Some new Hermite-Hadamard-type inequalities for geometric-arithmetically s-convex functions

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Abstract: In this paper, motivated by the concept of "geometric-arithmetically s-convex function", we establish some new Hermite-Hadamard-type inequalities for geometric-arithmetically s-convex functions, which not only recapture the recent results about Hermite-Hadamard-type inequalities for convex functions, but also give some new results as special cases.

Key–Words: Hermite-Hadamard-type inequalities, Hölder's inequality, s-GA-convexity

1 INTRODUCTION

Let I be on interval in $\mathbb{R}$. Then $f : I \to \mathbb{R}$ is said to be convex, if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

One of the most famous inequalities for convex functions is Hermite-Hadamard inequality. This double inequality is stated as follows: Let f be a convex function on some nonempty interval $[a,b]$ of real line $\mathbb{R}$, where $a \ne b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see [1, 2, 3, 6, 13, 14, 7, 7, 10, 11, 18, 20, 21, 22, 23, 24, 27, 29] and references therein). For example, Toader [26] defined the concept of m-convexity as the following:

Definition 1 *The function $f : [0,b] \to \mathbb{R}$ is said to be $m-$convex, where $m \in (0,1]$, if for every $x, y \in [0,b]$ and $t \in [0,1]$ we have:*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y).$$

In [4], the following inequality of Hermite-Hadamard type for $m-$ convex functions holds:

Theorem 2 *Let $f : [0,\infty) \to \mathbb{R}$ be a $m-$convex function with $m \in (0,1]$. If $0 \le a < b < \infty$ and $f \in L_1[a,b]$, then one has the inequality:*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \min\left\{\frac{f(a)+mf(\frac{b}{m})}{2}, \frac{f(b)+mf(\frac{a}{m})}{2}\right\}. \quad (1)$$

In [5], Dragomir proved the following theorem.

Theorem 3 *Let $f : [0,\infty) \to \mathbb{R}$ be a $m-$convex function with $m \in (0,1]$. If $f \in L_1[am,b]$ where $0 \le a < b$, then one has the inequality:*

$$\frac{1}{m+1}\left[\frac{1}{mb-a}\int_a^{mb} f(x)dx + \frac{1}{b-ma}\int_{ma}^b f(x)dx\right] \le \frac{f(a)+f(b)}{2}.$$

The s-convexity in the second sense, (α, m)- convexity and $GA-$ convexity are defined as follows:

Definition 4 ([9]) *The function* $f : [0, b] \to \mathbb{R}$, $b > 0$, *is said to be* s*-convex function in the second sense, where* $s \in (0, 1]$, *if*

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in [0, b]$ *and* $t \in [0, 1]$.

Definition 5 ([15]) *The function* $f : [0, b] \to \mathbb{R}$, $b > 0$, *is said to be* (α, m)*-convex, where* $(\alpha, m) \in [0, 1]^2$, *if*

$$f(tx + m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha)f(y)$$

holds for all $x, y \in [0, b]$ *and* $t \in [0, 1]$.

Definition 6 ([16, 17]) *The function* $f : I \subseteq R_0 \to \mathbb{R}$ *is said to be a GA-convex function on* I, *if*

$$f(x^t y^{1-t}) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ *and* $t \in [0, 1]$, *where* $x^t y^{1-t}$ *and* $tf(x) + (1-t)f(y)$ *are respectively called the weighted geometric mean of two positive numbers* x *and* y *and the weighted arithmetic mean of* $f(x)$ *and* $f(y)$.

Recently, Ji et al. [12] introduced the concepts of (α, m)-geometric-arithmetically-convex function as follows:

Definition 7 *The function* $f : [0, b] \to \mathbb{R}$ *and* $(\alpha, m) \in [0, 1]^2$, *if*

$$f(x^t y^{m(1-t)}) \le t^\alpha f(x) + m(1-t^\alpha)f(y)$$

holds for all $x, y \in [0, b]$ *and* $t \in [0, 1]$, *then* $f(x)$ *is said to be an* (α, m)*-geometric-arithmetically convex function, or simply speaking, an* (α, m)*-GA-convex function.*

Then, Ji et al. [12] obtained the Hermite-Hadamard type inequalities for (α, m)-GA-convex function as follows:

Theorem 8 *Let* $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ *be a differentiable function and* $f' \in L^1([a, b])$ *for* $0 < a < b < \infty$. *If* $|f'|^q$ *is an* (α, m)*-GA-convex function on* $[0, \max\{a^{\frac{1}{m}}, b\}]$ *for* $(\alpha, m) \in [0, 1]^2$ *and* $q \ge 1$, *then*

$$\begin{aligned}&\left|\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x)\mathrm{d}x\right| \\ \le&\frac{\ln b - \ln a}{2}[L(a^3, b^3)]^{1-\frac{1}{q}} \left\{m[L(a^3, b^3) \right.\\ &\left. -G(\alpha, 3)]|f'(a^{\frac{1}{m}})|^q + G(\alpha, 3)|f'(b)|^q\right\}^{\frac{1}{q}}, \quad (2)\end{aligned}$$

where

$$G(\alpha, l) = \int_0^1 t^\alpha a^{l(1-t)} b^{lt} \mathrm{d}t \quad (3)$$

and

$$L(x, y) = \frac{y - x}{\ln y - \ln x} \quad (4)$$

for all $x, y > 0$, $l \ge 0$ *with* $x \ne y$.

