}^{m,k}-\frac{1}{h}(\Phi^{m,k}-\Phi^{m-1,k})\\ &-\frac{w^{m,k-1}}{h}(\Phi^{m,k}-\Phi^{m,k-1})\\ &-\alpha\Phi^{m,k}+\beta\Phi^{m,k-1}\\ &\geq 2(\rho_\eta^{m,k})^2+2(r_\eta^{m,k})^2\\ &+(\frac{1-w^{m,k-1}}{h}-2d_1-\alpha)(\rho^{m,k})^2\\ &+(\frac{1-w^{m,k-1}}{h}-2d_1-\alpha)(r^{m,k})^2\\ &+(\beta+\frac{w^{m,k-1}}{h})[(\rho^{m,k-1})^2+(r^{m,k-1})^2]\\ &+\frac{1}{h}[(\rho^{m-1,k})^2+(r^{m-1,k})^2]\\ &-2c_2(\rho^{m,k}+r^{m,k})-\frac{2\rho^{m,k}}{h}\rho^{m-1,k}\\ &+\frac{2\rho^{m,k}}{h}[w^{m,k-1}(r^{m-1,k}-\rho^{m,k-1})\\ &-w^{m-1,k-1}(\rho^{m,k-1}-\rho^{m-1,k-1}+r^{m-2,k})]\end{aligned}$$

$$+\frac{2r^{m,k}}{h}(\rho^{m,k-1}-r^{m-1,k}-r^{m,k-1})$$
$$-\frac{2r^{m,k}}{h}w^{m,k-2}r^{m,k-1}.$$

If we choose $\alpha=\alpha(h)$ large enough, h small enough, such that $\frac{1-w^{m,k-1}}{h}-2d_1-\alpha>0$, this is possible because of (38) and $0\le w^{m,k-1}\le\frac{1}{2}$. Then by Cauchy inequality, i.e.

$$ab\le\varepsilon a^2+c(\varepsilon)b^2,$$

we have

$$\begin{aligned}&\Phi^{m,k}_{\eta\eta}-\frac{1}{h}(\Phi^{m,k}-\Phi^{m-1,k})\\&-\frac{w^{m,k-1}}{h}(\Phi^{m,k}-\Phi^{m,k-1})-\alpha\Phi^{m,k}+\beta\Phi^{m,k-1}\\&\ge 2(\rho^{m,k}_\eta)^2+2(r^{m,k}_\eta)^2\\&+(\frac{1-w^{m,k-1}-\varepsilon}{h}-2d_1-\alpha-\varepsilon)(\rho^{m,k})^2\\&+(\frac{1-w^{m,k-1}-\varepsilon}{h}-2d_1-\alpha-\varepsilon)(r^{m,k})^2\\&+(\beta+\frac{w^{m,k-1}-\varepsilon}{h}-\varepsilon)[(\rho^{m,k-1})^2+(r^{m,k-1})^2]\\&+\frac{1-\varepsilon}{h}[(\rho^{m-1,k})^2+(r^{m-1,k})^2]-c(\varepsilon)(1+\frac{1}{h}).\end{aligned}$$

Let $\Phi_1=\Phi+1$. Then

$$\begin{aligned}&\Phi^{m,k}_{1\eta\eta}-\frac{1}{h}(\Phi^{m,k}_1-\Phi^{m-1,k})\\&-\frac{w^{m,k-1}}{h}(\Phi^{m,k}_1-\Phi^{m,k-1}_1)-\alpha\Phi^{m,k}_1+\beta\Phi^{m,k-1}_1\\&\ge 2(\rho^{m,k}_\eta)^2+2(r^{m,k}_\eta)^2\\&+(\frac{1-w^{m,k-1}-2d_1-\varepsilon}{h}-\alpha-\varepsilon)(\rho^{m,k})^2\\&+(\frac{1-w^{m,k-1}-\varepsilon}{h}-2d_1-\alpha-\varepsilon)(r^{m,k})^2\\&+(\beta+\frac{w^{m,k-1}-\varepsilon}{h}-\varepsilon)[(\rho^{m,k-1})^2+(r^{m,k-1})^2]\\&+\frac{1-\varepsilon}{h}[(\rho^{m-1,k})^2+(r^{m-1,k})^2]\\&+(\beta-\alpha-c(\varepsilon)(1+\frac{1}{h})).\end{aligned}$$

