Completeness and Separability of the Space of a Class of Integrable Fuzzy Valued Functions Based on the tK-Integral Norm Metric

WEIWEI SONG
Tianjin Normal University
School of Mathematics
Tianjin 300387
CHINA
359230610@qq.com

GUIJUN WANG
Tianjin Normal University
School of Mathematics
Tianjin 300387
CHINA
Corresponding author:tjwgj@126.com

QUANJIE HAN
Chinese academy of Sciences
Academy of Mathematics and
Systems Science, Beijing 100190
CHINA
1275816016@qq.com

Abstract: When a class of fuzzy value functions constitute a metric space, the completeness and separability is an important problem that must be considered to discuss the approximation of fuzzy systems. In this paper, Firstly, a new tK-integral norm is defined by introducing two induced operators, and prove that the class of tK-integrable fuzzy value functions is a metric space. And then, the integral transformation theorems and tK-integrable Borel-Cantelli Lemma are applied to study the completeness of the space, furthermore, its separability is discussed by means of the approximation of fuzzy valued simple functions and fuzzy valued Bernstein polynomials. The results show that the space of the tK-integrable fuzzy valued functions constitutes a complete separable metric space in the sense of the tK-integral norm.

Key–Words: K-Additive Measure, tK-Integrable Function, tK-Integral Norm, Completeness, Separability

1 Introduction

It is well known that the separability is a basic concept to study the topological structures and the topological properties of a metric space, and the completeness is an important characterization used to describe a kind of perfect degree of the metric space. In fact, the completeness can ensure that the objects being approximation still belong to the space, and the separability make sure that the object of researches can be limited to someone countable dense subset. Thus, when the approximation of the fuzzy systems and fuzzy neural networks is studied $[1, 2, 4]$, it is an important theoretical problem to explore the completeness and separability of the spaces of a kind of fuzzy valued functions.

In 1987, Sugeno [12] introduced the concepts of the pseudo-addition and pseudo-multiplication for the first time, and set up the theoretical framework for the pseudo-additive measures and integrals. In 1993, Jiang [7] suggested a generalized addition and two kinds of generalized multiplications, thus obtained the tK-integral and the Kt-integral, this moment, the formed as Lebesgues integral transformation theorems were being given. Wang [13] had carried on the limits to the operators $t = K$ to unify the above two kinds of integrals in 1998, furthermore, Wang [13] established the K-pseudo-additive fuzzy integral on fuzzy measure space, and systematically discussed some properties and the convergence theorems of the fuzzy integral $[14, 15, 16]$. In 2000, Liu [8] firstly introduced the concept of the integral norm, and aimed at the forward regular fuzzy neural networks, studied the universal approximations for a class of integrable real functions, in the meantime, the approximations of a generalized Mamdani and the T-S fuzzy systems were being systematically discussed in $[9, 10]$. Since 2004, the universal approximations of the forward regular fuzzy neural networks to a class of fuzzy valued functions had been studied in the sense of the L-integral norm as well as the S-integral norm in [19]. As for multi (or single) layer regular fuzzy neural networks, In 2009-2011, Huang $[5, 6]$ discussed the approximation capability and its algorithms of the fuzzy valued functions.

In 2012, Wang [16] firstly regarded the K-integral norm as a metric, and relied on the polygonal fuzzy numbers to explore the completeness and separability of the space of a class of integrable polygonal fuzzy valued functions. In 2013, the universal approximations of forward regular fuzzy neural networks to a class of integrable bounded fuzzy valued functions were studied in [17] by using the K-integral norm metric. These many useful results have important theoretical value for further design of fuzzy inference networks and fuzzy controllers.

Unfortunately, the completeness and separability being discussed by [16] is only for the space of the polygonal fuzzy valued functions, however, the polygonal fuzzy number is a special fuzzy number. In addition, the completeness and separability of the space of the integrable functions did not be involved in [17]. Hence, in the sense of more generalized K-integral norm, the discussion to the completeness and separability of the space of a class of integrable fuzzy valued functions is very important and necessary. This paper is based on the $[16, 17]$, by strengthening the conditions of the t-operator and the K-operator to give the definition of the tK-integral, we prove that the space of the tK-integrable fuzzy valued functions constitutes a complete separable metric space in the sense of the tK-integral norm.

Present paper is organized as follows: After the introduction, some basic notions of fuzzy numbers and the induced operators are briefly summarized in Section 2. In Section 3, a new integral norm is firstly defined, and it is obtained that the space of the fuzzy valued integrable functions is a metric space. The completeness and separability of this space is proved in detail in section 4. In the final section, we make a conclusion of the meaning of our all work.

2 Preliminaries

Throughout this paper, let $\mathbb{R}$ be the set of all real numbers, $\mathbb{R}^+ = [0, +\infty), \mathbb{N}$ denote a set of natural numbers, d be an Euclidean distance on $\mathbb{R}^n$. Besides, the$(X, \mathfrak{R})$ represents an arbitrary measurable space, where $\mathfrak{R}$ is a σ-algebra on X. For each $A, B \in \mathbb{R}^n$, define

$$d_H(A, B) = \max\{\sup_{x\in A} \inf_{y\in B} d(x, y), \sup_{y\in B} \inf_{x\in A} d(x, y)\}.$$

Then $d_H(A, B)$ is called an Hausdorff distance between A and B. In particular, when A and B are bounded closed intervals, namely, $A = [a, b]$, $B = [c, d]$, their Hausdorff distance

$$d_H([a, b], [c, d]) = |a - c| \vee |b - d|.$$

Definition 1. *([3]) Let a mapping $\widetilde{A} : \mathbb{R} \to [0, 1]$, if the following conditions (1)-(3) are satisfied:*

(1) Ker$(\widetilde{A}) = \{x \in \mathbb{R} | \widetilde{A}(x) = 1\} \neq \varnothing$;

(2) For each $\alpha \in (0, 1]$, the cut-set $\widetilde{A}_\alpha = \{x \in \mathbb{R} | \widetilde{A}(x) \geq \alpha\}$ is a bounded closed interval;

(3) The $\mathrm{Supp}\widetilde{A} = \{x \in \mathbb{R} | \widetilde{A}(x) > 0\}$ *is a bounded closed set.*

Then $\widetilde{A}$ is called a fuzzy number on $\mathbb{R}$.

Let $F_0(\mathbb{R})$ denote the family of fuzzy numbers on $\mathbb{R}$, for each $\widetilde{A}, \widetilde{B} \in F_0(\mathbb{R})$, define

$$D(\widetilde{A}, \widetilde{B}) = \bigvee_{\lambda\in(0,1]} d_H(\widetilde{A}_\lambda, \widetilde{B}_\lambda). \qquad (1)$$

Obviously, D is metric on $F_0(\mathbb{R})$. By [3], we know that $(F_0(\mathbb{R}), D)$ is a complete metric space.

Remark 2. *In order to obtain the better properties of a fuzzy number space, we should restrict the space for a smaller class such that it can constitute a complete separable metric space. Let $F_0^*(\mathbb{R}) \subset F_0(\mathbb{R})$ in this paper, and make $F_0^*(\mathbb{R})$ be a complete separable metric space. For example, we may choose $F_0^*(\mathbb{R})$ to be a space formed by all the triangular or trapezoidal fuzzy numbers.*

In the sequel of this paper, the space $F_0^*(\mathbb{R})$ will play an important role. In fact, this is the reason why we choose the stronger fuzzy numbers than the traditional fuzzy numbers.

