An Extended Newton's Method with Free Second-order Derivatives

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Abstract: We propose the cubic-order numerical method free of second derivatives and derive the asymptotic error constant in terms of control parameters. Applying this proposed scheme to various test functions, numerical results show a good agreement with the theory analyzed in this paper and are proven using Mathematica with its high-precision computability.

Key–Words: Newton's method, Multiple root, Nonlinear equation, Order of convergence, Root finding

1 Introduction

The iteration methods to find the roots of nonlinear equations have various applications in many science problems. Among them, the Newton's method is one of the most well-known iteration schemes and is modified by many researchers[1,2,3,4].

Assume that a function $f : \mathbb{C} \to \mathbb{C}$ has a multiple root α with integer multiplicity $m \geq 1$ and is analytic in a small neighborhood of α. We express the given equation $f(x) = 0$ in the form $x - g(x) = 0$ where $g : \mathbb{C} \to \mathbb{C}$ is analytic in a sufficiently small neighborhood of α. Then We find an approximated α by a scheme

$$x_{n+1} = g(x_n),\ n = 0,\ 1,\ 2, \cdots, \tag{1}$$

where $g : \mathbb{C} \to \mathbb{C}$ is an iteration function and $x_0 \in \mathbb{C}$ is given. Then we find an approximated α using an iterative method[5,6,7,8]. The roots of the equation are obtained using the following scheme:

$$g(x_n) = x_n - \lambda \frac{f(x_n - \mu h(x_n))}{f'(x_n)} \tag{2}$$

where

$$h(x) = \begin{cases} f(x)/f'(x), & if\ x \neq \alpha \\ \lim_{x\to\alpha} f(x)/f'(x), & if\ x = \alpha. \end{cases} \tag{3}$$

For a given $p \in \boldsymbol{N}$, we suppose that

$$\begin{cases} \left|\frac{d^p}{dx^p} g(x)\right|_{x=\alpha} = |g^{(p)}(\alpha)| < 1, & \text{if } p = 1. \\ g^{(i)}(\alpha) = 0 \text{ for } 1 \leq i \leq p-1 \text{ and } g^{(p)}(\alpha) \neq 0, & \text{if } p \geq 2. \end{cases} \tag{4}$$

Let $z(x) = x - \mu h(x)$ and $F(x) = \frac{f(x-\mu h(x))}{f'(x)}$. Since $g(x)$ is continuous at $x = \alpha$, $g(x)$ is represented by

$$g(x) = \begin{cases} x - \lambda F(x), & if\ x \neq \alpha \\ x - \lambda \lim_{x\to\alpha} F(x), & if\ x = \alpha. \end{cases} \tag{5}$$

By Corollary 1 and Corollary 2, we have $[f(z)]^{(k)}_{x=\alpha} = 0,\ 0 \leq k \leq m-1$ and $f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0,\ f^{(m)} \neq 0$. Using L'Hospital's rule repeatedly[5,6,9], we have

$$\lim_{x\to\alpha} F(x) = \frac{[f(z)]^{(m-1)}_{x=\alpha}}{[f'(x)]^{(m-1)}} = 0 \tag{6}$$

The next corollary is useful to calculate $g'(\alpha)$, $g''(\alpha)$ and $g'''(\alpha)$.

Corollary 1 *Suppose $f : \mathbb{C} \to \mathbb{C}$ has a multiple root α with a given integer multiplicity $m \geq 1$ and is analytic in a small neighborhood of α. Then the function $h(x)$ and its derivatives up to order 3 evaluated at α has the following properties with $\theta_j = \frac{f^{(m+j)}(\alpha)}{f^{(m)}(\alpha)}, j \in \mathbb{N}$:*

(i) $h(\alpha) = 0$

(ii) $h'(\alpha) = \frac{1}{m}$

(iii) $h''(\alpha) = -\frac{2}{m^2(m+1)}\theta_1$

(iv) $h^{(3)}(\alpha) = \frac{6}{m^3(m+1)}\left\{{\theta_1}^2 - \frac{2m}{m+2}\theta_2\right\}$.