If we choose β large enough such that

$$\beta-\alpha-c(\varepsilon)(1+\frac{1}{h})>0,$$

then

$$\begin{aligned}&\Phi^{m,k}_{1\eta\eta}-\frac{1}{h}(\Phi^{m,k}_1-\Phi^{m-1,k})\\&-\frac{w^{m,k-1}}{h}(\Phi^{m,k}_1-\Phi^{m,k-1}_1)-\alpha\Phi^{m,k}_1+\beta\Phi^{m,k-1}_1\\&=\Phi^{m,k}_{1\eta\eta}-\frac{1}{h}(\Phi^{m,k}_1-\Phi^{m-1,k})\\&+(\beta+\frac{w^{m,k-1}}{h})(\Phi^{m,k}_1-\Phi^{m,k-1}_1)\\&-(\alpha-\beta)\Phi^{m,k}_1>0.\end{aligned}\tag{48}$$

Clearly, Φ_1 has the same maximum or minimum point as Φ.

(i). If at the maximum point of $\Phi^{m,k}$, suppose $\Phi^{m,k}-\Phi^{m,k-1}\ge 0$. Now, we have two cases. The first case is, at the maximum point of $\Phi^{m,k}_1$, $\Phi^{m,k}_1-\Phi^{m-1,k}_1\ge 0$, then by maximum principle, $\Phi^{m,k}_1$ (also $\Phi^{m,k}$) can not attain its maximum in the interior of $(0,R)$. The second case is, at the maximum point of $\Phi^{m,k}_1$, $\Phi^{m,k}_1-\Phi^{m-1,k}_1\le 0$, let $\tilde{\Phi}_1=e^{-\gamma kh}\Phi_1$. Then by (48)

$$\begin{aligned}&\tilde{\Phi}_{1\eta\eta}^{m,k}-\frac{1}{h}(\tilde{\Phi}_1^{m,k}-\tilde{\Phi}_1^{m-1,k})\\&-\gamma e^{\gamma h_1}(\beta h-w^{m,k-1})\tilde{\Phi}_1^{m-1,k}\\&-(\beta+\frac{w^{m,k-1}}{h})(\tilde{\Phi}_1^{m,k}-\tilde{\Phi}_1^{m,k-1})\\&-(\alpha-\beta)\tilde{\Phi}_1^{m,k}>0,\end{aligned}$$

$$\begin{aligned}&\tilde{\Phi}^{m,k}_{1\eta\eta}-\frac{1}{h}\tilde{\Phi}_1^{m,k}\\&+(\frac{1}{h}-\gamma e^{\gamma h_1}(\beta h+w^{m,k-1}))\tilde{\Phi}_1^{m-1,k}\\&-(\beta+\frac{w^{m,k-1}}{h})(\tilde{\Phi}_1^{m,k}-\tilde{\Phi}_1^{m,k-1})\\&-(\alpha-\beta)\tilde{\Phi}_1^{m,k}>0,\end{aligned}$$

where $h_1<h$. If we choose $\gamma=\gamma(h)>\frac{1}{h}$ large enough, then $\tilde{\Phi}_1^{m,k}$ can not attain its maximum in the interior of $(0,R)$. Thus $\Phi_1(\eta)=e^{\gamma mh}\tilde{\Phi}_1(\eta)$ (also $\Phi(\eta)$) can not attain its maximum in the interior of $(0,R)$.