Remark 3. *In (1), whenever $\widetilde{B} = \widetilde{0}$, we define*

$$||\widetilde{A}|| = D(\widetilde{A}, \widetilde{0}) = \bigvee_{\alpha\in(0,1]} (|A_\alpha^-| \bigvee |A_\alpha^+|),$$

then $||\widetilde{A}||$ is called a module of the fuzzy number $\widetilde{A}$, where $\widetilde{A}_\alpha = [A_\alpha^-, A_\alpha^+]$ for any $\alpha \in (0, 1]$, $\widetilde{0}$ is a zero fuzzy number, namely,

$$\widetilde{0}(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Lemma 4. *([10]) Let $\widetilde{A}, \widetilde{A}_1, \widetilde{A}_2 \in F_0(\mathbb{R})$, $\{\widetilde{W}_k\}_{k=1}^m$, $(\widetilde{V}_k)_{k=1}^m \subset F_0(\mathbb{R})$, then the following inequalities (1) and (2) hold:*

(1) $D(\widetilde{A} \cdot \widetilde{A}_1, \widetilde{A} \cdot \widetilde{A}_2) \leq ||\widetilde{A}|| D(\widetilde{A}_1, \widetilde{A}_2)$;

(2) $D(\sum_{k=1}^m \widetilde{W}_k, \sum_{k=1}^m \widetilde{V}_k) \leq \sum_{k=1}^m D(\widetilde{W}_k, \widetilde{V}_k)$.

Definition 5. *([7]) Let $K : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function with strictly increasing, and $K(0) = 0$, $\lim_{x\to+\infty} K(x) = +\infty$, then K is called a K-induced operator on $\mathbb{R}^+$.*

Definition 6. *Let $t : \mathbb{R}^+ \to \mathbb{R}^+$ be a differentiable concave function with strictly increasing, and satisfying $t(0) = 0$, $t(1) = 1$, then t is called a t-induced operator on $\mathbb{R}^+$.*

Remark 7. *It is worth noting that the t-induced operator in definition 6 has a big change with respect to that in [7]. In this paper, we assume that t is a differentiable concave function in order to obtain the* *following Theorem 10 (4), which is very important in the proof of the following Theorem 18. Without confusion, we simply call them induced operators. In addition, since K is strictly increasing, it is obvious that K^{-1} is also an induced operator, but the t^{-1} is not necessary the induced operator, this is because the inverse operator t^{-1} need not to be differentiable concave function.*

For example, it is easy to get the induced operators, such as $K(x) = \sqrt{x}$, $K^{-1}(x) = x^2$; $t(x) = \log_2(1 + x)$, for any $x \in \mathbb{R}^+$.

Definition 8. *([7]) Let K and t are the given induced operators, for arbitrary $a, b \in \mathbb{R}^+$, if define*

$$a \perp b \triangleq K^{-1}(K(a) + K(b));$$

$$a \odot b \triangleq K^{-1}(t(a)K(b)).$$

Then operation $\perp$ and $\odot$ are called separately the K-sum and tK-product of a and b .

Remark 9. *By Definition 8, we can see that K induces the sum operation $\perp$, K and t induce another multiplication operation $\odot$. Similarly, if we change the order of K and t , we may obtain another multiplication*

$$a \otimes b = K^{-1}(K(a)t(b)),$$

and it can induce another K-additive integral like the integral defined in Definition 13. However, we just consider the operation $\odot$ in this paper, the another operation $\otimes$ will not be mentioned.

Theorem 10. *([7, 13, 16]) Suppose K and t are the induced operators, for arbitrary $a, b, c \in \mathbb{R}^+$,then the following properties hold:*

(1) $(a \perp b) \perp c = a \perp (b \perp c)$;

(2) $a \perp b = b \perp a$, but $a \odot b \neq b \odot a$;

(3) $a \perp 0 = a, a \odot 0 = 0 \odot a = 0, 1 \odot a = a$;

(4) $t(a + b) \leq t(a) + t(b)$;

(5) If $a \leq b, c \leq d$, then $a \perp c \leq b \perp d, a \odot c \leq b \odot d$;

(6) $a \odot (b \perp c) = (a \odot b) \perp (a \odot c)$;

(7) $K(a \odot b) = t(a) \cdot K(b), K(a \perp b) = K(a) + K(b)$;

(8) $K^{-1}(a \cdot b) = t^{-1}(a) \odot K^{-1}(b), K^{-1}(a + b) = K^{-1}(a) \perp K^{-1}(b)$.

Lemma 11. *([7, 13]) Suppose K and t are the given induced operators, for arbitrary two groups of finite real numbers $\{a_i\}_{i=1}^m, \{b_i\}_{i=1}^m \subset [0, +\infty)$, then*

$$\perp \sum_{i=1}^{m} (a_i \odot b_i) = K^{-1}(\sum_{i=1}^{m} t(a_i)K(b_i)),$$

where $\perp \sum\limits_{i=1}^{m} (a_i \odot b_i) = (a_1 \odot b_1) \perp (a_2 \odot b_2) \perp \cdots \perp (a_m \odot b_m)$.

Because the induced operator t is limited to a few conditions, a new definition of tK-integral and its integral norm different from the ones given in [13,14,15,16] will be presented in the next section, which lay the foundation for the further to study the completeness and separability of the space of fuzzy valued integrable functions .

3 tK-additive integral and tK- integral norm

In this section, we give the concepts of integrable space $L^1(T, \mu)$ and integral norm in $F_0^*(\mathbb{R})$. Firstly, the definitions of K-additive measure and tK-additive integral are given. After that, on the basis of the two induced operators, we define a new tK-integral norm, this integral norm provides us a new tool to handle with the approximation problems of fuzzy neural network and fuzzy system.

Definition 12. *([7, 12]) Let $(X, \mathfrak{R})$ be measurable space, K a given induced operator, if a set function $\mu : \mathfrak{R} \to [0, +\infty)$ satisfies the following conditions (1)-(4) hold*

(1) $\mu(\varnothing) = 0$;

(2) If $A, B \in \mathfrak{R}, and A \cap B = \varnothing$, then $\mu(A \cup B) = \mu(A) \perp \mu(B)$;

(3) If $A_n \subset \mathfrak{R}, A_n \uparrow A$, then $\mu(A_n) \uparrow \mu(A)$;

(4) If $A_n \subset \mathfrak{R}, A_n \downarrow A$, and there exists $n_0 \in \mathbb{N}$ such that $\mu(A_{n_0}) < +\infty$, then $\mu(A_n) \downarrow \mu(A)$.

Then μ is called a K-additive measure, the corresponding triple $(X, \mathfrak{R}, \mu)$ is said to be a K-additive measure space.

Definition 13. *Let $(X, \mathfrak{R}, \mu)$ be a K-additive measure space, K and t are the induced operators, $f : X \to \mathbb{R}^+$ is an nonnegative measurable function, $T \in \mathfrak{R}$. Let $P_T = \{T_1, T_2, \cdots, T_m\}$ denote an arbitrary finite measurable partition of T, i.e., $\bigcup\limits_{i=1}^{m} T_i = T, T_i \bigcap T_j = \varnothing (i \neq j)$,* if define

$$S_{tK}(f, P_T, T) \triangleq \perp \sum_{i=1}^{m} (\inf_{x \in T \cap T_i} f(x) \odot \mu(T \cap T_i)),$$

$$\int_T^{tK} f(x) \mathrm{d}\mu \triangleq \sup_{P_T} S_{tK}(f, P_T, T).$$

Then $\int_T^{tK} f(x) \mathrm{d}\mu$ is called the tK-additive integral of f on T with respect to μ.

For simplicity, we call it tK-additive integral of f. In particular, if $\int_T^{tK} f(x) \mathrm{d}\mu < +\infty$, then we say that f is tK-integrable.

Lemma 14. *Let $(X, \mathfrak{R}, \mu)$ be a K-additive measure space, K and t are the given induced operators . $f : X \to \mathbb{R}^+$ is a nonnegative measurable function,for $T \in \mathfrak{R}$, then*

$$\int_T^{tK} f(x)\mathrm{d}\mu = K^{-1}(\int_T t(f(x))\mathrm{d}\mu^*),$$

where $\mu^*(\cdot) = K(\mu(\cdot))$ is a Lebesgue measure.