Corollary 2 *Let f stated in Corollary 1 have a multiple root α with a given multiplicity $m \geq 1$. Let $z(x) = x - \mu h(x)$ and $h(x)$ be defined by (3). Then the following hold:*

$$\left.\frac{d^k}{dx^k} f(z)\right|_{x=\alpha} = \begin{cases} 0, & if\ 0 \leq k \leq m-1 \\ f^{(m)}(\alpha)t^m, & if\ k = m \\ f^{(m)}(\alpha)\cdot\theta_1\cdot t^{m-1}(1-t+t^2), & if\ k = m+1 \\ f^{(m)}(\alpha)\cdot t^{m-2}\cdot\{q_1\theta_1^2 + q_2\theta_2\}, & if\ k = m+2 \end{cases}$$

where $q_1 = \frac{(m+2)\lambda}{2m(m+1)}(t-1)^2\{2(m+1)t - m + 1\}$, $q_2 = t(t^3 - 2t + 2)$ *and* $t^0 \equiv 1$ *for any* $t \in \mathbb{C}$.

In this paper, our aim is to establish some relationships between $\lambda, m, g'(\alpha)$, $g''(\alpha)$ and $g'''(\alpha)$ for cubic order of convergence[8,9] and derive the corresponding asymptotic error constant. Various numerical experiments are presented to confirm the validity of the suggested method.

2 Convergence

We analyze the convergent properties of this proposed scheme (2) and investigate the order of convergence and the asymptotic error constant[10] in terms of parameter λ and μ. From the definition of $g(x)$ as described in (2), we rewrite

$$(g - x) \cdot f'(x) = -\lambda f(z). \tag{7}$$

where $f = f(x), f' = f'(x), z = x - \mu h(x)$ are used for concise and the symbol $'$ denotes the derivative with respect to x.
Differentiating both sides of (7) with respect to x, we obtain

$$(g' - 1) \cdot f' + (g - x) \cdot f''(x) = -\lambda[f(z)]^{(1)} \tag{8}$$

Since g' is continuous at α, we have

$$g'(x) - 1 = \begin{cases} F_1(x), & if\ x \neq \alpha \\ \lim_{x\to\alpha} F_1(x), & if\ x = \alpha, \end{cases} \tag{9}$$

where $F_1(x) = \frac{-(g-x)f''(x) - \lambda[f(z)]^{(1)}}{f'}$.
Using Corollary 2 and $g(\alpha) = \alpha$, we have the following:

$$(g-x)f''(x)]^{(k)}_{x=\alpha} = \begin{cases} 0, & if\ 0 \le k \le m-2, m \ge 2 \\ (m-1)(g'-1)f^{(m)}(\alpha), & if\ k = m-1, \end{cases} \tag{10}$$

$$[f(z)]^{(1)}\Big]^{(k)}_{x=\alpha} = \begin{cases} 0, & if\ 0 \le k \le m-2,\ m \ge 2 \\ f^{(m)}(\alpha)(1 - \frac{\mu}{m})^m, & if\ k = m-1, \end{cases} \tag{11}$$

Substituting (10) and (11) into (9) leads

$$g'(\alpha) - 1 = -(m-1)(g'(\alpha) - 1) - \lambda(1 - \frac{\mu}{m})^m$$

To obtain $g'(\alpha) = 0$, we get

$$m = \lambda t^m \tag{12}$$

where $t = 1 - \frac{\mu}{m}$.
Differentiate both sides of Eq(8) with respect to x, we have

$$g'' + 2(g\prime - 1) \cdot f'' + (g - x) \cdot f^{(3)} = -\lambda[f(z)]^{(2)} \tag{13}$$

We rewrite

$$g''(x) = \begin{cases} F_2(x), & if\ x \neq \alpha \\ \lim_{x\to\alpha} F_2(x), & if\ x = \alpha, \end{cases} \tag{14}$$

where $F_2(x) = \frac{-2(g'-1)\cdot f'' - (g-x)\cdot f^{(3)} - \lambda[f(z)]^{(2)}}{f'}$.
Applying L'Hospital's rule with Corollary 2, the numerator of $F_2(x)$ yields

$$-2(g'-1)f'' - (g-x)f^{(3)} - \lambda[f(z)]^{(2)}$$

$$= \begin{cases} 0, & if\ 0 \le k \le m-3 \\ f^{(m)}(\alpha)(m - \lambda t^m), & if\ k = m-2 \\ f^{(m)}(\alpha)[\theta_1\{(m+1) - \lambda(t^{m+1} - t^m + t^{m-1})\} \\ -g''(\alpha)\frac{(m+2)(m-1)}{2}], & if\ k = m-1, \end{cases} \tag{15}$$

From (14) and (15), we obtain

$$g'' = \frac{2\theta_1}{m(m+1)}\{(m+1) - \lambda(t^{m+1} - t^m + t^{m-1})\} \tag{16}$$