(ii). If at the maximum point of $\Phi^{m,k}$, $\Phi^{m,k}-\Phi^{m,k-1}\le 0$, also let $\Phi_1=\Phi+1$. We rewrite (48) as

$$\begin{aligned}&\Phi^{m,k}_{1\eta\eta}-\frac{1}{h}\Phi^{m,k}_1-\frac{w^{m-1,k}}{h}(\Phi^{m,k}_1-\Phi^{m,k-1}_1)\\&-\alpha\Phi^{m,k}_1+\beta\Phi^{m,k-1}_1\\&=\Phi^{m,k}_{1\eta\eta}-\frac{1}{h}\Phi^{m,k}_1-\frac{w^{m-1,k}}{h}(\Phi^{m,k}_1-\Phi^{m,k-1}_1)\\&-(\alpha-\beta)(\Phi^{m,k}_1-\Phi^{m,k-1}_1)-\beta\Phi^{m,k}_1>0.\end{aligned}$$

By the maximum principle, $\Phi^{m,k}_1$ can not attain its maximum in the interior of $(0,R)$. Thus $\Phi^{m,k}$ can not attain its maximum in the interior of $(0,R)$ too.

When $\eta = R, \Phi^{m,k} = 0$. When $\eta = 0$, since $w_1(\eta,\xi,\tau)$ is a smooth function on $\overline{\Omega \times (0,T)}$, clearly we have $\Phi^{m,k}(0) \leq c$, so

$$\Phi^{m,k}(\eta) \leq c, \forall \eta \in [0,R]. \tag{49}$$

Now by (31), $| w^{m,k}_{\eta\eta} | \leq c$. This implies that $| w^{m,k}_{\eta} | \leq c$. So Lemma 6 is proved.

Theorem 7 *Under the assumptions of Lemma 5, problem (17)-(20) admits the solution w with the following properties: w is continuous,*

$$0 \leq w \leq K_1(R-\eta) \leq \frac{1}{2}, \tag{50}$$

in the domain $\Omega \times (0,T)$, w has bounded weak derivatives w_η, w_ξ, w_t, and

$$| w_\xi | \leq c(R-\eta), | w_t | \leq c(R-\eta). \tag{51}$$

Moreover, the weak derivative $w_{\eta\eta}$ exist and is bounded, equation (17) holds almost everywhere.

Proof: The solutions $w^{m,k}$ of problem (31)-(32) should be linearly extended to the domain $\Omega \times (0,T)$. Firstly, when $N-(k-1)h \geq \xi > N-kh$, $k = 1,2,\cdots,k(h)$, $k(h) = [N/H]$, let

$$w^m_h(\eta,\xi) = w^m_h(\eta, N-(k-1)h\lambda-(1-\lambda)kh) \\ = (1-\lambda)w^{m,k}(\eta) + \lambda w^{m,k-1}(\eta). \tag{52}$$

Secondly, when $(m-1)h < t < mh$, $m = 1,2,\cdots,m(h)$, $m(h) = [T/h]$, let

$$w_h(\eta,\xi,t) = w_h(\eta,\xi,mh(1-\sigma)+(m-1)h\sigma) \\ = (1-\sigma)w^m_h(\eta,\xi) + \sigma w^{m-1}_h(\eta,\xi). \tag{53}$$

According to Lemma 5, Lemma 6, the functions $w_h(\eta,\xi,t)$ from this family satisfy the Lipschitz condition with respect to ξ, t, and have uniformly (in h) bounded first derivative in η for $0 \leq \xi \leq N, 0 \leq \eta \leq R$. By Arzela Theorem, there is a sequence $h_i \to 0$ such that w_h uniformly converge to some $w(\eta,\xi,t)$. It follows from Lemma 5, Lemma 6 that $w(\eta,\xi,t)$ has bounded weak derivatives $w_t, w_\xi, w_\eta, w_{\eta\eta}$ in $\Omega \times (0,T)$. Moreover,