Definition 15. *Let $F : T \to F_0^*(\mathbb{R})$ be a fuzzy valued function, where subset $T \subset \mathbb{R}^n$. For any $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in T$, there exists an nonnegative tK-integrable function $\rho(\boldsymbol{x})$ satisfies $||F(\boldsymbol{x})|| \le \rho(\boldsymbol{x})$. Then F is called a fuzzy valued tK-integrable function on T.*

Remark 16. *Let $L^1(T, \mu)$ denote the set of all the tK-integrable fuzzy valued functions on T, we call it a tK-integrable fuzzy valued function space. Notice that, the difference between f and F is that the former is a real function, the later is a fuzzy valued function. Besides, if $F_1, F_2 \in L^1(T, \mu)$, we might as well agree that $F_1 = F_2$ iff $F_1(\boldsymbol{x}) = F_2(\boldsymbol{x})$ μ- a.e. on T.*

Definition 17. *Let $(\mathbb{R}^n, \mathfrak{R}, \mu)$ be a K-additive measure space, $T \in \mathfrak{R}$, K and t are the given induced operators. For any $F_1, F_2 \in L^1(T, \mu)$, let*

$$H(F_1, F_2) \triangleq \int_T^{tK} D(F_1(\boldsymbol{x}), F_2(\boldsymbol{x}))\mathrm{d}\mu.$$

Then H is called a tK-integral norm on $L^1(T, \mu)$, for short, H a tK-integral norm.

Below, we will prove the tK-integral norm H is a metric on $L^1(T, \mu)$, i.e., the pair $(L^1(T, \mu)$,$H)$ constitute a metric space.

Theorem 18. *All the condition with the same in Definition 17, for every $F_1, F_2 \in L^1(T, \mu)$, then integral norm H satisfies the triangular inequality under the operation $\perp$.*

Proof: For any $F_1, F_2 \in L^1(T, \mu)$, by Definition 15, there exist real nonnegative tK-integrable functions $\rho_i(\mathbf{x})(i = 1, 2)$ with

$$||F_1(\mathbf{x})|| \le \rho_1(\mathbf{x}), ||F_2(\mathbf{x})|| \le \rho_2(\mathbf{x}),$$

for all $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in T \subset \mathbb{R}^n$.

In fact, for any $a, b \in \mathbb{R}^+$, and $0 < a < b$, since t is differentiable on $\mathbb{R}^+$. We consider separately $t(x)$ on $[0, a]$ and $[a, a+b]$. By Lagrange Theorem of mean value, it follows that there exist $\xi_1 \in (0, a)$ and $\xi_2 \in (a, a + b)$ such that

$$t(a) = t(a) - t(0) = t'(\xi_1)a,$$

$$t(a + b) - t(b) = t'(\xi_2)a.$$

Because the function $t(x)$ is a differentiable concave function which is equivalent to $t'(x)$ is decreasing, at this time, by $0 < \xi_1 < a < b < \xi_2$, then $t'(\xi_2) \le t'(\xi_1)$. Thus, this show

$$t(a + b) \le t(a) + t(b). \quad (2)$$

On the other hand, for any

$$\mathbf{x} = (x_1, x_2, \cdots, x_n) \in T,$$

it is easy to get that

$$D(F_1(\mathbf{x}), F_2(\mathbf{x})) \le D(F_1(\mathbf{x}), \widetilde{0}) + D(\widetilde{0}, F_2(\mathbf{x}))$$
$$= ||F_1(\mathbf{x})|| + ||F_2(\mathbf{x})|| \le \rho_1(\mathbf{x}) + \rho_2(\mathbf{x}).$$

This implies $D(F_1(\mathbf{x}), F_2(\mathbf{x}))$ still is a real tK- integrable function.

Applying Lemma 14 and the above (2), for arbitrary $F_1, F_2, F_3 \in L^1(T, \mu)$, we have

$$K(H(F_1, F_3)) = \int_T t(D(F_1(\mathbf{x}), F_3(\mathbf{x}))\mathrm{d}\mu^*$$
$$\le \int_T (t(D(F_1(\mathbf{x}), F_2(\mathbf{x}))) + t(D(F_2(\mathbf{x}), F_3(\mathbf{x}))))\mathrm{d}\mu^*$$
$$= \int_T t(D(F_1, F_2))\mathrm{d}\mu^* + \int_T t(D(F_2, F_3))\mathrm{d}\mu^*.$$

By Theorem 10 (8), it is easy to obtain that

$$H(F_1, F_3)) \le K^{-1}(\int_T t(D(F_1(\mathbf{x}), F_2(\mathbf{x})))\mathrm{d}\mu^*)$$
$$\perp K^{-1}(\int_T t(D(F_2(\mathbf{x}), F_3(\mathbf{x})))\mathrm{d}\mu^*)$$
$$= H(F_1, F_2) \perp H(F_2, F_3).$$

Theorem 19. *The $(L^1(T, \mu), H)$ is a metric space under the addition operation $\perp$.*

Proof: According to Theorem 18, the tK-integral norm $H : L^1(T, \mu) \times L^1(T, \mu) \to \mathbb{R}^+$ satisfies triangular inequality. Next, we will show H satisfies the positive definiteness and symmetry.

In fact, for any $F_1, F_2 \in L^1(T, \mu)$, for arbitrary $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in T$, by Hausdorff distance (1) and symmetry of D, obviously,

$$H(F_1, F_2) = H(F_2, F_1),$$

and $H(F_1, F_2) \ge 0$. Besides, if $H(F_1, F_2) = 0$, It is easy to get that

$$K^{-1}(\textstyle\int_T t(D(F_1(\mathbf{x}), F_2(\mathbf{x})))d\mu^*) = 0.$$

Hence

$$\int_T t(D(F_1(\mathbf{x}), F_2(\mathbf{x})))d\mu^* = K(0) = 0.$$

By Lebesgue integral properties, it is clear to see that $t(D(F_1(\mathbf{x}), F_2(\mathbf{x}))) = 0$ μ^* a. e. to T. Thus,

$$D(F_1(\mathbf{x}), F_2(\mathbf{x})) = t^{-1}(0) = 0 \ \ \mu^* \text{ a. e. to } T.$$

According to the agreement in Remark 16, this implies $F_1 = F_2$.

On the other hand, if $F_1 = F_2$, it is obvious that $H(F_1, F_2) = 0$, the positive definiteness is proved. Synthesizing the above the conclusions, therefore, $(L^1(T, \mu), H)$ is a metric space.

4 Completeness and separability

In above section, we have proved that $(L^1(T, \mu), H)$ constitute a metric space in the sense of the tK-integral norm. In this section, firstly, we will use the Borel-Cantelli Lemma to prove the completeness of the space $(L^1(T, \mu), H)$. Second, through constructing fuzzy valued simple functions and Bernstein polynomials with fuzzy value coefficients, we may obtain the separability of $(L^1(T, \mu), H)$.

Lemma 20. *If one-variable function $\varphi(x)$ is differentiable at $x = 0$ and $\varphi'(0) > 0$, $\varphi(0) = 0$. Then there exists $N \in \mathbb{N}$ such that $\varphi(\frac{1}{4^n})/\varphi(\frac{1}{2^n}) \leq \frac{3}{2^n}$, whenever $n > N$.*

Proof: Since $\varphi(x)$ is differentiable at $x = 0$, then

$$\varphi'(0) = \lim_{x\to 0} \frac{\varphi(x) - \varphi(0)}{x} = \lim_{x\to 0} \frac{\varphi(x)}{x} > 0.$$

According to Heine's Theorem, we can separately choose two sequences of numbers $x_n = 1/2^n$ and $y_n = 1/4^n$ which satisfy

$$\lim_{n\to\infty} \frac{\varphi(1/2^n)}{1/2^n} = \lim_{n\to\infty} \frac{\varphi(x_n)}{x_n} = \varphi'(0) > \frac{1}{2}\varphi'(0),$$

$$\lim_{n\to\infty} \frac{\varphi(1/4^n)}{1/4^n} = \lim_{n\to\infty} \frac{\varphi(y_n)}{y_n} = \varphi'(0) < \frac{3}{2}\varphi'(0).$$

By definition of the limit of sequence of numbers, there exist $N_1, N_2 \in \mathbb{N}$, respectively, such that $\frac{\varphi(1/2^n)}{1/2^n} > \frac{1}{2}\varphi'(0)$, whenever $n > N_1$;

$\frac{\varphi(1/4^n)}{1/4^n} < \frac{3}{2}\varphi'(0)$, whenever $n > N_2$.