More recently, Shuang et al. [25] introduced the following concept of geometric-arithmetically s-convex function, based on which some inequalities of Hermite-Hadamard type for geometric-arithmetically s-convex functions are established.

Definition 9 *Let* $f : [0, b] \to \mathbb{R}$, $b > 0$, *if*

$$f(x^t y^{1-t}) \le t^s f(x) + (1-t)^s f(y) \quad (5)$$

holds for all $x, y \in [0, b]$ *and* $t \in [0, 1]$, *then* $f(x)$ *is said to be geometric-arithmetically* s*-convex function or, simply speaking, an* s*-GA-convex function. If*

$$f(x^t y^{1-t}) \ge t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in [0, b]$ *and* $t \in [0, 1]$, *then* $f(x)$ *is said to be an* s*-GA-concave function.*

Motivated by the above works, the purpose of the present paper is to use the above concept of "geometric-arithmetically s-convex function" to establish some new inequalities of Hermite-Hadamard-type for geometric-arithmetically s-convex functions. These inequalities not only recapture the recent results about Hermite-Hadamard-type inequalities for convex functions, but also give some new results as special cases.

2 Some new Hermite-Hadamard-type inequalities

To establish some new Hermite-Hadamard type inequalities for geometric-arithmetically s-convex functions, we need the following lemma.

Lemma 10 *Let* $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ *be a differentiable function on* I. $a, b \in I$ *with* $a < b$. *If* $f' \in L^1([a, b])$, *then*

$$\begin{aligned}&\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1} f(x)\mathrm{d}x \\ =&\frac{\ln b - \ln a}{n} \int_0^1 a^{(n+1)(1-t)} b^{(n+1)t} f'(a^{1-t}b^t)\mathrm{d}t. \quad (6)\end{aligned}$$

Proof: Let $x = a^{1-t}b^t$ for $t \in [0,1]$, then

$$
\begin{aligned}
&(\ln b - \ln a)\int_0^1 a^{(n+1)(1-t)}b^{(n+1)t}f'(a^{1-t}b^t)\mathrm{d}t \\
=&\int_a^b x^n f'(x)\mathrm{d}x \\
=&b^n f(b) - a^n f(a) - n\int_a^b x^{n-1}f(x)\mathrm{d}x.
\end{aligned}
$$

Thus, Lemma 10 is proved. □

Remark 11 *When $n = 1$, identity* (6) *becomes*

$$
\begin{aligned}
&bf(b) - af(a) - \int_a^b f(x)\mathrm{d}x \\
=&(\ln b - \ln a)\int_0^1 a^{2(1-t)}b^{2t}f'(a^{1-t}b^t)\mathrm{d}t;
\end{aligned}
$$

and when $n = 2$, identity (6) *becomes*

$$
\begin{aligned}
&\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x)\mathrm{d}x \\
=&\frac{\ln b - \ln a}{2}\int_0^1 a^{3(1-t)}b^{3t}f'(a^{1-t}b^t)\mathrm{d}t.
\end{aligned}
$$

Thus, the identities in [28, Lemma 2.1] and [12, Lemma 7] are recaptured, respectively.

Now we turn our attention to establish inequalities of Hermite-Hadamard type for s-GA-convex functions.

Theorem 12 *Let $f : \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ be a differentiable function and $f' \in L^1([a,b])$ for $0 < a < b < \infty$. If $|f'|^q$ is an s-GA-convex function on $[0,b]$ for $s \in (0,1]$ and $q \geq 1$, then*

$$
\begin{aligned}
&\left|\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1}f(x)\mathrm{d}x\right| \\
\leq&\frac{\ln b - \ln a}{n}[L(a^{n+1},b^{n+1})]^{1-\frac{1}{q}}[G(s,n+1)|f'(b)|^q \\
&+ H(s,n+1)|f'(a)|^q]^{\frac{1}{q}}, \qquad (7)
\end{aligned}
$$

where $G(s,l)$ and $L(x,y)$ are given in (3) *and* (4), *respectively, and*

$$
H(s,l) = \int_0^1 (1-t)^s a^{l(1-t)}b^{lt}\mathrm{d}t \qquad (8)
$$

for all $x, y > 0$, $l \geq 0$ with $x \neq y$.