$$| w_\xi | \leq c(R-\eta), | w_t | \leq c(R-\eta). \tag{54}$$

The sequence w_{h_i} may be assumed such that the derivatives $w_t, w_\xi, w_\eta, w_{\eta\eta}$ in the domain $\Omega \times (0,T)$ coincide with weak limits in $L^2(\Omega \times (0,T))$ of the respective functions

$$\frac{w_{h_i}(\eta,\xi,t+h_i) - w_{h_i}(\eta,\xi,t)}{h_i},$$

$$\frac{w_{h_i}(\eta,\xi+h_i,t) - w_{h_i}(\eta,\xi,t)}{h_i}, \ w_{h_i\eta}, w_{h_i\eta\eta}.$$

Let us show that the equation (17) holds for $w(\eta,\xi,t)$ almost everywhere. Denoting $w^{m,k}_h = w_h(\eta,\xi,t) = w(\eta,kh,mh)$, by (31),

$$w^{m,k}_{h\eta\eta} - \frac{w^{m,k}_h - w^{m-1,k}_h}{h} \\ -w^{m-1,k}_h \frac{w^{m,k}_h - w^{m,k-1}_h}{h} \\ -f(\eta, N-kh, mh, w^{m,k}_h) = 0. \tag{55}$$

Now, suppose that $\varphi(\eta,\xi,t)$ is a smooth function, its support set is compact in $\Omega \times (0,T)$. Let

$$\varphi^{m,k}(\eta) = \varphi(\eta, N-kh, mh).$$

Multiplying with $h\varphi^{m,k}(\eta)$ at the two sides of (55), integrating the resulting equation in η from 0 to R, and taking the sum over k,m from 1 to $k(h), m(h)$ respectively, we obtain

$$\sum_{m,k} h \int_{-R}^{R} \varphi^{m,k}[w^{m,k}_{h\eta\eta} - \frac{w^{m,k}_h - w^{m-1,k}_h}{h} \\ -w^{m-1,k}_h \frac{w^{m,k}_h - w^{m,k-1}_h}{h} \\ -f(\eta, N-kh, mh, w^{m,k}_h)]d\eta = 0. \tag{56}$$

Denoting the function $\bar{f}(\eta,\xi,t,w^{m,k})$ on $\Omega \times [0,T]$ as: for $(m-1)h < \tau < mh, N-(k-1)h \geq \xi > N-kh$,

$$\overline{f(\eta,\xi,t,w^{m,k})} = f(\eta, N-kh, mh, w^{m,k}_h), \tag{57}$$

and denoting

$$(\frac{\Delta w_h}{h})^m_1 = \frac{w^{m,k}_h - w^{m-1,k}}{h},$$

$$(\frac{\Delta w_h}{h})^k_2 = \frac{w^{m,k}_h - w^{m,k-1}}{h},$$

then we can rewrite (56) to

$$\int_0^T \int_\Omega [\bar{w}_{h\eta\eta}\bar{\varphi} - (\overline{\frac{\Delta w_h}{h}})^m_1 \bar{\varphi} - (\overline{\frac{\Delta w_h}{h}})^k_2 \bar{\varphi}\bar{w}_h \\ -\overline{f(\eta,\xi,t,w^{m,k})}\bar{\varphi}]dtd\xi d\eta = 0. \tag{58}$$