Let $N = \max\{N_1, N_2\}$, whenever $n > N$, it is immediately to obtain that

$$\varphi(\frac{1}{4^n})/\varphi(\frac{1}{2^n}) < (\frac{3}{2}\frac{1}{4^n}\varphi'(0)) \cdot (\frac{2^{n+1}}{\varphi'(0)}) = \frac{3}{2^n}.$$

Lemma 21. *(Borel-Cantelli Lemma) Let $(\mathbb{R}^n, \Re, \nu)$ be a finite Lebesgue measure space, if there exists a sequence of measurable sets $\{A_n\} \subset \Re$ with $\sum\limits_{n=1}^{\infty} \nu(A_n) < +\infty$, then $\nu(\bigcap\limits_{i=1}^{\infty} \bigcup\limits_{n=i}^{\infty} A_n) = 0$.*

Theorem 22. *Let $(\mathbb{R}^d, \Re, \mu)$ be a finite K-additive measure space, $T \in \Re$, K and t are the given induced operators, $t'(0) > 0$ and $K(x) = O(t(x))$ whenever $x \to 0^+$, then the fuzzy valued tK-integrable function space $L^1(T, \mu)$ is complete.*

Proof: Suppose $\{F_n\}$ is an arbitrary Cauchy sequence in $L^1(T, \mu)$. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $H(F_m, F_n) < \varepsilon$ whenever $m, n > N$.

For each $k \in \mathbb{N}$, let $\varepsilon_k = 1/4^k > 0$, there exists $N_k \in \mathbb{N}$, in particular, $m = n + 1$, we have

$$H(F_{n+1}, F_n) < \varepsilon_k = 1/4^k. \tag{3}$$

Let $E_k = \{\mathbf{x} \in T | D(F_{n+1}(\mathbf{x}), F_n(\mathbf{x})) > \frac{1}{2^k}\} \subseteq T$, where $n > N_k, k = 1, 2, \cdots, \mathbf{x} = (x_1, x_2, \cdots, x_d)$.

On the one hand, whenever $n > N_k$, by the Definition 17, Lemma 14 and the strictly increasing of K and t, it is clear to see that

$$H(F_{n+1}, F_n) = K^{-1}(\int_T t(D(F_{n+1}(\mathbf{x}), F_n(\mathbf{x})))\mathrm{d}\mu^*)$$

$$\geq K^{-1}(\textstyle\int_{E_k} t(D(F_{n+1}(\mathbf{x}), F_n(\mathbf{x})))\mathrm{d}\mu^*)$$

$$\geq K^{-1}(\textstyle\int_{E_k} t(\frac{1}{2^k})\mathrm{d}\mu^*) = K^{-1}(t(\frac{1}{2^k})\mu^*(E_k)).$$

By the increasing of K and (3), if make

$$K^{-1}(t(\frac{1}{2^k})\mu^*(E_k)) < \frac{1}{4^k}.$$

Consequently,

$$\mu^*(E_k) \leq K(\frac{1}{4^k})/t(\frac{1}{2^k}), \tag{4}$$

where μ^* is a Lebesgue measure, and satisfying

$$\mu^*(E_n) = K(\mu(E_n)) \leq K(\mu(T)) < +\infty.$$

On the other hand, since $t(0) = 0$ and $t'(0) > 0$, according to Lemma 20, there exists a sufficiently large $k_0 \in \mathbb{N}$, for each $k > k_0$, we have

$$t(\frac{1}{4^k})/t(\frac{1}{2^k}) \leq \frac{3}{2^k}. \tag{5}$$

Besides, due to the $K(x) = O(t(x))$ whenever $x \to 0^+$, there exists $b \in \mathbb{R}$ such that

$$\lim_{x\to 0^+} K(x)/t(x) = b \neq 0.$$

Utilizing Heines theorem, choose $x_k = 1/4^k \to 0$ $(k \to \infty)$, then

$$\lim_{k\to\infty} K(1/4^k)/t(1/4^k) = b.$$

Let $\varepsilon_0 = 1 > 0$, there exists $N' \in \mathbb{N}$ such that whenever $k > N'$, we have

$$|\frac{K(1/4^k)}{t(1/4^k)} - b| < 1,$$

or

$$0 < K(\frac{1}{4^k})/t(\frac{1}{4^k}) < 1 + |b| \tag{6}$$

Let $N = \max\{k_0, N'\}$, whenever $k > N$, the (5) and (6) are both satisfied. By (4), therefore,

$$\mu^*(E_k) \leq K(\frac{1}{4^k})/t(\frac{1}{2^k})$$

$$= (K(\frac{1}{4^k})/t(\frac{1}{4^k})).(t(\frac{1}{4^k})/t(\frac{1}{2^k}))$$

$$\leq 3(1 + |b|)/2^k.$$

Since the positive series $\sum\limits_{k=1}^{\infty} 3(1 + |b|)/2^k$ is convergent, based on the comparison test of a series convergence criterion, the positive series $\sum\limits_{k=1}^{\infty} \mu^*(E_k)$ is also convergent, i. e. ,

$$\sum_{k=1}^{\infty} \mu^*(E_k) < +\infty.$$

According to the Borel-Cantelli Lemma (Lemma 21), we can immediately acquire that

$$\mu^*(\bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} E_k) = 0 \text{ or } \mu^*(\overline{\lim_{k\to\infty}} E_k) = 0.$$

Let $E_0 = \bigcap\limits_{i=1}^{\infty} \bigcup\limits_{k=i}^{\infty} E_k$, then $\mu^*(E_0) = 0$, where $E_0 \subset T \subset \mathbb{R}^d$. Hence, for any $\mathbf{x} = (x_1, \cdots, x_d) \in E_0^c = \bigcup\limits_{i=1}^{\infty} \bigcap\limits_{k=i}^{\infty} E_k^c$, there exists $i(\mathbf{x}) \in \mathbb{N}$, whenever $k \geq i(\mathbf{x})$, we have $\mathbf{x} \in E_k^c$, i.e.,

$$D(F_{n+1}(x), F_n(x)) \leq \frac{1}{2^k},\ \ n > N_k.$$

Hence, for arbitrary $\varepsilon > 0$, for any $l \in \mathbb{N}$, when $\mathbf{x} \in E_0^c = T - E_0$ and $n > N_k$, in order to make

$$D(F_{n+l}(\mathbf{x}), F_n(\mathbf{x})) \leq \sum_{j=n}^{n+l-1} D(F_{j+1}(\mathbf{x}), F_j(\mathbf{x}))$$

$$\leq \sum_{j=k}^{k+l-1} \frac{1}{2^j} \leq \sum_{j=k}^{\infty} \frac{1}{2^j}$$

$$= \frac{1}{2^k}(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots) = \frac{1}{2^{k-1}} < \varepsilon. \tag{7}$$

We only need to take $k > k'$, and $k' \geq [1 + \log_2 \frac{1}{\varepsilon}]$.

Thus, the sequence $\{F_n(\mathbf{x})\}$ of the fuzzy valued functions is a Cauchy sequence of $F_0^*(\mathbb{R})$, and because $(F_0^*(\mathbb{R}), D)$ is a complete metric space. For any $\mathbf{x} = (x_1, x_2, \cdots, x_d) \in T - E_0$, then $\{F_n(\mathbf{x})\}$ converge to a fuzzy number in $F_0^*(\mathbb{R})$.