Proof. Making use of the s-GA-convexity of $|f'|^q$ on $[0,b]$, Lemma 10 and Hölder's inequality, we get

$$
\begin{aligned}
&\left|\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1}f(x)\mathrm{d}x\right| \\
\leq&\frac{\ln b - \ln a}{n}\int_0^1 a^{(n+1)(1-t)}b^{(n+1)t}\left|f'(a^{1-t}b^t)\right|\mathrm{d}t \\
\leq&\frac{\ln b - \ln a}{n}\left[\int_0^1 a^{(n+1)(1-t)}b^{(n+1)t}\mathrm{d}t\right]^{1-\frac{1}{q}} \\
&\times\left[\int_0^1 a^{(n+1)(1-t)}b^{(n+1)t}\left|f'(a^{1-t}b^t)\right|^q\mathrm{d}t\right]^{\frac{1}{q}} \\
\leq&\frac{\ln b - \ln a}{n}\left[L(a^{n+1},b^{n+1})\right]^{1-\frac{1}{q}}\left[\int_0^1 a^{(n+1)(1-t)}\right. \\
&\left.\times\ b^{(n+1)t}\left(t^s\left|f'(b)\right|^q + (1-t)^s\left|f'(a)\right|^q\right)\mathrm{d}t\right]^{\frac{1}{q}} \\
=&\frac{\ln b - \ln a}{n}[L(a^{n+1},b^{n+1})]^{1-\frac{1}{q}} \\
&\times[G(s,n+1)|f'(b)|^q + H(s,n+1)|f'(a)|^q]^{\frac{1}{q}}.
\end{aligned}
$$

As a result, the inequality (7) follows. □

Corollary 13 *Under the conditions of Theorem 12, if $q = 1$, then*

$$
\begin{aligned}
&\left|\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1}f(x)\mathrm{d}x\right| \\
\leq&\frac{\ln b - \ln a}{n}[G(s,n+1)|f'(b)| \\
&+ H(s,n+1)|f'(a)|]. \qquad (9)
\end{aligned}
$$

Corollary 14 *Under the conditions of Theorem 12, if $q = 1$, $n = 1$, then*

$$
\begin{aligned}
&\left|bf(b) - af(a) - \int_a^b f(x)\mathrm{d}x\right| \\
\leq&(\ln b - \ln a)[G(s,2)|f'(b)| + H(s,2)|f'(a)|]. \qquad (10)
\end{aligned}
$$

Corollary 15 *Under the conditions of Theorem 12, if $q = 1$, $n = 2$, then*

$$
\begin{aligned}
&\left|\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x)\mathrm{d}x\right| \\
\leq&\frac{\ln b - \ln a}{2}[G(s,3)|f'(b)| + H(s,3)|f'(a)|]. \qquad (11)
\end{aligned}
$$

Corollary 16 *Under the conditions of Theorem 12, if $s = 1$, then*

$$
\begin{aligned}
&\left|\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1}f(x)\mathrm{d}x\right| \\
\leq&\frac{\ln b - \ln a}{n}[L(a^{n+1},b^{n+1})]^{1-\frac{1}{q}}[G(1,n+1)|f'(b)|^q \\
&+ H(1,n+1)|f'(a)|^q]^{\frac{1}{q}}. \qquad (12)
\end{aligned}
$$

Corollary 17 *Under the conditions of Theorem 12, if* $s = 1$*,* $n = 1$*, then*

$$\left|bf(b) - af(a) - \int_a^b f(x)\mathrm{d}x\right|$$
$$\leq (\ln b - \ln a)[L(a^2, b^2)]^{1-\frac{1}{q}}[G(1,2)|f'(b)|^q + H(1,2)|f'(a)|^q]^{\frac{1}{q}}. \quad (13)$$

Thus, the inequality in [28, Theorem 3.1] is recaptured.

Corollary 18 *Under the conditions of Theorem 12, if* $s = 1$*,* $n = 2$*, then*

$$\left|\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b xf(x)\mathrm{d}x\right|$$
$$\leq \frac{(b^3 - a^3)^{1-\frac{1}{q}}}{6}\{[b^3 - L(a^3, b^3)]|f'(b)|^q + [L(a^3, b^3) - a^3]|f'(a)|^q\}^{\frac{1}{q}}. \quad (14)$$

Proof. By

$$G(1,3) = \int_0^1 ta^{3(1-t)}b^{3t}\mathrm{d}t = \frac{b^3 - L(a^3, b^3)}{\ln b^3 - \ln a^3}$$

and

$$H(1,3) = L(a^3, b^3) - G(1,3) = \frac{L(a^3, b^3) - a^3}{\ln b^3 - \ln a^3}.$$

The corollary can be proved easily. □

Theorem 19 *Let* $f : \mathbb{R}_0 \to \mathbb{R}$ *be a differentiable function and* $f' \in L^1([a,b])$ *for* $0 < a < b < \infty$*. If* $|f'|^q$ *is an s-GA-convex function on* $[0,b]$ *for* $s \in (0,1]$ *and* $q \geq 1$*, then*

$$\left|\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1}f(x)\mathrm{d}x\right|$$
$$\leq \frac{\ln b - \ln a}{n}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}[L(a^{\frac{(n+1)q}{q-1}}, b^{\frac{(n+1)q}{q-1}})]^{1-\frac{1}{q}} \times [|f'(b)|^q + |f'(a)|^q]^{\frac{1}{q}}, \quad (15)$$

where L *is defined by* (4).