Since

$$| \bar{w}_h - w | \leq | \bar{w}_h - w_h | + | w_h - w | \leq ch + | w_h - w |,$$

when $h \to 0$, $\bar{w}_h \rightrightarrows w$, i.e. $\bar{w}_h$ convergent to w uniformly. Just likely, $\bar{\varphi} \rightrightarrows \varphi$, $\overline{f(\eta,\xi,t,w^{m,k})\varphi} \rightrightarrows f(\eta,\xi,t,w)\varphi$. At the same time,

$$(\overline{\frac{\Delta w_h}{h}})_1 \rightharpoonup w_\tau, \ -(\overline{\frac{\Delta w_h}{h}})_2 \rightharpoonup w_\xi, \ \bar{w}_{h\eta\eta} \rightharpoonup w_{\eta\eta},$$

in $L^2((0,R)\times(0,N)\times(0,T))$, so, if let $h\to 0$ in (58), then

$$\int_0^T\int_0^R\int_0^N [w_{\eta\eta}-w_\tau+ww_\xi -f(\eta,\xi,t,w)]\varphi dtd\xi d\eta=0. \quad (59)$$

By the arbitrary of φ, we get ours result. □

The proof of Theorem 1: Let w be the solution of (17)-(20). From Theorem 7, we know that w is a strong solution. To prove Theorem 1, it only remains to prove that the solution is in the classical sense provided that (27)-(29) are true.

Due to the solution w in Theorem 7 is a strongly global solution, as well as it is a locally classical solution, we can make any partial derivatives on (17).

(i) By making derivation on (17) with respect to η, and denoting $u=\partial_\eta w$, then we get

$$\partial_{\eta\eta}u+u\partial_\xi w+w\partial_\xi u-\partial_t u =\frac{\partial f}{\partial\eta}+\frac{\partial}{\partial w}f(\eta,\xi,t,w)u.$$

Let

$$g(\eta,\xi,t,u)=\frac{\partial f(\eta,\xi,t,w)}{\partial\eta}+u(\frac{\partial f(\eta,\xi,t,w)}{\partial w}-\partial_\xi w).$$

If the assumptions (27)-(29) are true, in particular,

$$|w_\xi|\le c_5, c_1-c_5>0.$$

where the constant c_1 appears in the condition (24),

$$c_2(u-v)\ge f(\eta,\xi,t,u)-f(\eta,\xi,t,v)\ge c_1(u-v),$$

it implies that

$$c_2\ge\frac{\partial}{\partial w}f(\eta,\xi,t,w)\ge c_1.$$

Then, if $u-v\ge 0$,

$$\begin{aligned}&(c_2+c_5)(u-v)\ge g(\eta,\xi,t,u)-g(\eta,\xi,t,v)\\&=(\tfrac{\partial}{\partial w}f(\eta,\xi,t,w)-\partial_\xi w)(u-v)\\&\ge(c_1-c_5)(u-v).\end{aligned} \quad (60)$$

Consider the following problem: when $(\eta,\xi,t)\in\Omega\times(0,T)$,

$$\partial_{\eta\eta}u+w\partial_\xi u-\partial_t u=g(\eta,\xi,t,u), \quad (61)$$
$$u|_{t=0}=w_{0\eta}(\eta,\xi,0),\quad u|_{\xi=N}=w_{2\eta}(\eta,N,t), \quad (62)$$
$$u\mid_{\{\eta=0\}\times[0,T)}=w_{1\eta}(0,\xi,t),\quad u\mid_{\{\eta=R\}\times[0,T)}=0. \quad (63)$$

$g(\cdot,u)$ satisfies (60). Similarly, by Oleinik's line method, as we have discussed equation (17) to get Theorem 7, we are able to get the boundedness of the first weak order derivatives of u and get the boundedness of $u_{\eta\eta}$. Then $\partial_{\eta\eta}w=\partial_\eta u, \partial_{\eta t}w=\partial_t u, \partial_{\eta\eta}u=\partial_{\eta\eta\eta}w$ are bounded.