Without loss of generality, for every $\mathbf{x} \in T - E_0$, let $F(\mathbf{x}) \in F_0^*(\mathbb{R})$, and it satisfies

$$\lim_{n\to\infty} D(F_n(\mathbf{x}), F(\mathbf{x})) = 0 \tag{8}$$

By definition the limit of sequence of numbers, for arbitrary $\varepsilon > 0$, there exists $N_3 \in \mathbb{N}$ such that

$$D(F_n(\mathbf{x}), F(\mathbf{x})) < \varepsilon,\ \ \mathbf{x} \in E_0^c,\ n > N_3.$$

Specially, whenever $\mathbf{x} \in E_0$, let $F(\mathbf{x}) = \widetilde{0} \in F_0^*(\mathbb{R})$. By Definition 17 and Lemma 14, whenever $n > N_3$, we can get that

$$H(F, F_n) = K^{-1}(\int_T t(D(F(\mathbf{x}), F_n(\mathbf{x})))\mathrm{d}\mu^*)$$

$$= K^{-1}(\int_{T-E_0} t(D(F(\mathbf{x}), F_n(\mathbf{x})))\mathrm{d}\mu^*$$

$$+ \int_{E_0} t(D(F(\mathbf{x}), F_n(\mathbf{x})))\mathrm{d}\mu^*)$$

$$< K^{-1}(t(\varepsilon)\mu^*(T - E_0) + 0)$$

$$\leq K^{-1}(t(\varepsilon)\mu^*(T)).$$

As $\mu^*(T)$ is finite. For any $\varepsilon > 0$, the expression $K^{-1}(t(\varepsilon)\mu^*(T))$ may be still arbitrarily small. Therefore, in the sense of tK-integral norm, there exist a fuzzy valued function $F : T \to F_0^*(\mathbb{R})$ such that $\lim\limits_{n\to\infty} H(F, F_n) = 0$.

Consequently, every Cauchy sequence $\{F_n\}$ in $L^1(T, \mu)$ is convergent to in $L^1(T, \mu)$.

Next, we further will show the $F \in L^1(T, \mu)$. According to (8), given $\varepsilon = 1$, fixed $N_0 > N_3$ and $N_0 \in \mathbb{N}$, we have $D(F_{N_0}(\mathbf{x}), F(\mathbf{x})) < 1$ for every $\mathbf{x} = (x_1, x_2, \cdots, x_d) \in T - E_0$.

By Definition 15, there exists a real nonnegative tK-integrable function $\rho_0(\mathbf{x})$ such that $||F_{N_0}(\mathbf{x})|| \leq \rho_0(\mathbf{x})$. According to Remark 2 the following conclusions hold

$$||F(\mathbf{x})|| = D(F(\mathbf{x}), \widetilde{0})$$

$$\leq D(F(\mathbf{x}), F_{N_0}(\mathbf{x})) + D(F_{N_0}(\mathbf{x}), \widetilde{0})$$

$$< 1 + ||F_{N_0}(\mathbf{x})|| \leq 1 + \rho_0(\mathbf{x}).$$

Let $\rho(\mathbf{x}) = 1 + \rho_0(\mathbf{x})$, it is easy to check that $\rho(\mathbf{x})$ is a real nonnegative tK-integrable function. Assume $F(\mathbf{x}) = \widetilde{0}$, for every $\mathbf{x} \in E_0$, then

$$||F(\mathbf{x})|| = D(\widetilde{0}, \widetilde{0}) = 0 \leq \rho(\mathbf{x}).$$

Hence, $||F(\mathbf{x})|| \le \rho(\mathbf{x})$, for all $\mathbf{x} = (x_1, x_2, \cdots, x_d) \in T \subset \mathbb{R}^d$. By Definition 15, we can obtain that the $F \in L^1(T, \mu)$.

Up to now, we have proved every cauchy sequence $\{F_n\}$ in $L^1(T, \mu)$ is convergent, and converges to a fuzzy number in $L^1(T, \mu)$. Synthesizing the above conclusions, it is straightforward to see that the fuzzy valued tK-integrable function space $L^1(T, \mu)$ is complete. □

Definition 23. *Let $S : T \to F_0(\mathbb{R})$, $T \subset \mathbb{R}^d$, if there exist a group of fuzzy numbers $\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_m \in F_0(\mathbb{R})$ and a measurable partition $\{T_1, T_2, \cdots, T_m\}$ of T, such that $S(\boldsymbol{x}) = \sum\limits_{k=1}^{m} \chi_{T_k}(\boldsymbol{x}) \cdot \widetilde{A}_k$, for each $\boldsymbol{x} = (x_1, x_2, \cdots, x_d) \in T$, where $\chi_{T_k}(\boldsymbol{x})$ is a characteristic function. Then we call that $S(\boldsymbol{x})$ is a fuzzy valued simple function on T.*

Let $S(T)$ denote the family of all fuzzy valued simple functions on T.

Theorem 24. *Let $(\mathbb{R}^d, \mathfrak{R}, \mu)$ be a finite K-additive measure space, K and t are the given induced operators, $T \subset \mathbb{R}^d$ and $T \in \mathfrak{R}$, then $S(T)$ is dense in $L^1(T, \mu)$, i.e., for each $\varepsilon > 0$, for every $F \in L^1(T, \mu)$, there exists $S_0 \in S(T)$ such that $H(F, S_0) < \varepsilon$.*

Proof: Since $(F_0^*(\mathbb{R}), D)$ is a complete separable metric space, let $\{\widetilde{A}_i\}$ be a countable dense set in $F_0^*(\mathbb{R})$. For each $\varepsilon > 0$, putting

$$T_1 = \{\mathbf{x} \in T | D(F(x), \widetilde{A}_1) < \varepsilon\};$$

$$T_2 = \{x \in T | D(F(\mathbf{x}), \widetilde{A}_1) \ge \varepsilon, D(F(\mathbf{x}), \widetilde{A}_2) < \varepsilon\};$$

$$\cdots\cdots\cdots\cdots$$

$$T_n = \{\mathbf{x} \in T | D(F(\mathbf{x}), \widetilde{A}_i) \ge \varepsilon (i = 1, 2, \ldots, n-1), \quad D(F(\mathbf{x}), \widetilde{A}_n) < \varepsilon\};$$

$$\cdots\cdots\cdots\cdots$$

Obviously, the sequence $\{T_i\}$ of sets satisfies $T_i \cap T_j = \varnothing (i \ne j)$ and $\cup_{i=1}^{\infty} T_i = T$.

In fact, by definition of $\{T_i\}$, then $\cup_{i=1}^{\infty} T_i \subset T$. Vise versa, if $T \not\subset \cup_{i=1}^{\infty} T_i$, then there exist $\mathbf{x}_0 = (x_1', x_2', \cdots, x_d') \in T$, but $\mathbf{x}_0 \notin T_i, i = 1, 2, 3, \cdots$, where $F(\mathbf{x}_0) \in F_0^*(\mathbb{R})$.

Hence $D(F(\mathbf{x}_0), \widetilde{A}_i) \ge \varepsilon$, this contradicts the density of $\{\tilde{A}_i\}$ in $F_0^*(\mathbb{R})$, so $T \subset \cup_{i=1}^{\infty} T_i$.