Proof. Since $|f'|^q$ is an s-GA-convex function on $[0,b]$, from Lemma 10 and Hölder's inequality, we have

$$\left|\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1}f(x)\mathrm{d}x\right|$$
$$\leq \frac{\ln b - \ln a}{n}\int_0^1 a^{(n+1)(1-t)}b^{(n+1)t}\left|f'(a^{1-t}b^t)\right|\mathrm{d}t$$
$$\leq \frac{\ln b - \ln a}{n}\left[\int_0^1 a^{\frac{(n+1)q(1-t)}{q-1}}b^{\frac{(n+1)qt}{q-1}}\mathrm{d}t\right]^{1-\frac{1}{q}} \times \left[\int_0^1 \left(t^s\left|f'(b)\right|^q + (1-t)^s\left|f'(a)\right|^q\right)\mathrm{d}t\right]^{\frac{1}{q}}$$
$$= \frac{\ln b - \ln a}{n}[L(a^{\frac{(n+1)q}{q-1}}, b^{\frac{(n+1)q}{q-1}})]^{1-\frac{1}{q}} \times \left[\frac{|f'(b)|^q}{s+1} + \frac{|f'(a)|^q}{s+1}\right]^{\frac{1}{q}}$$
$$= \frac{\ln b - \ln a}{n}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}[L(a^{\frac{(n+1)q}{q-1}}, b^{\frac{(n+1)q}{q-1}}]^{1-\frac{1}{q}} \times \left[|f'(b)|^q + |f'(a)|^q\right]^{\frac{1}{q}}.$$

The proof of Theorem 19 is completed. □

Corollary 20 *Under the conditions of Theorem 19, if* $s = 1$*, then*

$$\left|\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1}f(x)\mathrm{d}x\right|$$
$$\leq \frac{\ln b - \ln a}{n}\left(\frac{1}{2}\right)^{\frac{1}{q}}[L(a^{\frac{(n+1)q}{q-1}}, b^{\frac{(n+1)q}{q-1}})]^{1-\frac{1}{q}} \times [|f'(b)|^q + |f'(a)|^q]^{\frac{1}{q}}. \quad (16)$$

Corollary 21 *Under the conditions of Theorem 19, if* $n = 1$*, then*

$$\left|bf(b) - af(a) - \int_a^b f(x)\mathrm{d}x\right|$$
$$\leq (\ln b - \ln a)\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \times [L(a^{\frac{2q}{q-1}}, b^{\frac{2q}{q-1}})]^{1-\frac{1}{q}}[|f'(b)|^q + |f'(a)|^q]^{\frac{1}{q}}. \quad (17)$$

Corollary 22 *Under the conditions of Theorem 19, if* $s = 1$*,* $n = 1$*, then*

$$\left|bf(b) - af(a) - \int_a^b f(x)\mathrm{d}x\right|$$
$$\leq (\ln b - \ln a)\left(\frac{1}{2}\right)^{\frac{1}{q}} \times [L(a^{\frac{2q}{q-1}}, b^{\frac{2q}{q-1}})]^{1-\frac{1}{q}}[|f'(b)|^q + |f'(a)|^q]^{\frac{1}{q}}. \quad (18)$$

So, the inequality in [28, Theorem 3.3] is recaptured.

Corollary 23 *Under the conditions of Theorem 19, if $n = 2$, then*

$$\left|\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x)\mathrm{d}x\right| \le \frac{\ln b - \ln a}{2}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} [L(a^{\frac{3q}{q-1}}, b^{\frac{3q}{q-1}})]^{1-\frac{1}{q}} \times [|f'(b)|^q + |f'(a)|^q]^{\frac{1}{q}}. \tag{19}$$

Corollary 24 *Under the conditions of Theorem 19, if $s = 1$, $n = 2$, then*

$$\left|\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x)\mathrm{d}x\right| \le \frac{\ln b - \ln a}{2}\left(\frac{1}{2}\right)^{\frac{1}{q}} [L(a^{\frac{3q}{q-1}}, b^{\frac{3q}{q-1}})]^{1-\frac{1}{q}} \times [|f'(b)|^q + |f'(a)|^q]^{\frac{1}{q}}. \tag{20}$$

Theorem 25 *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be a differentiable function and $f' \in L^1([a,b])$ for $0 < a < b < \infty$. If $|f'|^q$ is an s-GA-convex function on $[0,b]$ for $s \in (0,1]$ and $q \ge 1$, then*

$$\left|\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1} f(x)\mathrm{d}x\right| \le \frac{\ln b - \ln a}{n}[G(s,(n+1)q)|f'(b)|^q + H(s,(n+1)q)|f'(a)|^q]^{\frac{1}{q}}, \tag{21}$$

where G and H are respectively defined by (3) *and* (8).

Proof. Since $|f'|^q$ is an s-GA-convex function on $[0,b]$, from Lemma 10 and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1} f(x)\mathrm{d}x\right| \\
\le&\frac{\ln b - \ln a}{n}\int_0^1 a^{(n+1)(1-t)} b^{(n+1)t}\left|f'(a^{1-t}b^t)\right|\mathrm{d}t \\
\le&\frac{\ln b - \ln a}{n}\left(\int_0^1 1^p \mathrm{d}t\right)^{\frac{1}{P}} \\
&\times\left[\int_0^1\left[a^{(n+1)(1-t)} b^{(n+1)t}\left|f'(a^{1-t}b^t)\right|\right]^q \mathrm{d}t\right]^{\frac{1}{q}} \\
=&\frac{\ln b - \ln a}{n}[G(s,(n+1)q)|f'(b)|^q \\
&+ H(s,(n+1)q)|f'(a)|^q]^{\frac{1}{q}}.
\end{aligned}$$