(ii) By making derivation on (17) with respect to ξ, and denoting $p=\partial_\xi w$, then we get

$$\partial_{\eta\eta}p+p\partial_\xi w+w\partial_\xi p-\partial_t p =\frac{\partial f}{\partial\xi}+\frac{\partial}{\partial w}f(\eta,\xi,t,w)p.$$

Let

$$h(\eta,\xi,t,p) =\frac{\partial f(\eta,\xi,t,w)}{\partial\xi}+p(\frac{\partial}{\partial w}f(\eta,\xi,t,w)-\partial_\xi w).$$

If the assumptions (27)-(29) are true, in particular,

$$|w_\xi|\le c_5, c_1-c_5>0.$$

Then, if $u-v\ge 0$,

$$\begin{aligned}&(c_2+c_5)(u-v)\\&\ge h(\eta,\xi,t,u)-h(\eta,\xi,t,v)\\&=(\frac{\partial}{\partial w}f(\eta,\xi,t,w)-\partial_\xi w)(u-v)\\&\ge(c_1-c_5)(u-v).\end{aligned}$$

Consider the following problem

$$\partial_{\eta\eta}p+w\partial_\xi p-\partial_t p=h(\eta,\xi,t,p),$$
$$\begin{aligned}&p|_{t=0}=w_{0\xi}(\eta,\xi,0),\\&p|_{\xi=N}=w_{2\xi}(\eta,N,t),\\&p\mid_{\{\eta=0\}\times[0,T)}=w_{1\xi}(0,\xi,t),\\&p\mid_{\{\eta=R\}\times[0,T)}=0.\end{aligned}$$

Similarly, by Oleinik's line method, as we have discussed equation (17) to get Theorem 7, we are able to get the boundedness of the first weak order derivatives of p and get the boundedness of $p_{\eta\eta}$. Then $\partial_{\eta\xi}w=\partial_\eta p, \partial_{\xi t}w=\partial_t p, \partial_{\eta\xi}p=\partial_{\eta\eta\xi}w$ are bounded.

(iii) By making derivation on (17) with respect to t, and denoting $q=\partial_t w$, then we get

$$\partial_{\eta\eta}q+q\partial_\xi w+w\partial_\xi q-\partial_t q =\frac{\partial f}{\partial t}+\frac{\partial}{\partial w}f(\eta,\xi,t,w)q.$$

Let

$$l(\eta,\xi,t,q) =\frac{\partial f(\eta,\xi,t,w)}{\partial t}+q(\frac{\partial}{\partial w}f(\eta,\xi,t,w)-\partial_\xi w).$$

If the assumptions (27)-(29) are true, in particular,

$$|w_\xi| \le c_5, c_1 - c_5 > 0.$$

Then, if $u - v \ge 0$,

$$\begin{aligned}&(c_2 + c_5)(u - v) \ge l(\eta, \xi, t, u) - l(\eta, \xi, t, v)\\&= (\frac{\partial}{\partial w} f(\eta, \xi, t, w) - \partial_\xi w)(u - v)\\&\ge (c_1 - c_5)(u - v).\end{aligned}$$

Consider the following problem

$$\begin{aligned}&\partial_{\eta\eta} q + w\partial_\xi q - \partial_t q = l(\eta, \xi, t, q),\\&q|_{t=0} = w_{0t}(\eta, \xi, 0),\\&p|_{\xi=N} = w_{2t}(\eta, N, t),\\&q\mid_{\{\eta=0\}\times[0,T)} = w_{1t}(0, \xi, t),\\&q\mid_{\{\eta=R\}\times[0,T)} = 0.\end{aligned}$$

Similarly, by Oleinik's line method, as we have discussed equation (17) to get Theorem 7, we are able to get the boundedness of the first weak order derivatives of q and get the boundedness of $q_{\eta\eta}$. Then $\partial_{\eta t} w = \partial_\eta q, \partial_{tt} w = \partial_t q, \partial_{\eta t} p = \partial_{\eta\eta t} w$ are bounded.

The above discussion (i) - (iii) means that $\partial_\eta w$, $\partial_{\eta\eta} w$, $\partial_\xi w$, $\partial_t w$ are actually continuous functions. So (17)-(20) has the solution in classical sense. □

5 The stability of the solution

If one notices that all the constants of the estimations in Lemma 5-6 and in Theorem 7 are independent of the time T, one knows that Theorem 1 is true for $t \in (0, \infty)$ actually. At the end of the paper, we give a theorem to show the stability of the solution for (17).