Combining these two results, we can get that $\cup_{i=1}^{\infty} T_i = T$. Because the partition $\{T_i\}$ is pairwise disjoint, then

$$\sum_{i=1}^{\infty} \mu^*(T_i) = \mu^*(\bigcup_{i=1}^{\infty} T_i) = \mu^*(T) < +\infty.$$

In the light of the definition of convergent series, for arbitrary $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\mu^*(\bigcup_{i=N+1}^{\infty} T_i) = \sum_{i=N+1}^{\infty} \mu^*(T_i) < \varepsilon,$$

whenever $n \ge N$ (take $n = N$)

Let $T_0 = \cup_{i=N+1}^{\infty} T_i$, then $\mu^*(T_0) < \varepsilon$. Write $T = T_0 \cup (\bigcup_{i=1}^{N} T_i)$, then $\{T_0, T_1, T_2, \cdots, T_N\}$ is a finite measurable partition of T. If we define $\widetilde{A}_0 = \widetilde{0}$, let $S_0(\mathbf{x}) = \sum\limits_{i=0}^{N} \chi_{T_i}(\mathbf{x}) \cdot \widetilde{A}_i$, for arbitrarily $\mathbf{x} = (x_1, x_2, \cdots, x_d) \in T$. Then S_0 is a fuzzy valued simple function on T. By the absolute continuity of Lebesgue integral, for arbitrarily $\varepsilon > 0$, take $\delta = \varepsilon > 0$, whenever $\mu^*(T_0) < \delta$, we have

$$\int_{T_0} t(D(F(\mathbf{x}), S_0(\mathbf{x}))) \mathrm{d}\mu^* < \varepsilon.$$

By the definition of T_i, for every $\mathbf{x} = (x_1, x_2, \cdots, x_d) \in T_i$, $i = 1, 2, \cdots, N$, $D(F(\mathbf{x}), S_0(\mathbf{x})) < \varepsilon$.

In accordance with the Lemma 14 and definition of the tK-integral norm, then

$$\begin{aligned}
&\int_T^{tK} D(F(\mathbf{x}), S_0(\mathbf{x})) \mathrm{d}\mu \\
&= K^{-1}\Big(\int_{\cup_{i=0}^{N} T_i} t(D(F(\mathbf{x}), S_0(\mathbf{x}))) \mathrm{d}\mu^*\Big) \\
&= K^{-1}\Big(\sum_{i=1}^{N} \int_{T_i} t(D(F(\mathbf{x}), S_0(\mathbf{x}))) \mathrm{d}\mu^* + \\
&\qquad \int_{T_0} t(D(F(\mathbf{x}), S_0(\mathbf{x}))) \mathrm{d}\mu^*\Big) \\
&< K^{-1}\Big(\sum_{i=1}^{N} \int_{T_i} t(\varepsilon) \mathrm{d}\mu^* + \varepsilon\Big) \\
&= K^{-1}\Big(\sum_{i=1}^{N} t(\varepsilon)\mu^*(T_i) + \varepsilon\Big) \\
&= K^{-1}(t(\varepsilon)\mu^*(T) + \varepsilon).
\end{aligned}$$

Because of the arbitrariness of ε and finiteness of $\mu^*(T)$, then $K^{-1}(t(\varepsilon)\mu^*(T) + \varepsilon)$ can be arbitrarily small. Thus, $S(T)$ is dense in $L^1(T, \mu)$. □

Definition 25. *([10]) Let $f : [0, 1]^d \to F_0(\mathbb{R})$ be a d-variable fuzzy valued function, given $m \in \mathbb{N}$, for every $\boldsymbol{x} = (x_1, x_2, \cdots, x_d) \in [0, 1]^d$, define a fuzzy valued Bernstein polynomial $B_m(f)$ as follows*

$$B_m(f; \boldsymbol{x}) \triangleq$$

$$\sum_{i_1,\cdots,i_d=0}^{m} G_{m;i_1,\cdots,i_d}(x_1,\cdots,x_d)\cdot f(\frac{i_1}{m},\cdots,\frac{i_d}{m}),$$

where

$$G_{m;i_1,\cdots,i_d}(x_1,x_2,\cdots,x_d) \triangleq C_m^{i_1}C_m^{i_2}\cdots C_m^{i_d}x_1^{i_1}x_2^{i_2}$$
$$\cdots x_d^{i_d}(1-x_1)^{m-i_1}(1-x_2)^{m-i_2}\cdots(1-x_d)^{m-i_d},$$

and satisfies

$$\sum_{i_1,i_2,\cdots,i_d=0}^{m} G_{m;i_1,i_2,\cdots,i_d}(\boldsymbol{x}) = 1.$$

Lemma 26. *([10]) Suppose $f : [a_1,b_1]\times\cdots\times[a_d,b_d] \to F_0(\mathbb{R})$ is a continuous fuzzy valued function, then for arbitrary $\varepsilon > 0$, there exists $m \in \mathbb{N}$ and a fuzzy valued Bernstein polynomial $B_m(f)$ such that $D(f(\boldsymbol{x}), B_m(f;\boldsymbol{x})) < \varepsilon$, for any $\boldsymbol{x} \in [a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_d,b_d]$.*

Theorem 27. *Let $(\mathbb{R}^d,\mathfrak{R},\mu)$ be a finite K-additive measure space, K and t are the given induced operators, T is a bounded set in $\mathfrak{R}$, the $\{\widetilde{A}_i\}$ is a countable dense subset of $F_0^*(\mathbb{R})$. Let*

$$P(T) = \Big\{F : T \to F_0^*(\mathbb{R}) \mid$$
$$F(\boldsymbol{x}) = \sum_{i_1,i_2,\cdots,i_d=0}^{m} \widetilde{A}_{i_1,i_2,\cdots,i_d} G_{m;i_1,i_2,\cdots,i_d}(\boldsymbol{x}),$$
$$\widetilde{A}_{i_1,i_2,\cdots,i_d} \in \{\widetilde{A}_i\}\Big\}.$$

Then $P(T)$ is dense in $S(T)$.

Proof: Let a set $C(T) = \{F : T \to F_0^*(\mathbb{R})|\ F$ is a continuous fuzzy valued function $\}$.

We divide our proof into two steps: (1) $C(T)$ is dense in $S(T)$; (2) $P(T)$ is dense in $C(T)$.

(1) For every $S \in S(T)$, by Definition 23, there exists a group of fuzzy numbers $\widetilde{A}_1,\widetilde{A}_2,\cdots,\widetilde{A}_m \in F_0(\mathbb{R})$ and a measurable partition $\{T_1,T_2,\cdots,T_m\}$ of T such that

$$S(\mathbf{x}) = \sum_{i=1}^{m} \chi_{T_i}(\mathbf{x})\cdot\widetilde{A}_i,$$

for any $\mathbf{x} = (x_1,x_2,\cdots,x_d) \in T$.

For each measurable subset T_i , for given $\varepsilon > 0$, let $\delta_i = \varepsilon/m > 0$, there exists a closed subset $E_i \subset T_i$ $(i = 1,2,\cdots,m)$ with

$$\mu^*(T_i - E_i) < \varepsilon/m = \delta_i.$$

Now we construct a sequence $\{\beta_n^i(x)\}$ of the measurable functions on T as follows:

$$\beta_n^i(\mathbf{x}) = e^{-nd(\mathbf{x},E_i)},$$

for each $\mathbf{x} \in T, i = 1,2,\cdots,m, n = 1,2,3,\cdots$, where each $\beta_n^i(\mathbf{x})$ is continuous on T, and

$$d(\mathbf{x},E_i) = \inf_{y\in E_i} d(\mathbf{x},\mathbf{y}),$$

and satisfy $0 \le \beta_n^i(\mathbf{x}) \le 1,$ for all $\mathbf{x} \in T$.