The proof of Theorem 25 is completed. □

Corollary 26 *Under the conditions of Theorem 25, if $s = 1$, then*

$$\begin{aligned}
&\left|\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1} f(x)\mathrm{d}x\right| \\
\le&\frac{(\ln b - \ln a)^{1-\frac{1}{q}}}{n}\left[\frac{1}{(n+1)q}\right]^{\frac{1}{q}} \\
&\times\left\{[b^{(n+1)q} - L(a^{(n+1)q}, b^{(n+1)q})]|f'(b)|^q\right. \\
&+[L(a^{(n+1)q}, b^{(n+1)q}) - a^{(n+1)q}] \\
&\left.\times|f'(a)|^q\right\}^{\frac{1}{q}}.
\end{aligned} \tag{22}$$

Proof. By

$$G(1,(n+1)q) = \frac{b^{(n+1)q} - L(a^{(n+1)q}, b^{(n+1)q})}{\ln b^{(n+1)q} - \ln a^{(n+1)q}}$$

and

$$H(1,(n+1)q) = L(a^{(n+1)q}, b^{(n+1)q}) - G(1,(n+1)q).$$

The corollary can be proved easily. □

Corollary 27 *Under the conditions of Theorem 25, if $s = 1$, $n = 1$, then*

$$\left|bf(b) - af(a) - \int_a^b f(x)\mathrm{d}x\right| \le (\ln b - \ln a)\left[G(1,2q)|f'(b)|^q + H(1,2q)|f'(a)|^q\right]^{\frac{1}{q}}, \tag{23}$$

So, the inequality in [28, Theorem 3.4] is recaptured.

Corollary 28 *Under the conditions of Theorem 25, if $s = 1$, $n = 2$ then*

$$\begin{aligned}
&\left|\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x)\mathrm{d}x\right| \\
\le&\frac{(\ln b - \ln a)^{1-\frac{1}{q}}}{2}\left(\frac{1}{3q}\right)^{\frac{1}{q}} \\
&\times\left\{[b^{3q} - L(a^{3q}, b^{3q})]|f'(b)|^q\right. \\
&\left.+[L(a^{3q}, b^{3q}) - a^{3q}]|f'(a)|^q\right\}^{\frac{1}{q}}.
\end{aligned} \tag{24}$$

Proof. By

$$G(1,3q) = \frac{b^{3q} - L(a^{3q}, b^{3q})}{\ln b^{3q} - \ln a^{3q}}$$

and

$$H(1,3q) = \frac{L(a^{3q}, b^{3q}) - a^{3q}}{\ln b^{3q} - \ln a^{3q}}.$$

The corollary can be proved easily. □

Corollary 29 *Under the conditions of Theorem 25, if* $n = 1$*, then*

$$\left|bf(b) - af(a) - \int_a^b f(x)\mathrm{d}x\right| \leq (\ln b - \ln a)[G(s,2q)|f'(b)|^q + H(s,2q)|f'(a)|^q]^{\frac{1}{q}}. \quad (25)$$

Corollary 30 *Under the conditions of Theorem 25, if* $n = 2$*, then*

$$\left|\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b xf(x)\mathrm{d}x\right| \leq \frac{\ln b - \ln a}{2}[G(s,3q)|f'(b)|^q + H(s,3q)|f'(a)|^q]^{\frac{1}{q}}. \quad (26)$$

Theorem 31 *Let* $f : \mathbb{R}_0 \to \mathbb{R}$ *be a differentiable function and* $f' \in L^1([a,b])$ *for* $0 < a < b < \infty$*. If* $|f'|^q$ *is an s-GA-convex function on* $[0,b]$ *for* $s \in (0,1]$*,* $q > 1, q > p > 0$*, and* $\frac{1}{p} + \frac{1}{q} = 1$*, then*

$$\left|\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1} f(x)\mathrm{d}x\right| \leq \frac{\ln b - \ln a}{n}[L(a^{\frac{(n+1)(q-p)}{q-1}}, b^{\frac{(n+1)(q-p)}{q-1}})]^{1-\frac{1}{q}} \times [G(s,(n+1)p)|f'(b)|^q + H(s,(n+1)p)|f'(a)|^q]^{\frac{1}{q}}, \quad (27)$$

where G *and* H *are respectively defined by* (3) *and* (8).

Proof. Since $|f'|^q$ is an s-GA-convex function on $[0,b]$, from Lemma 10 and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1} f(x)\mathrm{d}x\right| \\
\leq &\frac{\ln b - \ln a}{n} \times \int_0^1 a^{(n+1)(1-t)} b^{(n+1)t} \left|f'(a^{1-t}b^t)\right| \mathrm{d}t \\
\leq &\frac{\ln b - \ln a}{n} \times \left[\int_0^1 a^{\frac{(n+1)(q-p)(1-t)}{q-1}} b^{\frac{(n+1)(q-p)t}{q-1}} \mathrm{d}t\right]^{1-\frac{1}{q}} \times \left[\int_0^1 a^{(n+1)p(1-t)} b^{(n+1)pt} \left|f'(a^{1-t}b^t)\right|^q \mathrm{d}t\right]^{\frac{1}{q}} \\
= &\frac{\ln b - \ln a}{n}[L(a^{\frac{(n+1)(q-p)}{q-1}}, b^{\frac{(n+1)(q-p)}{q-1}})]^{1-\frac{1}{q}} \times [G(s,(n+1)p)|f'(b)|^q + H(s,(n+1)p)|f'(a)|^q]^{\frac{1}{q}}.
\end{aligned}$$