From (16), to have $g''(\alpha) = 0$ we get the following relation,

$$m + 1 = \lambda(t^{m+1} - t^m + t^{m-1}) \tag{17}$$

Differentiate both sides of (13) with respect to x, we get

$$g^{(3)} \cdot f' + 3g'' \cdot f'' + 3(g'-1) \cdot f^{(3)} + (g-x) \cdot f^{(4)} = -\lambda[f(z)]^{(3)}. \tag{18}$$

We rewrite

$$g^{(3)}(x) = \begin{cases} F_3(x), & if \quad x \neq \alpha \\ \lim_{x\to\alpha} F_3(x), & if \quad x = \alpha, \end{cases} \tag{19}$$

where

$$F_3(x) = \frac{-3g''f'' - 3(g'-1)f^{(3)} - (g-x)f^{(4)} - \lambda[f(z)]^{(3)}}{f'}. \tag{20}$$

Using Corollary 2 and the fact that $g(\alpha) = \alpha, g^{\boldsymbol{P}}(\alpha) = 0, g''(\alpha) = 0$ for cubic order of convergence, we have the relation below:

$$\Big[-3g''\cdot f'' - 3(g'-1)\cdot f^{(3)}(g-x)\cdot f^{(4)} - \lambda[f(z)]^{(3)}\Big]^{(k)}_{x=\alpha}$$

$$= \begin{cases} 0, & if\ 0 \le k \le m-4 \\ f^{(m)}(\alpha)(m-\lambda t^m), & if\ k = m-3 \\ \theta_1 f^{(m)}(\alpha)\{m+1-\lambda(t^{m+1}-t^m+t^{m-1})\}, & if\ k=m-2 \\ f^{(m)}(\alpha)\{\phi_1\theta_1^2+\phi_2\theta_2-\frac{(m-1)(m^2+4m+6)}{6}g^{(3)}\}, & if\ k=m-1, \end{cases} \quad (21)$$

where

$$\phi_1 = \begin{cases} t^{m-2}q_1(t, & \text{if } m \ge 2 \\ 3(t-1)^2, & \text{if } m = 1, \end{cases}$$

$$\phi_2 = \begin{cases} m+2-\lambda t^{m-2}\cdot q_2(t), & \text{if } m \ge 2 \\ -t(t^2-3), & \text{if } m = 1, \end{cases}$$

$q_1 = \frac{(m+2)\lambda}{2m(m+1)}(t-1)^2\{2(m+1)t-m+1\}$ and $q_2 = t(t^3-2t+1)$.
From (19) and (21), we have

$$g^{(3)}(\alpha) = \frac{6}{m(m+1)(m+2)}\{\phi_1\theta_1^2+\phi_2\theta_2\}. \quad (22)$$

Theorem 3 *Let $f:\mathbb{C}\to\mathbb{C}$ have a multiple real zero α with integer multiplicity $m \ge 1$ and be analytic in a small neighborhood of α. Let θ_1, θ_2 be defined as in Corollary and ϕ_1, ϕ_2 be defined as in (21). Let t be a root of $\boldsymbol{R}(t)$. Let x_0 be an initial value chosen in a sufficiently small neighborhood of α. Then this proposed method stated in section 1 has order 3 and its asymptotic error constant η as follows:*

$$\eta = \frac{1}{6}|g^{(3)}(\alpha)| = \frac{1}{m(m+1)(m+2)}|\phi_1\theta_1^2+\phi_2\theta_2|, \quad (23)$$

provided that $\phi_1\theta_1^2+\phi_2\theta_2 \ne 0$.

From (12) and (17), we get

$$mt^2-(2m+1)t+m=0$$

Typical cases for $1 \le m \le 4$ are studied here and listed in Table 1 to confirm Theorem 2.1.

Table 1: Values ρ and η for $1 \le m \le 4$

m	$\rho(t)$	η
1	$t^2-3t+1=0$	$\frac{1}{6}[\theta_2(4-3t)+2\theta_1^2(1-t)]$
2	$2t^2-5t+2=0$	$\frac{1}{24}[\theta_2\frac{5t^2+2t+4}{t}+\theta_1^2\frac{7t^2-2t+2}{3t^2}]$
3	$3t^2-7t+3=0$	$\frac{1}{60}[\theta_2\frac{-7t^2+2t+6}{t}+5\theta_1^2\frac{4t^3+t^2-6t+1}{4t^2}]$
4	$4t^2-9t+4=0$	$\frac{1}{20}[\theta_2\frac{10t-8}{t}+\theta_1^2\frac{30t^3-49t^2+28t-9}{5t^2}]$

3 Conclusion

The symbolic and computational ability of *Mathematica*[11] leads us to a zero-finding algorithm based on the convergent behaviour studied in Sections 1 and 2.