Let $n \to \infty, i = 1,2,\cdots,m$, it is clearly to get

$$\lim_{n\to\infty}\beta_n^i(\mathbf{x}) = \chi_{E_i}(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in E_i \\ 0, & \mathbf{x} \notin E_i \end{cases} \quad (9)$$

Hence, for arbitrary $\varepsilon > 0$, for any $\mathbf{x} = (x_1,x_2,\cdots,x_d) \in T$, there exists $N \in \mathbb{N}$, whenever $n > N$,

$$|\beta_n^i(\mathbf{x}) - \chi_{E_i}(\mathbf{x})| < \varepsilon, i = 1,2,\cdots,m.$$

According to the strictly increasing of t, the following conclusion holds

$$t(\sum_{i=1}^{m}||\widetilde{A}_i|||\beta_n^i(\mathbf{x})-\chi_{E_i}(\mathbf{x})|) \le t(\sum_{i=1}^{m}||\widetilde{A}_i||\varepsilon). \quad (10)$$

Let $F_n(\mathbf{x}) = \sum\limits_{i=1}^{m}\widetilde{A}_i\cdot\beta_n^i(\mathbf{x})$ for every $\mathbf{x} = (x_1,x_2,\cdots,x_d) \in T$, $n = 1,2,\cdots$. Utilizing Lemma 4 (1) and (2), we can infer that

$$\begin{aligned} & D(F_n(\mathbf{x}),S(\mathbf{x})) \\ & = D(\sum_{i=1}^{m}\widetilde{A}_i\cdot\beta_n^i(\mathbf{x}),\sum_{i=1}^{m}\widetilde{A}_i\cdot\chi_{T_i}(\mathbf{x})) \\ & \le \sum_{i=1}^{m} D(\widetilde{A}_i\cdot\beta_n^i(\mathbf{x}),\widetilde{A}_i\cdot\chi_{T_i}(\mathbf{x})) \\ & \le \sum_{i=1}^{m}||\widetilde{A}_i||\cdot|\beta_n^i(\mathbf{x})-\chi_{T_i}(\mathbf{x})|. \end{aligned}$$

As $\mu^*(T_i - E_i) < \varepsilon/m = \delta_i, i = 1,2,\cdots,m$, by absolute continuity of Lebesgue integral, then

$$\int_{T_i-E_i} t(\sum_{i=1}^{m}||\widetilde{A}_i||\cdot|\beta_n^i(\mathbf{x})-\chi_{T_i-E_i}(\mathbf{x})|)d\mu^* < \frac{\varepsilon}{m}.$$

Let $\sum\limits_{i=1}^{m}||\widetilde{A}_i|| = a > 0$. For each $\varepsilon > 0$ and every $E_i \subset T_i, i = 1,2,\cdots,m$, whenever $n > N$, by inequality (10), we have

$$\int_{T_i} t(\sum_{i=1}^{m}||\widetilde{A}_i||\cdot|\beta_n^i(\mathbf{x})-\chi_{T_i}(\mathbf{x})|)\mathrm{d}\mu^*$$

$$= \int_{T_i-E_i} t(\sum_{i=1}^{m}||\widetilde{A}_i||\cdot|\beta_n^i(\mathbf{x})-\chi_{T_i-E_i}(\mathbf{x})|)\mathrm{d}\mu^*$$

$$+\int_{E_i} t(\sum_{i=1}^{m} ||\widetilde{A}_i|| \cdot |\beta_n^i(\mathbf{x}) - \chi_{E_i}(\mathbf{x})|)\mathrm{d}\mu^*$$
$$< \frac{\varepsilon}{m} + \int_{E_i} t(\sum_{i=1}^{m} ||\widetilde{A}_i|| \cdot \varepsilon)\mathrm{d}\mu^*$$
$$= \frac{\varepsilon}{m} + t(a\varepsilon)\mu^*(E_i). \tag{11}$$

Since $T = \cup_{i=1}^m T_i, T_i \cap T_j = \varnothing (i \neq j)$. By Lemma 14 and inequality (11), whenever $n > N$, then

$$K(H(F_n, S)) = \int_T t(D(F_n(\mathbf{x}), S(\mathbf{x})))\mathrm{d}\mu^*$$
$$\leq \int_{\cup_{i=1}^m T_i} t(\sum_{i=1}^{m} ||\widetilde{A}_i|| \cdot |\beta_n^i(\mathbf{x}) - \chi_{T_i}(\mathbf{x})|)\mathrm{d}\mu^*$$
$$= \sum_{i=1}^{m} \int_{T_i} t(\sum_{i=1}^{m} ||\widetilde{A}_i|| \cdot |\beta_n^i(\mathbf{x}) - \chi_{T_i}(\mathbf{x})|)\mathrm{d}\mu^*$$
$$< \sum_{i=1}^{m} (\frac{\varepsilon}{m} + t(a\varepsilon)\mu^*(E_i)) = \varepsilon + t(a\varepsilon)\sum_{i=1}^{m} \mu^*(E_i)$$
$$< \varepsilon + t(a\varepsilon)\mu^*(T).$$

By definition of induced operator K, then $H(F_n, S) < K^{-1}(\varepsilon + t(a\varepsilon)\mu^*(T))$.

Analogously, Since $\varepsilon > 0$ is arbitrary, then the expression $K^{-1}(\varepsilon + t(a\varepsilon)\mu^*(T))$ may be still arbitrarily small, of course, $H(F_n, S)$ be arbitrarily small. Hence, $C(T)$ is dense in $S(T)$.

(2) As $\{\widetilde{A}_i\}$ is a countable set in $F_0^*(\mathbb{R})$, then the set $P(T)$ of the fuzzy valued Bernstein polynomials generating with some coefficient in $\{\widetilde{A}_i\}$ is a countable set. In the following, we will show that the set $P(T)$ is dense in $C(T)$.

Since T is a bounded set, then there exists a d-dimension cubes $[a, b]^d$ such that $T \subset [a, b]^d$, naturally, the Lebesgue measure value $\mu^*(T) = K(\mu(T))$ is also bounded.

According to Lemma 26, for arbitrary $\varepsilon > 0$ and $F \in C(T)$, there exist $m \in \mathbb{N}$ and a fuzzy valued Bernstein polynomial $B_m(F; \mathbf{x}) \in P(T)$, where

$$B_m(F; \mathbf{x}) = \sum_{i_1, i_2, \cdots, i_d = 0}^{m} \widetilde{B}_{i_1, i_2, \cdots, i_d} G_{m; i_1, i_2, \cdots, i_d}(\mathbf{x})$$

such that

$$D(F(\mathbf{x}), B_m(F; \mathbf{x})) < \varepsilon, \tag{12}$$

for every $\mathbf{x} = (x_1, x_2, \cdots, x_d) \in [a, b]^d$.

Because $\{\widetilde{A}_i\}$ is dense in $F_0^*(\mathbb{R})$, for every $\varepsilon > 0$ and $\widetilde{B}_{i_1, i_2, \cdots, i_d} \in F_0^*(\mathbb{R})$, there exists a fuzzy number $\widetilde{A}'_{i_1, i_2, \cdots, i_d} \in \{\widetilde{A}_i\}$ with

$$D(\widetilde{A}'_{i_1, i_2, \cdots, i_d}, \widetilde{B}_{i_1, i_2, \cdots, i_d}) < \varepsilon, \tag{13}$$

for every $i_1, i_2, \cdots, i_d \in \{0, 1, 2, \cdots, m\}$. Let

$$Q(\mathbf{x}) = \sum_{i_1, i_2, \cdots, i_d = 0}^{m} \widetilde{A}'_{i_1, i_2, \cdots, i_d} G_{m; i_1, i_2, \cdots, i_d}(\mathbf{x}),$$

for all $\mathbf{x} = (x_1, x_2, \cdots, x_d) \in [a, b]^d$.

Clearly, the polynomial function $Q(\mathbf{x})$ is continuous, i. e. , $Q \in C(T)$.