The proof of Theorem 31 is completed. □

Corollary 32 *Under the conditions of Theorem 31, if* $s = 1$*, then*

$$\begin{aligned}
&\left|\frac{b^n f(b) - a^n f(a)}{n} - \int_a^b x^{n-1} f(x)\mathrm{d}x\right| \\
\leq &\frac{(\ln b - \ln a)}{n}\left[\frac{1}{(n+1)p}\right]^{\frac{1}{q}} \times L\left(a^{\frac{(n+1)(q-p)}{q-1}}, b^{\frac{(n+1)(q-p)}{q-1}}\right)^{1-\frac{1}{q}} \\
&\times \{[(b^{(n+1)p} - L(a^{(n+1)p}, b^{(n+1)p})]|f'(b)|^q + [L(a^{(n+1)p}, b^{(n+1)p}) - a^{(n+1)p}] \times |f'(a)|^q\}^{\frac{1}{q}}.
\end{aligned} \quad (28)$$

Proof. By

$$G(1,(n+1)p) = \int_0^1 t a^{(n+1)p(1-t)} b^{(n+1)pt} \mathrm{d}t = \frac{b^{(n+1)p} - L(a^{(n+1)p}, b^{(n+1)p})}{\ln b^{(n+1)p} - \ln a^{(n+1)p}}$$

and

$$H(1,(n+1)p) = L(a^{(n+1)p}, b^{(n+1)p}) - G(1,(n+1)p).$$

The corollary can be proved easily. □

Corollary 33 *Under the conditions of Theorem 31, if* $n = 1$*, then*

$$\left|bf(b) - af(a) - \int_a^b f(x)\mathrm{d}x\right| \leq (\ln b - \ln a)[L(a^{\frac{2(q-p)}{q-1}}, b^{\frac{2(q-p)}{q-1}})]^{1-\frac{1}{q}} \times [G(s,2p)|f'(b)|^q + H(s,2p)|f'(a)|^q]^{\frac{1}{q}}. \quad (29)$$

Corollary 34 *Under the conditions of Theorem 31, if* $s = 1$*,* $n = 1$*, then*

$$\left|bf(b) - af(a) - \int_a^b f(x)\mathrm{d}x\right| \leq (\ln b - \ln a)\left[L\left(a^{\frac{2(q-p)}{q-1}}, b^{\frac{2(q-p)}{q-1}}\right)\right]^{1-\frac{1}{q}} \times \left[G(1,2p)|f'(b)|^q + H(1,2p)|f'(a)|^q\right]^{\frac{1}{q}}. \quad (30)$$

Corollary 35 *Under the conditions of Theorem 31, if* $n = 2$*, then*

$$\begin{aligned}&\left|\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x)\mathrm{d}x\right| \\ \le&\frac{(\ln b - \ln a)}{2}[L(a^{\frac{(n+1)(q-p)}{q-1}}, b^{\frac{(n+1)(q-p)}{q-1}})]^{1-\frac{1}{q}} \\ &\times [G(s,(n+1)p)|f'(b)|^q \\ &+ H(s,(n+1)p)|f'(a)|^q]^{\frac{1}{q}}. \qquad (31)\end{aligned}$$

Corollary 36 *Under the conditions of Theorem 31, if* $s = 1$*,* $n = 2$*, then*

$$\begin{aligned}&\left|\frac{b^2 f(b) - a^2 f(a)}{2} - \int_a^b x f(x)\mathrm{d}x\right| \\ \le&\frac{(\ln b - \ln a)}{2}\left[\frac{1}{3p}\right]^{\frac{1}{q}} \\ &\times L\left(a^{\frac{3(q-p)}{q-1}}, b^{\frac{3(q-p)}{q-1}}\right)^{1-\frac{1}{q}} \\ &\times \{[(b^{3p} - L(a^{3p}, b^{3p})]|f'(b)|^q \\ &+ [L(a^{3p}, b^{3p}) - a^{3p}] \\ &\times |f'(a)|^q\}^{\frac{1}{q}}. \qquad (32)\end{aligned}$$

Theorem 37 *Let* $f, g : \mathbb{R}_0 \to \mathbb{R}_0$ *be a differentiable function and* $fg \in L^1([a,b])$ *for* $0 < a < b < \infty$*. If* f^q *is an* s_1*-GA-convex function on* $[0,b]$ *and* g^q *is an* s_2*-GA-convex function on* $[0,b]$ *for* $s_1, s_2 \in (0,1]$ *and* $q \ge 1$*, then*

$$\begin{aligned}&\int_a^b f(x)g(x)\mathrm{d}x \\ \le&(\ln b - \ln a)[L(a,b)]^{1-\frac{1}{q}}[G(s_1+s_2,1)f^q(b)g^q(b) \\ &+ H(s_1+s_2,1)f^q(a)g^q(a) \\ &+ M(s_1,s_2,1)f^q(b)g^q(a) \\ &+ M(s_2,s_1,1)f^q(a)g^q(b)]^{\frac{1}{q}}, \qquad (33)\end{aligned}$$