Theorem 8 *Let w, $\tilde{w}$ be solutions of (17)-(20) with given w_0, w_1, w_1 and $\tilde{w}_0, \tilde{w}_1, \tilde{w}_2$ respectively, and*

$$w\mid_{\{\eta=R\}\times[0,\infty)} = \tilde{w}\mid_{\{\eta=R\}\times[0,\infty)} = 0. \tag{64}$$

Then, there exist suitably large positive constants K, α_0 such that

$$\mid w - \tilde{w}\mid \le Ke^{-\alpha_0 t}, \tag{65}$$

where K, α_0 may be depend on the constants appearing in Theorem 1.

Proof: Let w, $\tilde{w}$ be solutions of (17)-(20). Then

$$|w| + |w_\eta| + |w_{\eta\eta}| + |w_t| + |w_\xi| \le c,$$

$$|\tilde{w}| + |\tilde{w}_\eta| + |\tilde{w}_{\eta\eta}| + |\tilde{w}_t| + |\tilde{w}_\xi| \le c,$$

where c is independent of the time T.

By the uniqueness of the solution of (17)-(20), it is only need to probe the stability of the solutions of (31)-(32). Let $w^{m,k}, \tilde{w}^{m,k}$ be the corresponding solutions of (31)-(32) to the solution $w, \tilde{w}$ respectively. Let $S^{m,k} = w^{m,k} - \tilde{w}^{m,k}$. Then

$$\begin{aligned}&S^{m,k}_{\eta\eta} - \frac{S^{m,k} - S^{m-1,k}}{h}\\&-\frac{w^{m-1,k}(S^{m,k} - S^{m-1,k})}{h}\\&-\frac{S^{m,k-1}(\tilde{w}^{m,k} - \tilde{w}^{m,k-1})}{h}\\&-f(\cdot, w^{m,k}) + f(\cdot, \tilde{w}^{m,k}) = 0,\end{aligned} \tag{66}$$

$$\begin{aligned}&S^{m,k}\mid_{\eta=R} = 0,\\&w^{m,k}\mid_{\eta=0} = 0,\\&S^{0,k} = w^{0,k} - \tilde{w}^{0,k}.\end{aligned} \tag{67}$$

Set

$$E^{m,k} = \frac{S^{m,k-1}(\tilde{w}^{m,k} - \tilde{w}^{m,k-1})}{h}$$

$$+f(\cdot, w^{m,k}) - f(\cdot, \tilde{w}^{m,k}),$$

and

$$J_{m,k}(\Phi) = \phi^{m,k}_{\eta\eta} - \frac{\phi^{m,k} - \phi^{m-1,k}}{h}$$

$$-w^{m-1,k}\frac{\phi^{m,k} - \phi^{m,k-1}}{h}.$$

Then $J_{m,k}(S) = -E^{m,k}$.

Let $H = Ke^{-\alpha_0 mh}$. Then

$$J_{m,k}(H) = -\alpha_0 Ke^{-\alpha_0 mh} e^{\alpha_0 h'}, \tag{68}$$

where $0 < h' < h$. From (68), if we choose α_0 large enough, h is small enough, then

$$J_{m,k}(H) + \mid E^{m,k}\mid < 0.$$

Since $\mid S^{0,k}\mid \le K, S^{m,k}(R) = 0$,

$$J_{m,k}(H \pm S) < 0, \quad 0 \le \eta \le R.$$

Thus

$$\mid S^{m,k}\mid \le Ke^{-\alpha_0 mh}.$$

Let $h \to 0$, we have

$$\mid w - \tilde{w}\mid \le Ke^{-\alpha_0 t}.$$

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