By Lemma 4 (2) and the inequality in (13), we may immediately obtain that

$$D(Q(\mathbf{x}), B_m(F; \mathbf{x}))$$
$$= D(\sum_{i_1, i_2, \cdots, i_d = 0}^{m} \widetilde{A}'_{i_1, i_2, \cdots, i_d} G_{m; i_1, i_2, \cdots, i_d}(\mathbf{x}),$$
$$\sum_{i_1, i_2, \cdots, i_d = 0}^{m} \widetilde{B}_{i_1, i_2, \cdots, i_d} G_{m; i_1, i_2, \cdots, i_d}(\mathbf{x}))$$
$$\leq \sum_{i_1, i_2, \cdots, i_d = 0}^{m} G_{m; i_1, \cdots, i_d}(\mathbf{x}) D(\widetilde{A}'_{i_1, \cdots, i_d}, \widetilde{B}_{i_1, \cdots, i_d})$$
$$\leq \sum_{i_1, i_2, \cdots, i_d = 0}^{m} G_{m; i_1, i_2, \cdots, i_d}(\mathbf{x}) \cdot \varepsilon = 1 \cdot \varepsilon$$
$$= \varepsilon. \tag{14}$$

In accordance with the theorem 19, we can know that H is a metric with respect to the operation $\perp$. Besides, by Lemma 14, and combining (9) and (11), it's easy to get that

$$H(F, Q) \leq H(F, B_m(F)) \perp H(B_m(F), Q)$$
$$= K^{-1}(\int_T t(D(F(\mathbf{x}), B_m(F; \mathbf{x})))d\mu^*) \perp$$
$$K^{-1}(\int_T t(D(Q(\mathbf{x}), B_m(F; \mathbf{x})))d\mu^*)$$
$$< K^{-1}(\int_T t(\varepsilon)d\mu^*) \perp K^{-1}(\int_T t(\varepsilon)d\mu^*)$$
$$= K^{-1}(t(\varepsilon)\mu^*(T)) \perp K^{-1}(t(\varepsilon)\mu^*(T))$$
$$= K^{-1}(2t(\varepsilon)\mu^*(T)).$$

Since ε is arbitrary, and $\mu^*(T)$ is bounded, then $K^{-1}(2t(\varepsilon)\mu^*(T))$ can be arbitrarily small, this shows that $P(T)$ is dense in $C(T)$. □

Theorem 28. *Let $(\mathbb{R}^d, \mathfrak{R}, \mu)$ be a finite K-additive measure space, K and t are the given induced operators, and $K(x) = O(t(x))$ whenever $x \to 0^+$, Moreover $t'(0) > 0$. Then $(L^1(T, \mu), H)$ is a complete separable metric space.*

Proof: Since every fuzzy valued Bernstein polynomial $B_m(F; \mathbf{x})$ is continuous, of course, its also tK-integrable, and satisfies

$$P(T) \subset C(T) \subset L^1(T, \mu).$$

In the light of Theorem 27, $P(T)$ is dense in $C(T)$, and $C(T)$ is dense in $S(T)$, then $P(T)$ is dense in $S(T)$. By Theorem 24, $P(T)$ is dense in $L^1(T,\mu)$. Therefore, $P(T)$ is a countable dense subset of $L^1(T,\mu)$. This shows that the fuzzy valued tK-integrable function space $L^1(T,\mu)$ is separable.

In addition, Theorem 19 shows $(L^1(T,\mu),H)$ is a metric space, in the meantime,Theorem 22 shows that $(L^1(T,\mu),H)$ is a complete space. By Theorem 27, we can immediately get that $(L^1(T,\mu),H)$ is a complete separable metric space. So far, the main conclusion of this paper is proved in Theorem 28. □

5 Conclusion

In this paper, we give a new tK-integral norm through combining the induced operators K and t, and show that the fuzzy valued tK-integrable function space is a complete separable metric space in the sense of the tK-integral norm. In fact, tK-additive integral is a generalization of Lebesgue integral, namely, when the operators K and t are identity. Hence, its important to use this norm to express the approximation capability of a fuzzy neural network to a given integrable function. This conclusion is a generalization of main results of [16,17], and it is also a development of [8,9,10] . Since integrable function system is widely spread in some fuzzy network, such as the multiple (or single) input and single output regular fuzzy neural network and polygonal fuzzy neural networks. Undoubtedly, research on these methods will have important theoretical value in further realization of fuzzy inference network and design of fuzzy controller.

Acknowledgements: This work was supported by the National Natural Science Foundation, P. R. China (Grant No. 61374009).

References:

[1] J. J. Buckley, Y. Hayashi, Can neural nets be universal approximators for fuzzy functions, *Fuzzy Sets and Systems,* Vol.58,1999, pp. 273-278.

[2] F. L. Chung, J. C. Duan, On multistage fuzzy neural network modeling, *IEEE Transaction on Fuzzy Systems,*Vol.8, 2000, pp.125-142.

[3] P. Diamond, P. Kloeden, Metric Spaces of Fuzzy Sets.*World Scientific Press,* 1994.

[4] T. Feuring, W. M. Lippe,The fuzzy neural network approximation lemma. *Fuzzy Sets and Systems,* Vol.102 ,No.2, 1999, pp. 227-237.

[5] H. Huang, C. X. Wu, Approximation capabilities of multilayer fuzzy neural networks on the set of fuzzy valued functions, *Information Sciences,*Vol.179, No.16, 2009, pp. 2762-2773.

[6] H. Huang, C. X. Wu, Approximation of fuzzy functions by regular fuzzy neural networks, *Fuzzy Sets and Systems,* Vol.117, No.1, 2011, pp. 60-79.

[7] X. Z. Jiang, tK-integral and Kt-integral, *Journal Sichuan Normal Univercity,* Vol.16, No.2, 1993, pp. 31-39.

[8] P. Y. Liu, Approximation analyses for fuzzy valued functions in $L^1(\mu)$ by regular fuzzy neural network, *Journal of Electron Science,*Vol.17, No.2, 2000, pp.132 -138.

[9] P. Y. Liu, H. X. Li, Approximation of generalized fuzzy system to integrable function. *Science in China. Series (E),*Vol.43, No.6, 2000, pp. 618-628.

[10] P. Y. Liu, H. X. Li, Analyses for $L_p(\mu)$-norm approximation capability of the generalized Mamdani fuzzy systems. *Information Sciences,*Vol. 138, No.2, 2001, pp. 195-210.

[11] H. L. Royden and P. M. Fitzpatrick, Real Analysis,the*Macmillan company*, New York ,2010.

[12] M. Sugeno and T. Murofushi, Pseudo-additive measures and integrals,*Journal of Mathematical Analysis and Applications,*Vol.122, 1987, pp. 197-222.

[13] G. J. Wang, X. P. Li, Absolute continuity of K-quasi-additive fuzzy integrals. *Journal of Sichuan Normal University,* Vol. 21, No. 3, 1998, pp. 251-255.

[14] G. J. Wang, X. P. Li, K-quasi-additive fuzzy integrals of set-valued mappings, *Progress in Natural Science,* Vol.16, No.2, 2006, pp.125-132.

[15] G. J. Wang, X. P. Li, K-quasi-additive fuzzy number valued integral and its convergence, *Advances in Mathematics,*Vol.35, No.1, 2006, pp.109-119.

[16] G. J. Wang, X. P. Li, Universal approximation of polygonal fuzzy neural networks in sense of K-integral norms, *Science China. Information Science,* Vol.54, No.11, 2011, pp.2307-2323.

[17] G. J. Wang, D. Li, Capability of universal approximation of feedforward regular fuzzy neural networks in K-integral norm, *Acta Mathematicae Applicatae Sinica,* Vol. 36, No.1, 2013, pp. 141-152.

[18] M. G. Voskoglou, Fuzzy measures of a system's effectiveness-an application to problem solving, *WSEAS Transactions on Mathematics*, Vol. 12, No. 4, 2013, pp. 459-471.

[19] F. X. Zhao, H. X. Li, Universal approximation of regular fuzzy neural networks to Sugeno-integrable functions, *Acta Mathematicae Applicatae Sinica,* Vol. 29, No.1, 2006,pp.39-45.

[20] R. M. Rzaev, A. B. Imanova, Some boundary properties of Cauchy type integral in terms of mean oscillation, *WSEAS Transactions on Mathematics* , Vol.11, No. 2, 2012, pp.135-145.