where G*,* L*, and* H *are respectively defined by* (3)*,* (4) *and* (8) *and*

$$M(m,n,l) = \int_0^1 t^m(1-t)^n a^{l(1-t)} b^{lt}\mathrm{d}t. \qquad (34)$$

Proof. Since f^q is an s_1-GA-convex function on $[0,b]$ and g^q is an s_2-GA-convex function on $[0,b]$, we have

$$f^q(a^{1-t}b^t) \le t^{s_1} f^q(b) + (1-t)^{s_1} f^q(a)$$

and

$$g^q(a^{1-t}b^t) \le t^{s_2} g^q(b) + (1-t)^{s_2} g^q(a)$$

for $t \in [0,1]$. Letting $x = a^{1-t}b^t$ for $t \in [0,1]$, and using Hölder's inequality, we figure out

$$\begin{aligned}&\int_a^b f(x)g(x)\mathrm{d}x \\ =&\int_0^1 (\ln b - \ln a)a^{1-t}b^t f(a^{1-t}b^t)g(a^{1-t}b^t)\mathrm{d}t \\ \le&(\ln b - \ln a)\left(\int_0^1 a^{1-t}b^t\mathrm{d}t\right)^{1-\frac{1}{q}} \\ &\times \left\{\int_0^1 a^{1-t}b^t[t^{s_1}f^q(b) + (1-t)^{s_1}f^q(a)]\right. \\ &\left.\times [t^{s_2}g^q(b) + (1-t)^{s_2}g^q(a)]\mathrm{d}t\right\}^{\frac{1}{q}} \\ =&(\ln b - \ln a)[L(a,b)]^{1-\frac{1}{q}}[G(s_1+s_2,1)f^q(b)g^q(b) \\ &+ H(s_1+s_2,1)f^q(a)g^q(a) \\ &+ M(s_1,s_2,1)f^q(b)g^q(a) \\ &+ M(s_2,s_1,1)f^q(a)g^q(b)]^{\frac{1}{q}}.\end{aligned}$$

The proof of Theorem 37 is completed. □

Corollary 38 *Under the conditions of Theorem 37, if* $q = 1$*, then*

$$\begin{aligned}&\int_a^b f(x)g(x)\mathrm{d}x \\ \le&(\ln b - \ln a)[G(s_1+s_2,1)f(b)g(b) \\ &+ H(s_1+s_2,1)f(a)g(a) \\ &+ M(s_1,s_2,1)f(b)g(a) \\ &+ M(s_2,s_1,1)f(a)g(b)]. \qquad (35)\end{aligned}$$

Corollary 39 *Under the conditions of Theorem 37, if* $q = 1$ *and* $s_1 = s_2 = 1$*, then*

$$\begin{aligned}&\int_a^b f(x)g(x)\mathrm{d}x \\ \le&(\ln b - \ln a)\{G(2,1)[f(b)g(b) + f(a)g(a) \\ &- f(b)g(a) - f(a)g(b)] \\ &+ G(1,1)[f(b)g(a) + f(a)g(b) - 2f(a)g(a)] \\ &+ L(a,b)f(a)g(a)\}. \qquad (36)\end{aligned}$$

Proof. By

$$M(1,1,1) = G(1,1) - G(2,1)$$

and

$$H(2,1) = L(a,b) - 2G(1,1) + G(2,1).$$

The corollary can be proved easily. □

Theorem 40 *Let $f, g : \mathbb{R}_0 \to \mathbb{R}_0$ be a differentiable function and $fg \in L^1([a, b])$ for $0 < a < b < \infty$. If f^q is an s_1-GA-convex function on $[0, b]$ and $g^{\frac{q}{q-1}}$ is an s_2-GA-convex function on $[0, b]$ for $s_1, s_2 \in (0, 1]$ and $q \geq 1$, then*

$$\begin{aligned}&\int_a^b f(x)g(x)\mathrm{d}x\\ \leq&(\ln b - \ln a)[G(s_1, 1)f^q(b) + H(s_1, 1)f^q(a)]^{\frac{1}{q}}\\ &\times [G(s_2, 1)g^{\frac{q}{q-1}}(b)\\ &+ H(s_2, 1)g^{\frac{q}{q-1}}(a)]^{1-\frac{1}{q}}, \qquad (37)\end{aligned}$$

where G and H are respectively defined by (3) *and* (8).