Algorithm 3.1 (Zero-Finding Algorithm)

Step 1. For $k \in \boldsymbol{N} \cup \{0\}$, construct iteration scheme (1) with the given function f at a multiple zero α as stated in Section 1.
Step 2. Set the minimum number of precision digits. With exact zero α or most accurate zero, supply the theoretical asymptotic error constant η. Set the error range ϵ, the maximum iteration number n_{max} and the initial value x_0. Compute $f(x_0)$ and $|x_0-\alpha|$.
Step 3. Compute x_{n+1} in (1) for $0 \le n \le n_{max}$ and display the computed values of $n, x_n, f(x_n), |x_n-\alpha|, |e_{n+1}/e_n{}^p|$ and η.

In these experiments, we choose 300 as the minimum number of digits of precision by assigning *$MinPrecision=300* in Mathematica to achieve the specified accuracy. We set the error bound ϵ to 0.5×10^{-235} for $|x_n-\alpha|<\epsilon$ and evaluate the n^{th} order derivative of the complicated nonlinear functions using the Mathematica command ***D***[f, $\{x,n\}$].

As an example for the convergence, we first illustrate the order of convergence and the asymptotic error constant with a function

$$f(x) = (x^2-x+3)^4/(x^4+\sin x)$$

having a real zero $\alpha = \frac{1-i\sqrt{11}}{2}$ of multiplicity 4. We choose $x_0 = 0.468-1.58i$ as an initial guess. Table 2 verifies cubic convergence apparently.

Table 2: Convergence for $f(x)=(x^2-x+3)^4/(x^4+\sin x)$ with $m=4,\ \alpha=\frac{1-i\sqrt{11}}{2}$

$$(t,\mu,\lambda) = (\tfrac{9+\sqrt{17}}{8}, -2.56155, 1)$$

n	x_n	$\vert x_n-\alpha\vert$	$e_{n+1}/e_n{}^3$	η
0	0.468000000000000 - 1.58000000000000i	0.0845981		0.2554068175
1	0.500178290031692 - 1.65834669787011i	0.000181560	0.2998740289	
2	0.500000000001344 - 1.65831239517843i	1.52868×10^{-12}	0.2554204016	
3	0.500000000000000 - 1.65831239517770i	9.12388×10^{-37}	0.2554068175	
4	0.500000000000000 - 1.65831239517770i	1.93986×10^{-109}	0.2554068175	
5	0.500000000000000 - 1.65831239517770i	0.0×10^{-299}		

We choose an analytic function $f(x) = (x-\pi)\log^2(x+1-\pi)\sin^5 x\cdot e^x$ near a multiple root $\alpha=\pi$

of multiplicity 8. The extra informations regarding cubic convergence are used as a initial value $x_0 = 3.29$, $\mu = -3.37228$ and $\lambda = 0.479765623518$. We select a complex $t = \frac{17+\sqrt{33}}{16}$ that is approximated as one of 2 solutions to a polynomial equation $\rho(t)$ numerically. From the lists of Table 3, it can be confirmed that the computed asymptotic error constant coincides with the investigated one using Theorem 2 and this iteration method has cubic convergence.

The current study can be applied to the effiective variations to develop the higher order numerical schemes to find the multiple roots of nonlinear equations[12,13,14].

Table 3: Convergence for $f(x) = (x - \pi)\log^2(x + 1 - \pi)\sin^5 x \cdot e^x$ with $m = 8,\ \alpha = \pi$

$$(t, \mu, \lambda) = (\tfrac{17+\sqrt{33}}{16}, -3.37228, 0.479765623518)$$

n	x_n	$\lvert x_n - \alpha \rvert$	$e_{n+1}/{e_n}^3$	η
0	3.29000000000000	0.148407		0.1272715659
1	3.14213337664892	0.000540723	0.1654278750	
2	3.14159265360994	2.01430×10^{-11}	0.1274087393	
3	3.14159265358979	1.04017×10^{-33}	0.1272715660	
4	3.14159265358979	1.43232×10^{-100}	0.1272715659	
5	3.14159265358979	$0. \times 10^{-299}$		

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