Proof. By the s_1-GA-convexity of f^q and s_2-GA-convexity of $g^{\frac{q}{q-1}}$, we have

$$f^q(a^{1-t}b^t) \leq t^{s_1} f^q(b) + (1 - t)^{s_1} f^q(a)$$

and

$$g^{\frac{q}{q-1}}(a^{1-t}b^t) \leq t^{s_2} g^{\frac{q}{q-1}}(b) + (1 - t)^{s_2} g^{\frac{q}{q-1}}(a)$$

for $t \in [0, 1]$, letting $x = a^{1-t}b^t$ for $t \in [0, 1]$, and employing Hölder's inequality yield

$$\begin{aligned}&\int_a^b f(x)g(x)\mathrm{d}x\\ \leq&\left[\int_a^b f^q(x)\mathrm{d}x\right]^{\frac{1}{q}}\left[\int_a^b g^{\frac{q}{q-1}}(x)\mathrm{d}x\right]^{1-\frac{1}{q}}\\ \leq&(\ln b - \ln a)\left[\int_0^1 a^{1-t}b^t[t^{s_1} f^q(b)\right.\\ &\left.+ (1 - t)^{s_1} f^q(a)]\mathrm{d}t\right]^{\frac{1}{q}}\\ &\times\left[\int_0^1 a^{1-t}b^t[t^{s_2} g^{\frac{q}{q-1}}(b)\right.\\ &\left.+(1 - t)^{s_2} g^{\frac{q}{q-1}}(a)]\mathrm{d}t\right]^{1-\frac{1}{q}}\\ =&(\ln b - \ln a)[G(s_1, 1)f^q(b) + H(s_1, 1)f^q(a)]^{\frac{1}{q}}\\ &\times [G(s_2, 1)g^{\frac{q}{q-1}}(b) + H(s_2, 1)g^{\frac{q}{q-1}}(a)]^{1-\frac{1}{q}}.\end{aligned}$$

The proof of Theorem 40 is completed. □

Corollary 41 *Under the conditions of Theorem 40, if $s_1 = s_2 = 1$, then*

$$\begin{aligned}&\int_a^b f(x)g(x)\mathrm{d}x\\ \leq&\{[b - L(a, b)]f^q(b) + [L(a, b) - a]f^q(a)\}^{\frac{1}{q}}\\ &\times \{[b - L(a, b)]g^{\frac{q}{q-1}}(b)\\ &+ [L(a, b) - a]g^{\frac{q}{q-1}}(a)\}^{1-\frac{1}{q}}. \qquad (38)\end{aligned}$$

Proof. If $s_1 = s_2 = 1$, we have

$$G(1, 1) = \frac{b - L(a, b)}{\ln b - \ln a}$$

and

$$H(1, 1) = \frac{L(a, b) - a}{\ln b - \ln a}.$$

The corollary can be obtained easily. □

Theorem 42 *Let $f, g : \mathbb{R}_0 \to \mathbb{R}_0$ be a differentiable function and $fg \in L^1([a, b])$ for $0 < a < b < \infty$. If f is an s_1-GA-concave function on $[0, b]$ and g is an s_2-GA-concave function on $[0, b]$ for $s_1, s_2 \in (0, 1]$ and $q \geq 1$, then*

$$\begin{aligned}&\int_a^b f(x)g(x)\mathrm{d}x\\ \geq&(\ln b - \ln a)[G(s_1 + s_2, 1)f(b)g(b)\\ &+ H(s_1 + s_2, 1)f(a)g(a)\\ &+ M(s_1, s_2, 1)f(b)g(a)\\ &+ M(s_2, s_1, 1)f(a)g(b)], \qquad (39)\end{aligned}$$

where G, M and H are respectively defined by (3), (34) *and* (8).

Proof. Using the f is an s_1-GA-concave function on $[0, b]$ and g is an s_2-GA-concave function on $[0, b]$, we have

$$f(a^{1-t}b^t) \geq t^{s_1} f(b) + (1 - t)^{s_1}(a)$$

and

$$g(a^{1-t}b^t) \geq t^{s_2} g(b) + (1 - t)^{s_2} g(a)$$

for $t \in [0, 1]$, letting $x = a^{1-t}b^t$ for $t \in [0, 1]$ reveals

$$\begin{aligned}&\int_a^b f(x)g(x)\mathrm{d}x\\ =&\int_0^1 (\ln b - \ln a)a^{1-t}b^t f(a^{1-t}b^t)g(a^{1-t}b^t)\mathrm{d}t\\ \geq&(\ln b - \ln a)\left\{\int_0^1 a^{1-t}b^t[t^{s_1} f(b) + (1 - t)^{s_1}(a)]\right.\\ &\left.[t^{s_2} g(b) + (1 - t)^{s_2} g(a)]\mathrm{d}t\right\}\\ =&(\ln b - \ln a)[G(s_1 + s_2, 1)f(b)g(b)\\ &+ H(s_1 + s_2, 1)f(a)g(a)\\ &+ M(s_1, s_2, 1)f(b)g(a)\\ &+ M(s_2, s_1, 1)f(a)g(b)].\end{aligned}$$

The proof of Theorem 42 is completed. □

Corollary 43 *Under the conditions of Theorem 42, if* $s_1 = s_2 = 1$*, then we have*

$$\begin{aligned}
&\int_a^b f(x)g(x)\mathrm{d}x \\
&\geq \left[b - \frac{2b - 2L(a,b)}{\ln b - \ln a}\right] f(b)g(b) \\
&\quad + \frac{a + b - 2L(a,b)}{\ln b - \ln a}[f(b)g(a) + f(a)g(b)] \\
&\quad + \left[L(a,b) - a - \frac{a + b - 2L(a,b)}{\ln b - \ln a}\right] \\
&\quad \times f(a)g(a). \qquad (40)
\end{aligned}$$

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